Infinite quantum permutations

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Introduction

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The definition of quantum symmetry for finite graphs is due to Banica and Bichon, and this notion has been studied extensively.

It is natural to look for an analogue of these constructions for infinite graphs.

The aim of my talk is to describe certain *discrete* quantum automorphism groups associated to arbitrary infinite graphs.

Quantum permutations

Quantum permutations

Definition

Let I be a set. A *quantum permutation* of I is a pair $(\mathcal{H}, u)$ of a Hilbert space $\mathcal{H}$ and a family $u = (u_{ij})_{i,j\in I}$ of projections $u_{ij} \in B(\mathcal{H})$ such that

- For every $i \in I$ the projections u_{ij} for $j \in I$ are pairwise orthogonal,
- For every $j \in I$ the projections u_{ij} for $i \in I$ are pairwise orthogonal,
- We have
$$\sum_{k\in I} u_{ik} = 1 = \sum_{k\in I} u_{kj}$$
for all $i, j \in I$, with convergence in the strong operator topology.

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for all $i, j \in I$, with convergence in the strong operator topology.

Lemma

Quantum permutations of a set I whose underlying Hilbert space is $\mathbb{C}$ are the same thing as permutations of I.

Some further definitions

Some further definitions

Fix a set I throughout.

- If $\sigma = (\mathcal{H}, u)$ and $\tau = (\mathcal{K}, v)$ are quantum permutations then an *intertwiner* from σ to τ is a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ such that $Tu_{ij} = v_{ij}T$ for all $i, j \in I$.
- The *direct sum* of quantum permutations $\sigma = (\mathcal{H}, u)$ and $\tau = (\mathcal{K}, v)$ is defined by $\sigma \oplus \tau = (\mathcal{H} \oplus \mathcal{K}, u \oplus v)$, where $(u \oplus v)_{ij} = u_{ij} \oplus v_{ij}$ for all $i, j \in I$.
- The *tensor product* of quantum permutations $\sigma = (\mathcal{H}_\sigma, u^\sigma)$ and $\tau = (\mathcal{H}_\tau, u^\tau)$ is defined by $\sigma \otimes \tau = (\mathcal{H}_\sigma \otimes \mathcal{H}_\tau, u^\sigma ⫪ u^\tau)$ where $(u^\sigma ⫪ u^\tau)_{ij} = \sum_{k \in I} u^\sigma_{ik} \otimes u^\tau_{kj}$ for all $i, j \in I$.
- The *contragredient* $\overline{\sigma} = (\mathcal{H}_{\overline{\sigma}}, u^{\overline{\sigma}})$ of a quantum permutation $\sigma = (\mathcal{H}_\sigma, u^\sigma)$ is defined by taking $\mathcal{H}_{\overline{\sigma}}$ to be the conjugate Hilbert space of $\mathcal{H}_\sigma$ and the family of projections $u^{\overline{\sigma}} = (u^{\overline{\sigma}}_{ij})$ determined by $u^{\overline{\sigma}}_{ij}(\overline{\xi}) = \overline{u^\sigma_{ji}(\xi)}$ for $\xi \in \mathcal{H}_\sigma$.

Upshot

Quantum permutations of a set I form naturally a concrete C^*-tensor category.

Some further definitions

Some further definitions

We keep the set I.

- By the *dimension* of a quantum permutation $\sigma = (\mathcal{H}, u)$ we mean the dimension of $\mathcal{H}$.
- A quantum permutation $(\mathcal{H}, u)$ is called *irreducible* if every intertwiner $(\mathcal{H}, u) \to (\mathcal{H}, u)$ is a scalar multiple of the identity.
- A quantum permutation $(\mathcal{H}, u)$ is called *classical* if the C^*-algebra generated by the projections u_{ij} is abelian.

Examples

Examples

Lemma

Every quantum permutation of a set I with $|I| = 1, 2, 3$ is classical.

Proof.

