On Hochschild cohomology of uniform Roe algebras with coefficients in uniform Roe bimodules

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Harbin Functional Analysis Seminar, July 6, 2022

Uniform Roe algebras and bimodules

Let X be a countable discrete metric space with a metric d_X. Let $l^2(X)$ denote the Hilbert space of square-summable functions on X with the fixed orthonormal basis consisting of the delta functions δ_x, $x \in X$, and let $\mathbb{B}(l^2(X))$ (resp., $\mathbb{K}(l^2(X))$) denote the C^*-algebra of all bounded (resp., compact) operators on $l^2(X)$.

For an operator $a \in \mathbb{B}(l^2(X))$, we denote its matrix elements by a_{xy}, $x, y \in X$, where $a_{x,y} = \langle \delta_x, a\delta_y \rangle$. An operator $a \in \mathbb{B}(l^2(X))$ has propagation $\leq r$ if $a_{xy} = 0$ whenever $d_X(x, y) > r$. The norm closure $C^*_u(X)$ of the set of all bounded operators of finite propagation is a C^*-algebra called the uniform Roe algebra of X.

Uniform Roe algebras play an important role in noncommutative geometry.

Coarse equivalence

Recall that two metrics, d_X and b_X, on X are coarsely equivalent if there exists a homeomorphism φ on $[0, \infty)$ such that

$$d_X(x, y) \leq \varphi(b_X(x, y)) \quad \text{and} \quad b_X(x, y) \leq \varphi(d_X(x, y))$$

for any $x, y \in X$. Clearly, the uniform Roe algebra of X does not depend on the metric within its coarse equivalence class.

Example

The sets $\mathbb{N}$ and $X = \{n^2 : n \in \mathbb{N}\}$ (with the standard metric) are not coarsely equivalent. The sets X and $Y = \{2^n : n \in \mathbb{N}\}$ are coarsely equivalent.

Asymptotic dimension is a coarse invariant.

Bounded geometry condition

A metric space X is of bounded geometry if for any $r > 0$, the number of points in any ball of radius r is bounded by some constant depending on r. Bounded geometry condition also does not depend on the metric within its coarse equivalence class.

Example

A finitely generated group with the word length metric is of bounded geometry.

Example

The set $\{\log n : n \in \mathbb{N}\}$ is not of bounded geometry.

Uniform Roe bimodules

Let (X, d_X) and (Y, d_Y) be discrete metric spaces, and let ρ be a metric on $X \sqcup Y$ such that $\rho|_X$ and $\rho|_Y$ are coarsely equivalent to d_X and d_Y, respectively. We call such metric ρ *compatible* with d_X and d_Y. Let $M_\rho \subset \mathbb{B}(l^2(Y), l^2(X))$ be the norm closure of the set of all bounded operators from $l^2(Y)$ to $l^2(X)$ of finite propagation with respect to ρ. Clearly, M_ρ is a right $C^*_u(Y)$-module and a left $C^*_u(X)$-module, and the two module structures commute. When $(Y, d_Y) = (X, d_X)$, M_ρ is a $C^*_u(X)$-bimodule. We call it the uniform Roe bimodule determined by (the coarse equivalence class of) ρ.

Expanding sequences of subsets

Let $\mathcal{D} = (D_n)_{n\in\mathbb{N}}$ be a sequence of subsets in X such that

(1) D_n is non-empty for some $n \in \mathbb{N}$;

(2) there exists $r > 0$ such that $N_{kr}(D_n) \subset D_{n+k}$ for any $n, k \in \mathbb{N}$, where $N_r(D)$ denotes the r-neighborhood of the set $D \subset X$.

We call such sequences expanding sequences. Note that $\cup_{n\in\mathbb{N}} D_n = X$. Two expanding sequences, $\mathcal{D}$ and $\mathcal{D}'$, are equivalent if there exists a monotone self-map φ of $\mathbb{N}$ such that $D_n \subset D'_{\varphi(n)}$ and $D'_n \subset D_{\varphi(n)}$ for any $n \in \mathbb{N}$.

Metrics on doubles and expanding sequences

In what follows, it may be useful to distinguish between the two copies of X in $X \sqcup X = X \times \{0,1\}$. For shortness' sake, we write x for $(x, 0)$, and x' for $(x, 1)$.

Let ρ be a metric on $X \sqcup X$ compatible with d_X. Set

$$D_n = \{x \in X : \rho(x, x') \leq n\} \subset X.$$

Then clearly $N_{k/2}(D_n) \subset D_{n+k}$ for any $k, n \in \mathbb{N}$, where the distance is with respect to the metric $\rho|_X$. Replacing it by the metrix d_X, we can find $r > 0$ such that the r-neighborhood with respect to d_X is contained in the $1/2$-neighborhood with respect to the metric $\rho|_X$, hence $\mathcal{D} = (D_n)_{n\in\mathbb{N}}$ is an expanding sequence. For a metric ρ' coarsely equivalent to ρ (and, therefore, compatible with d_X), let $\mathcal{D}' = (D'_n)_{n\in\mathbb{N}}$ be its expanding sequence. Then $\mathcal{D}$ and $\mathcal{D}'$ are equivalent.

