Commutators in Harmonic Analysis: optimal estimates and asymptotic behavior

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Commutator of Riesz transform and multiplier: Euclidean setting

As usual, D_j is the (self-adjoint) j-th derivative and Δ is the (positive) Laplacian. Riesz transform R_j is defined as $D_j\Delta^{-\frac{1}{2}}$.
The following result was suggested by Connes, Sullivan and Teleman and eventually proved by Lord, McDonald, Sukochev and Zanin.

Theorem

We have $[R_j, M_f] \in \mathcal{L}_{d,\infty}(L_2(\mathbb{R}^d))$ *iff* $f \in \dot{W}^1_d(\mathbb{R}^d)$. *We have*

$$c_d^{-1}\|f\|_{\dot{W}^1_d(\mathbb{R}^d)} \leq \|[R_j, M_f]\|_{d,\infty} \leq c_d\|f\|_{\dot{W}^1_d(\mathbb{R}^d)}.$$

The proof is based on Double Operator Integrals (for the estimate from above) and on Connes Trace Theorem (for the estimate from below).

Reduction to commutator with power

By the Leibniz rule,

$$[R_j, M_f] = [D_j\Delta^{-\frac{1}{2}}, M_f] = [D_j, M_f]\Delta^{-\frac{1}{2}} + D_j[\Delta^{-\frac{1}{2}}, M_f].$$

Using the obvious formula $[D_j, M_f]$ and the resolvent identity, we write

$$[R_j, M_f] = M_{D_jf}\Delta^{-\frac{1}{2}} - D_j\Delta^{-\frac{1}{2}}[\Delta^{\frac{1}{2}}, M_f]\Delta^{-\frac{1}{2}} = M_{D_jf}\Delta^{-\frac{1}{2}} - R_j\cdot[\Delta^{\frac{1}{2}}, M_f]\Delta^{-\frac{1}{2}}.$$

It, therefore, suffices to estimate separately

$$M_{D_jf}\Delta^{-\frac{1}{2}} \text{ and } [\Delta^{\frac{1}{2}}, M_f]\Delta^{-\frac{1}{2}}.$$

Cwikel estimate

Theorem

If $d > 2$, then

$$\|M_f \Delta^{-\frac{1}{2}}\|_{d,\infty} \leq c_d \|f\|_d.$$

Applying this theorem with $D_j f$ instead of f, we estimate the first summand on slide 3.

To estimate the second summand on slide 3, we need not only Cwikel estimates, but also the prominent technique in Harmonic Analysis due to Birman and Solomyak — the Double Operator Integrals.

Double Operator Integrals I

Let $\nu_1, \nu_2 : \mathfrak{B}(\mathbb{R}) \to B(H)$ be commuting, countably additive, projection-valued, Borel measures on $\mathbb{R}$. The mapping

$$\nu_1 \times \nu_2 : A \times B \to \nu_1(A)\nu_2(B), \quad A, B \in \mathfrak{B}(\mathbb{R}),$$

extends to a countably additive Borel measure on $\mathbb{R}^2$.
Let X, Y be self-adjoint operators on H and let ν_1 and ν_2 be their spectral measures. Define commuting, countably additive, projection-valued, Borel measures $\bar{\nu}_1$ and $\bar{\nu}_2$ on $\mathbb{R}$ acting on the Hilbert space $\mathcal{L}_2(H)$ by the formulae

$$\bar{\nu}_1(A) : V \to \nu_1(A)V, \quad \bar{\nu}_2(A) : V \to V\nu_2(A), \quad A \in \mathfrak{B}(\mathbb{R}).$$

We need measure $\bar{\nu}_1 \times \bar{\nu}_2$.