This is obvious for $|I| = 1$. For $|I| = 2$ notice that we must have

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

for some projection p.
For $|I| = 3$ write $I = \{1, 2, 3\}$. Enough to show that u_{ij} and u_{kl} commute provided $i \neq j$ and $k \neq l$. Consider e.g. u_{11} and u_{22}. We get

$$u_{11}u_{22}u_{13} = u_{11}(1 - u_{21} - u_{23})u_{13} = 0,$$

which implies $u_{11}u_{22} = u_{11}u_{22}(u_{11} + u_{12} + u_{13}) = u_{11}u_{22}u_{11}$. This yields

$$u_{11}u_{22} = u_{11}u_{22}u_{11} = (u_{11}u_{22}u_{11})^* = (u_{11}u_{22})^* = u_{22}u_{11}$$

as required. □

Examples

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As soon as I has more than 3 elements one can find non-classical quantum permutations.

Indeed, for arbitrary projections p, q the matrix

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Theorem (Banica-Bichon 2009)

For $|I| = 4$ the irreducible quantum permutations of I have dimension $1, 2$ or 4.

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Theorem (Banica-Bichon 2009)

For $|I| = 4$ the irreducible quantum permutations of I have dimension $1, 2$ or 4.

Combining quantum permutations of finite subsets of an infinite set I with classical permutations of I one obtains a basic supply of infinite quantum permutations.

Discrete quantum groups

Discrete quantum groups

Let Γ be a discrete group. Then the group structure of Γ can be encoded by the C^*-algebra $c_0(\Gamma)$ of functions on Γ together with its comultiplication, that is, the nondegenerate $*$-homomorphism

$$\Delta : c_0(\Gamma) \to M(c_0(\Gamma) \otimes c_0(\Gamma)) = c_b(\Gamma \times \Gamma)$$

given by

$$\Delta(f)(s, t) = f(st)$$

This clearly satisfies coassociativity, namely $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

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A discrete quantum group is given by a C^*-algebra

$$A = c_0\text{-}\bigoplus_{\pi \in \Pi} M_{n_\pi}(\mathbb{C})$$

together with a nondegenerate $*$-homomorphism $\Delta : A \to M(A \otimes A)$ such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and certain additional properties.

By slight abuse of notation, we write $A = c_0(\Gamma)$ and refer to (the non-existing object) Γ as a discrete quantum group.

Quantum permutation groups

Quantum permutation groups

Definition

The (discrete) *quantum permutation group* of I is the discrete quantum group $\mathrm{Sym}^+(I)$ associated to the (concrete) rigid C^*-tensor category $\mathbf{Sym}^+(I)$ of all finite dimensional quantum permutations of I via Tannaka-Krein reconstruction.

Explicitly, the underlying C^*-algebra of functions on $\mathrm{Sym}^+(I)$ can be written as the C^*-direct sum of matrix algebras

$$C_0(\mathrm{Sym}^+(I)) = \bigoplus_\sigma B(\mathcal{H}_\sigma),$$

indexed by the isomorphism classes of irreducible objects in $\mathbf{Sym}^+(I)$.

The coproduct is the uniquely determined nondegenerate $*$-homomorphism $\Delta : C_0(\mathrm{Sym}^+(I)) \to M(C_0(\mathrm{Sym}^+(I)) \otimes C_0(\mathrm{Sym}^+(I)))$ satisfying

$$\Delta(u_{ij}) = \sum_{k\in I} u_{ik} \otimes u_{kj}$$

for all $i, j \in I$, with the sum converging in the strict topology.

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Consider $I = \{1, \ldots, n\}$ and abbreviate $\mathrm{Sym}^+(I) = \mathrm{Sym}_n^+$.

If $n \geq 4$ then there exist uncountably many nonisomorphic quantum permutations of $\{1, \ldots, n\}$ in every positive dimension, and Sym_n^+ is *not* isomorphic to Wang's quantum permutation group S_n^+.

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In fact, Sym_n^+ is the *discretization* of S_n^+ in the following sense.

Definition (Sołtan)

Let G be a compact quantum group. Then the discretization of G is the discrete quantum group G_δ associated to the concrete (rigid) C^*-tensor category of finite dimensional unital $*$-representations of the universal C^*-algebra $C(G)$ of G.