Let A be a C^*-algebra, and let M be a Banach A-A-bimodule. Let $C^n(A,M)$ denote the space of all bounded multilinear maps from A^n to M. The Hochschild differential $\delta_n : C^n(A,M) \to C^{n+1}(A,M)$ is defined by

$$\begin{aligned}\delta_n(\varphi)(a_1,\ldots,a_{n+1}) &= a_1\varphi(a_2,\ldots,a_{n+1})\\ &+ \sum_{i=1}^{n}(-1)^i\varphi(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})\\ &+ (-1)^{n+1}\varphi(a_1,\ldots,a_n)a_{n+1},\end{aligned}$$

$a_1,\ldots,a_{n+1} \in A$, $\varphi \in C^n(A,M)$. The Hochschild cohomology groups $HH^n(A,M)$ are standardly defined by $HH^n(A,M) = \operatorname{Ker}\delta_n / \operatorname{Im}\delta_{n-1}$.

Derivations

When $n = 1$, elements of $C^1(A, M)$ are linear bounded functions $d : A \to M$, and

$$\operatorname{Ker} \delta_1 = \{d \in C^1(A, M) : d(ab) = d(a)b + ad(b) \text{ for any } a, b \in A\}.$$

Such maps are called derivations. A derivation d is inner if there exists $m \in M$ such that $d(a) = [a, m] = am - ma$ for any $a \in A$ (then $d = \delta_0(m)$). The set of all (resp., of all inner) derivations is denoted by $\operatorname{Der}(A, M)$ (resp., by $\operatorname{Inn}(A, M)$). The quotient $\operatorname{Out}(A, M) = \operatorname{Der}(A, M)/\operatorname{Inn}(A, M)$ is the space of the outer derivations, which is identified with $HH^1(A, M)$.

Some C^*-algebras of functions on X

Let $C_b(X)$ denote the algebra of all bounded continuous functions on X. For $f \in C_b(X)$, $r > 0$, set

$$\mathrm{var}_{x,r} f = \max_{y \in X : d_X(x, y \leq r)} |f(y) - f(x)|.$$

Given an expanding sequence $\mathcal{D}$ in X, let us define two C^*-algebras of functions on X related to $\mathcal{D}$, $C_0(X, \mathcal{D})$ and $C_h(X, \mathcal{D})$.

1. Let $f \in C_b(X)$. We say that f vanishes far from $\mathcal{D}$ if

$$\lim_{n \to \infty} \sup_{x \in X \setminus D_n} |f(x)| = 0.$$

Denote the C^*-algebra of all bounded functions vanishing far from $\mathcal{D}$ by $C_0(X, \mathcal{D})$.

Some C^*-algebras of functions on X

2. We say that $f \in C_b(X)$ satisfies the Higson condition with respect to $\mathcal{D}$ if

$$\lim_{n\to\infty} \sup_{x\in X\setminus D_n} \mathrm{var}_{x,r} f = 0$$

for any $r > 0$. Clearly, this condition depends only on the coarse equivalence class of d_X, but not on d_X itself.
Define the Higson algebra $C_h(X, \mathcal{D})$ with respect to $\mathcal{D}$ as the algebra of all bounded functions on X satisfying the Higson condition. Thus, $C_h(X, \mathcal{D})$ consists of bounded functions with variation vanishing far from $\mathcal{D}$.
Both $C_0(X, \mathcal{D})$ and $C_h(X, \mathcal{D})$ depend only on the equivalence class of expanding sequences.
If all D_n, $n \in \mathbb{N}$, are bounded then $C_0(X, \mathcal{D}) = C_0(X)$, and $C_h(X, \mathcal{D}) = C_h(X)$, the classical Higson algebra, i.e. the algebra of continuous functions on the Higson compactification of X.

Diagonal operators

For shortness' sake, for a function $f \in C_b(X)$, we use the same notation f for the operator of multiplication by f on $l^2(X)$.

Lemma

Let $f \in C_b(X)$, let ρ be a metric on $X \sqcup X$ compatible with d_X, and let $\mathcal{D} = \mathcal{D}(\rho)$ be the expanding sequence determined by ρ. Then $f \in M_\rho$ if and only if $f \in C_0(X, \mathcal{D})$.

Corollary

The following conditions are equivalent:

1. *the algebra $C_0(X, \mathcal{D})$ is unital;*
2. *$D_n = X$ for some $n \in \mathbb{N}$;*
3. *$M_\rho = C_u^*(X)$.*

Proof (of the Corollary): (1) implies that the unit function vanishes fer from $\mathcal{D}$, hence (2) holds. If (2) holds then $\rho(x, x') \leq n$ for any x, hence $|\rho(x, y) - \rho(x, y')| \leq n$ for any $x, y \in X$, i.e. finite propagation with respect to ρ is the same as finite propagation with respect to $\rho|_X$, hence $M_\rho = C^*_u(X)$. Finally, if $M_\rho = C^*_u(X)$ then $1 \in M_\rho$, so (3) implies (1). $\square$

Similarly, but more technically we can prove the following:

Lemma

Suppose that X is of bounded geometry, $f \in C_b(X)$. Let ρ be a metric on $X \sqcup X$ compatible with d_X, and let $\mathcal{D} = \mathcal{D}(\rho)$ be the expanding sequence determined by ρ. Then $[a, f] \in M_\rho$ for any $a \in C^*_u(X)$ if and only if $f \in C_h(X, \mathcal{D})$.