Double Operator Integrals II

Let $\bar{\nu}_1 \times \bar{\nu}_2$ be the countably additive, projection valued, Borel measure on $\mathbb{R}$ acting on the Hilbert space $\mathcal{L}_2(H)$ as defined on slide 5.
For every bounded Borel measurable function F on $\mathbb{R}$, define a bounded linear mapping $T_F^{X,Y}$ on the Hilbert space $\mathcal{L}_2(H)$ by setting

$$T_F^{X,Y} = \int_{\mathbb{R}^2} F d(\bar{\nu}_1 \times \bar{\nu}_2).$$

This mapping is called the Double Operator Integral.
It is of utmost importance in Harmonic Analysis to know when the mapping $T_F^{X,Y} : \mathcal{L}_2(H) \to \mathcal{L}_2(H)$ extends to (or restricts to) a bounded mapping $T_F^{X,Y} : \mathcal{L}_p(H) \to \mathcal{L}_p(H)$.

Why Double Operator Integrals are important in studying commutators?

Let

$$F(\lambda,\mu)=\frac{\lambda}{\lambda+\mu},\quad \lambda,\mu>0.$$

Double Operator Integral

$$T_F^{X,X}:\mathcal{L}_{p,\infty}(H)\to\mathcal{L}_{p,\infty}(H),\quad 1<p<\infty,$$

is bounded.
We have

$$[X,Y]X^{-1}=T_F^{X,X}(X^{-1}[X^2,Y]X^{-1}).$$

Estimate from the above: final step

We are now able to estimate the second summand on slide 3.

$$\|[\Delta^{\frac{1}{2}}, M_f]\Delta^{-\frac{1}{2}}\|_{d,\infty} \leq \|T_F^{\Delta^{\frac{1}{2}},\Delta^{\frac{1}{2}}}\|_{\mathcal{L}_{d,\infty}\to\mathcal{L}_{d,\infty}}\|\Delta^{-\frac{1}{2}}[\Delta, M_f]\Delta^{-\frac{1}{2}}\|_{d,\infty}.$$

Now, we write

$$[\Delta, M_f] = \sum_{i=1}^{d}[D_i^2, M_f] = \sum_{i=1}^{d} D_i M_{D_i f} + M_{D_i f} D_i.$$

Hence,

$$\Delta^{-\frac{1}{2}}[\Delta, M_f]\Delta^{-\frac{1}{2}} = \sum_{i=1}^{d} R_i \cdot M_{D_i f}\Delta^{-\frac{1}{2}} + \Delta^{-\frac{1}{2}} M_{D_i f} \cdot R_i.$$

Each of the summands can be estimated via Cwikel theorem.

Principal symbol calculus

Let $\mathcal{A}_1 = C_0(\mathbb{R}^d)$ and $\mathcal{A}_2 = C(\mathbb{S}^{d-1})$. Let π_1 and π_2 be the representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ on $L_2(\mathbb{R}^d)$ defined as follows:

$$\pi_1(f) = M_f, \quad \pi_2(g) = g(\frac{\nabla}{\sqrt{\Delta}}).$$

Theorem

Let Π be the C^*-algebra generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$. There exists a $*$-homomorphism $\mathrm{sym} : \Pi \to C_0(\mathbb{R}^d \times \mathbb{S}^{d-1})$ such that

$$\mathrm{sym}(\pi_1(f))(t,s) = f(t), \quad \mathrm{sym}(\pi_2(g))(t,s) = g(s), \quad t \in \mathbb{R}^d, \quad s \in \mathbb{S}^{d-1}.$$

Main Asymptotic Result

We say that $T \in B(L_2(\mathbb{R}^d))$ is compactly supported from the right if there exists $f \in C_c(\mathbb{R}^d)$ such that $T = TM_f$.

Theorem

Let $d \geq 3$. If $T \in \Pi$ is compactly supported from the right, then there exists a limit

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, T\Delta^{-\frac{1}{2}}) = c_d\|\mathrm{sym}(T)\|_{L_d(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

We now sketch the proof of this theorem.