In other words, the Pontrjagin dual (compact) quantum group of G_δ can be viewed as the *Bohr compactification* of the dual (discrete) quantum group of G.

Universal property

Universal property

Proposition

The quantum permutation group Sym_n^+ *is the universal discrete quantum group acting on* $\{1, \dots, n\}$.

That is, if H is any discrete quantum group and $\gamma : B \to M(C_0(H) \otimes B)$ is a coaction on $B = C(\{1, \dots, n\})$ preserving the uniform measure there exists a unique morphism of quantum groups $\iota^* : C_0(\mathrm{Sym}_n^+) \to M(C_0(H))$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & M(C_0(\mathrm{Sym}_n^+) \otimes B) \arrow[d, "\iota^* \otimes \mathrm{id}"] \\ & M(C_0(H) \otimes B) \end{tikzcd}$$

is commutative.

Finitary quantum permutation groups

Classically, one obtains a subgroup $\Sigma(I) \subset \mathrm{Sym}(I)$ by considering all *finitary* permutations, that is, permutations which move only finitely many points of I. Equivalently, one can view $\Sigma(I) = \varinjlim_{F \subset I} \Sigma(F)$ as the direct limit of the permutation groups $\mathrm{Sym}(F) = \Sigma(F)$ taken over the finite subsets $F \subset I$.

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This translates easily to $\mathrm{Sym}^+(I)$. More precisely, consider the full subcategory of the C^*-tensor category $\mathbf{Sym}^+(I)$ formed by all quantum permutations $\sigma = (\mathcal{H}_\sigma, u^\sigma)$ for which there exists a finite set $F \subset I$ such that $u^\sigma_{ij} \neq \delta_{ij}$ only for $i, j \in F$. In this case we say that σ is *finitary*.

Definition

The finitary quantum permutation group of a set I is the discrete quantum group $\Sigma^+(I)$ obtained from the concrete rigid C^*-tensor category of all finite dimensional finitary quantum permutations of I via Tannaka-Krein reconstruction.

We clearly have $\mathrm{Sym}^+(I) = \Sigma^+(I)$ iff I is finite. In the same way as in the classical case one can write $\Sigma^+(I) = \varinjlim_{F \subset I} \Sigma^+(F)$ as direct limit of the quantum permutation groups of the finite subsets $F \subset I$.

Non-amenability

Non-amenability

Theorem

For $|I| \geq 4$ the quantum permutation group $\Sigma^+(I)$ is non-amenable.

Proof.

Idea: It suffices to consider case $|I| = 4$. According to Banica-Bichon, there is a *matrix model* for S_4^+, giving an (explicit) injective $*$-homomorphism $C(S_4^+) \to M_4(C(SO(3)))$.
Using the fact that the discretization $SO(3)_\delta$ of $SO(3)$ contains free subgroups one can then cook up a finitely generated quantum subgroup Γ of Σ_4^+ and apply Kyed's Følner criterion to show that Γ is non-amenable. $\square$

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Remark

If $|I| \geq 4$ then neither $\mathrm{Sym}^+(I)$ nor $\Sigma^+(I)$ are finitely generated, so do not have property (T). If I is infinite the quantum group $\mathrm{Sym}^+(I)$ is not weakly amenable, and does not have the Haagerup property.

Quantum symmetries of graphs

Quantum symmetries of graphs

By a *graph* X we mean an undirected simple graph without self-edges. We write $X = (I_X, E_X)$ where I_X is the set of vertices and $E_X \subset I_X \times I_X$ is the set of edges of X.

Write rel for the function on pairs of vertices which describes the adjacency relation, taking the values *equal*, *adjacent*, *distinct and non-adjacent*.

Definition

Let $X = (I_X, E_X)$ be a graph. A *quantum automorphism* of X is a quantum permutation $\sigma = (\mathcal{H}, u)$ of I_X such that

$$u_{i_1 j_1} u_{i_2 j_2} = 0$$

if $\mathrm{rel}(i_1, i_2) \neq \mathrm{rel}(j_1, j_2)$.

Terminology

We say that X *has quantum symmetry* if X admits a non-classical quantum automorphism, and that X *has no quantum symmetry* otherwise.