Asymptotic result for monomials

Theorem

Let $d \geq 3$. If $f \in C_c(\mathbb{R}^d)$ and $g \in C(\mathbb{S}^{d-1})$, then

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, \pi_1(f)\pi_2(g)\Delta^{-\frac{1}{2}}) = c_d\|f\|_d\|g\|_d.$$

What we actually prove is

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t, \pi_1(f)\pi_2(|g|^2)\Delta^{-1}\pi_1(\bar{f})) = c_d^2\|f\|_d^2\|g\|_d^2.$$

Asymptotic result for monomials on torus

Theorem

Let $d \geq 3$. If $f \in C(\mathbb{T}^d)$ and $g \in C(\mathbb{S}^{d-1})$, then

$$\lim_{t\to\infty} t^{\frac{1}{d}} \mu(t, M_f g(\frac{\nabla_{\mathbb{T}^d}}{\Delta_{\mathbb{T}^d}^{\frac{1}{2}}})\Delta_{\mathbb{T}^d}^{-\frac{1}{2}}) = c_d \|f\|_d \|g\|_d.$$

Proof of this theorem is omitted due to the lack of time.

Proof of asymptotic result for monomials

Without loss of generality, f is supported on $[0,1]^d$. We can also view f as a function on torus $\mathbb{T}^d$. An elementary computation shows that

$$\pi_1(f)\pi_2(|g|^2)\Delta^{-1}\pi_1(\bar{f})\Big|_{L_2([0,1]^d)} = M_f a(\nabla_{\mathbb{T}^d})M_{\bar{f}}\Big|_{L_2([0,1]^d)}.$$

Another elementary computation shows that

$$a(n) = |g|^2\left(\frac{n}{|n|}\right)|n|^{-2} + O(|n|^{-4}).$$

The asymptotic result for monomials in the Euclidean setting follows now from the corresponding result in torical setting.

Asymptotic result for sum of monomials

Theorem

Let $d \geq 3$. If $f_k \in C_c(\mathbb{R}^d)$ and $g_k \in C(\mathbb{S}^{d-1})$, $1 \leq k \leq m$, then

$$\lim_{t \to \infty} t^{\frac{1}{d}} \mu(t, \sum_{k=1}^{m} \pi_1(f_k)\pi_2(g_k)\Delta^{-\frac{1}{2}}) = c_d \| \sum_{k=1}^{m} f_k \otimes g_k \|_d.$$

Proof of this theorem is omitted due to the lack of time.

Proof of the Main Asymptotic Result

Let $T \in \Pi$. There exist

$$T_n = \sum_{k=1}^{m_n} \pi_1(f_{n,k})\pi_2(g_{n,k})$$

such that $T_n \to T$ in the uniform norm as $n \to \infty$. If $T = TM_f$ for some $f \in C_c(\mathbb{R}^d)$, then we may assume without loss of generality that $f_{n,k} = f \cdot f_{n,k}$ for every $1 \leq k \leq m_n$ and for every $n \in \mathbb{N}$.
By theorem on the slide 14, for every $n \in \mathbb{N}$, there exists a limit

$$\lim_{t \to \infty} t^{\frac{1}{d}} \mu(t, T_n \Delta^{-\frac{1}{2}}) = c_d \|\mathrm{sym}(T_n)\|_{L_d(\mathbb{R}^d \times \mathbb{S}^{d-1})}.$$

The assertion follows now from the Birman-Solomyak approximation lemma.

Approximate expression for the commutator

If $f \in C_c^\infty(\mathbb{R}^d)$, then

$$[R_j, M_f] - T_j \Delta^{-\frac{1}{2}} \in \mathcal{L}_{\frac{d}{2},\infty}(\mathbb{R}^d),$$

where

$$T_j = M_{D_j f} - \sum_{k=1}^{d} R_k R_j M_{D_k f}.$$

Asymptotic result for the commutator

Theorem

Let $d \geq 3$. For every $f \in \dot{W}_d^1(\mathbb{R}^d)$, we have

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t,[R_j,M_f]) = c_d^{(1)}\||\nabla f|\|_d.$$

For $f \in C_c^\infty(\mathbb{R}^d)$, it follows from the Main Asymptotic Result and the approximate expression for the commutator on slide 16 that

$$\lim_{t\to\infty} t^{\frac{1}{d}}\mu(t,[R_j,M_f]) = c_d\|D_jf\otimes 1-\sum_{k=1}^{d} D_kf\otimes s_ks_j\|_{L_d(\mathbb{R}^d\times\mathbb{S}^{d-1})} = c_d^{(1)}\||\nabla f|\|_d.$$

In general case, the assertion follows from the Birman-Solomyak approximation lemma.