Quantum automorphism groups

Quantum automorphism groups

A quantum automorphism of a graph X is the same thing as a quantum permutation $\sigma = (\mathcal{H}, u)$ of I_X such that

$$A_X u = u A_X$$

as matrices in $M_{I_X}(B(\mathcal{H}))$, where A_X is the adjacency matrix of X. Note here that the entries of these matrix products in this formula make sense in the strong operator topology.

Definition

Let $X = (I_X, E_X)$ be a graph. The discrete quantum automorphism group $\mathrm{Qut}_\delta(X)$ is the quantum subgroup of $\mathrm{Sym}^+(I_X)$ corresponding to the concrete rigid C^*-tensor category of finite dimensional quantum automorphisms of X.

If X is finite then

$$\mathrm{Qut}_\delta(X) = \mathrm{Qut}(X)_\delta$$

is the discretization of Banica's quantum automorphism group $\mathrm{Qut}(X)$ of X.

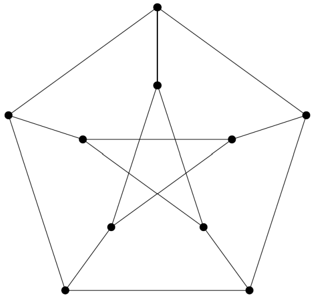
The same is true if X is a locally finite connected graph and $\mathrm{Qut}(X)$ is the quantum automorphism group in the sense of Rollier-Vaes.

Graphs with no quantum symmetry

Graphs with no quantum symmetry

Theorem (Schmidt 2018)

The Petersen graph has no quantum symmetry.



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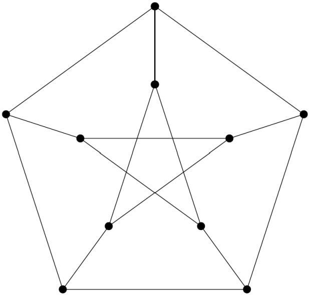

Theorem (Lupini-Mančinska-Roberson 2017)

Almost all finite graphs have no quantum symmetry.

Graphs with no quantum symmetry

Graphs with no quantum symmetry

The *infinite Johnson graph* $J(\infty, k)$ is the graph with vertices given by all k-element subsets of $\mathbb{N}$, such that two vertices are connected by an edge iff their intersection contains $k-1$ elements.

This graph has diameter k and is distance transitive.

Proposition

The Johnson graph $J(\infty, 2)$ *has no quantum symmetry.*

The proof is an easy adaption of a corresponding result for finite Johnson graphs due to Schmidt.

In fact, many of the criteria and techniques developed by Schmidt carry over to the infinite setting.

Examples of graphs with quantum symmetry

Two automorphisms σ, τ of a graph X are called *disjoint* iff $\sigma(i) \neq i$ implies $\tau(i) = i$ and $\tau(i) \neq i$ implies $\sigma(i) = i$.

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The following is again an easy generalisation of a result due to Schmidt.

Proposition

If X admits a pair of disjoint automorphisms then X has quantum symmetry.

This result allows one to give a range of examples of graphs with quantum symmetry.

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Theorem (Junk-Schmidt-Weber 2019)

Almost all finite trees have quantum symmetry.

An intriguing example

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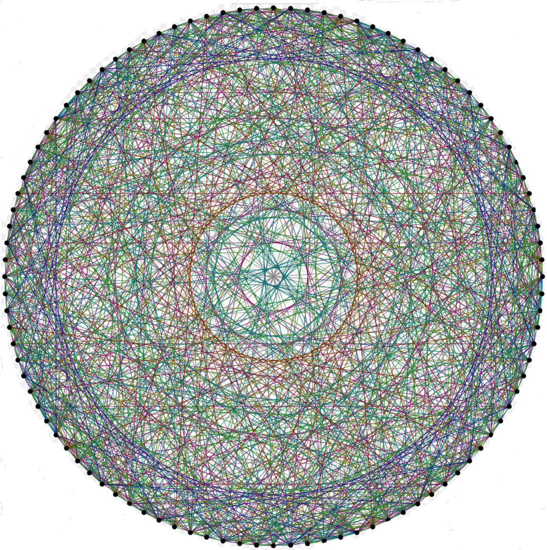

An intriguing example

The *Higman-Sims* graph HS is a graph with the following properties:

- HS has 100 vertices
- Every vertex has 22 neighbours
- HS is triangle-free
- If $a \neq b$ and $a \sim b$ then there are exactly 6 vertices c such that $a \sim c, b \sim c$
- If a, b, c, are distinct and not connected then they have two common neighbours.

Theorem (Jaeger-Kuperberg)

The quantum automorphism group $\mathrm{Qut}(HS)$ *is monoidally equivalent to* $SO_q(5)$ *where* $q = (\frac{1+\sqrt{5}}{2})^2$.

Disjoint unions

Disjoint unions

Let $(X_j)_{j \in J}$ be a collection of graphs labelled by some index set J and write $X = \bigcup_{j \in J} X_j$ for their disjoint union, so that $V_X = \bigcup_{j \in J} I_{X_j}$ and $E_X = \bigcup_{j \in J} E_{X_i}$.

Theorem

Let X be a connected graph. Then there is a canonical isomorphism

$$\mathrm{Qut}_\delta \left(\bigcup_{j \in J} X \right) \cong \mathrm{Qut}_\delta(X) \operatorname{Wr}^* \operatorname{Sym}^+(J)$$

of discrete quantum groups.

Here the *unrestricted free wreath product* $\Gamma \operatorname{Wr}^* \operatorname{Sym}^+(J)$ for a discrete quantum group Γ and a set J is constructed from a suitable C^*-tensor category.

If $\Gamma = G_\delta$ for a compact quantum group G and $J = \{1, \dots, n\}$ then

$$(G \wr^* S_n^+)_\delta \cong G_\delta \operatorname{Wr}^* \operatorname{Sym}_n^+$$

where $G \wr^* S_n^+$ is the free wreath product defined by Bichon.

The Rado graph

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Definition

The Rado graph is the graph R with vertex set $V_R = \mathbb{N}$ such that a pair of vertices (m, n) is an edge iff $m < n$ and the m-th digit in the binary expansion of n is odd, or the same with the roles of m and n reversed.

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Key property

For any pair of disjoint finite sets A, B of vertices in the Rado graph R there exists a vertex w in R outside $A \cup B$ such that $(x, w) \in E_R$ for all $x \in A$ and $(y, w) \notin E_R$ for all $y \in B$.

The Rado graph

In fact, the random graph R is *homogeneous*, which means that any finite partial automorphism of R can be extended to a global automorphism.

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- R contains *all* countable graphs as induced subgraphs.
- If one removes a finite number of vertices (and all adjacent edges) or a finite number edges from R the resulting graph is again isomorphic to R.
- The set of vertices of R can be split into infinitely many disjoint sets such that the induced subgraphs are isomorphic to R.

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The Rado graph does not admit any non-classical finite dimensional quantum automorphisms.

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Proposition

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That is, the quantum automorphism group $\mathrm{Qut}_\delta(R)$ is classical.

Question

Does the Rado graph have quantum symmetry?

Unit distance graphs

Unit distance graphs

A *unit distance graph* is a graph obtained by taking a subset of $\mathbb{R}^d$ as vertex set and connecting two vertices iff their Euclidean distance is equal to 1.

Examples of finite unit distance graphs in the plane include cycle graphs, hypercube graphs, and the Petersen graph.

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Consider the unit distance graph U_d associated to $\mathbb{R}^d$.

Proposition

The quantum automorphism group $\mathrm{Qut}_\delta(U_1)$ *is isomorphic to the free wreath product* $\mathrm{Aut}(L) \mathrm{Wr}^* \mathrm{Sym}(\mathbb{R}/\mathbb{Z})$*, where* L *is the "infinite line" graph, i.e. the Cayley graph of* $\mathbb{Z}$ *with respect to the standard generating set* $\{\pm 1\}$*.*

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Question

Does U_d for $d \geq 2$ have quantum symmetry?