Examples

Recall that in order to distinguish points in the two copies of X we write x for the point x in the first copy of X and x' for the same point in the second copy of X.
Let $A, B \subset X$, and let $\alpha : A \to B$ be an isometric bijection. Then we can define a metric $\rho^{A,\alpha,B}$ on $X \sqcup X$ by

$$\rho^{A,\alpha,B}(x, y) = \rho^{A,\alpha,B}(x', y') = d_X(x, y);$$

$$\rho^{A,\alpha,B}(x, y') = \inf_{z \in A} [d_X(x, z) + 1 + d_X(\alpha(z), y)],$$

$x, y \in X$. If $B = A$ and $\alpha = \mathrm{id}_A$ then we abbreviate $\rho^{A,\mathrm{id},A}$ to ρ^A.

Examples

There are two extreme cases for metrics of the form ρ^A.

If $A = X$ then $M_{\rho^X} = C^*_u(X)$, the corresponding expanding sequence is constant, $D_n = X$ for any $n \in \mathbb{N}$. Both algebras, $C_0(X, \mathcal{D})$ and $C_h(X, \mathcal{D})$, coincide with the whole $C_b(X)$.

If $A = \{x_0\}$ for some point $x_0 \in X$ then $M_{\rho^{x_0}} = \mathbb{K}(l^2(X))$, the corresponding expanding sequence $D_n = B_{\frac{n-1}{2}}(x_0)$ is the sequence of balls of radii $\frac{n-1}{2}$ centered at x_0. The algebras $C_0(X, \mathcal{D})$ and $C_h(X, \mathcal{D})$ equal $C_0(X)$ and the Higson algebra $C_h(X)$ of functions with variation vanishing at infinity, respectively.

Examples

The set of all uniform Roe bimodules over $C^*_u(X)$ is partially ordered by inclusion. If $M_{\rho_1} \subset M_{\rho_2}$, and $\mathcal{D}_i$ corresponds to ρ_i, $i = 1, 2$, then it is easy to see that $C_h(X, \mathcal{D}_2)/C_0(X, \mathcal{D}_2) \subset C_h(X, \mathcal{D}_1)/C_0(X, \mathcal{D}_1)$. The uniform Roe bimodule $\mathbb{K}(l^2(X))$ is the minimal one.

Given an expanding sequence $\mathcal{E} = \{E_n\}_{n\in\mathbb{N}}$ of non-empty sets, we can define a metric $\rho^{\mathcal{E}}$ compatible with d_X by setting $\rho^{\mathcal{E}}|_X = d_X$ and

$$\rho^{\mathcal{E}}(x, y') = \inf_{n\in\mathbb{N}} \inf_{z\in E_n} [d_X(x, z) + n + d_X(z, y)].$$

It can be shown that this is a metric and that its expanding sequence is equivalent to $\mathcal{E}$.

Invariant submodules in uniform Roe bimodules

Lemma

$$HH^0(C_u^*(X), M_\rho) \cong \begin{cases} \mathbb{C} & \text{if } M_\rho = C_u^*(X), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition,
$HH^0(A, M) = \{m \in M : am - ma = 0$ for any $a \in A\}$. As $C_u^*(X)$ contains all compact operators, $am = ma$ for any $a \in \mathbb{K}(l^2(X))$ implies that m is scalar. By Corollary 5, only $M = C_u^*(X)$ contains non-zero scalars. $\square$

Derivations of uniform Roe algebras with values in uniform Roe bimodules

We shall write $C_0(X,\mathcal{D})^\sim$ for $C_0(X,\mathcal{D})$ if it is already unital, and for the unitalization of $C_0(X,\mathcal{D})$ otherwise.

Theorem

Let (X,d_X) be a discrete metric space of bounded geometry, let ρ be a metric on $X\sqcup X$ compatible with d_X, and let $\mathcal{D}=\mathcal{D}(\rho)$ be the corresponding expanding sequence. Then
$\mathrm{Out}(C_u^*(X),M_\rho)\cong C_h(X,\mathcal{D})/C_0(X,\mathcal{D})^\sim$.

Proof

A classical result claims that if A is a C^*-algebra, $\mathbb{K}(H) \subset A \subset \mathbb{B}(H)$, where H is a Hilbert space, then $HH^n(A, \mathbb{B}(H)) = 0$ for any $n \in \mathbb{N}$. In particular, all derivations of $C_u^*(X)$ taking values in $\mathbb{B}(l^2(X))$ are inner.
This means that if $d : C_u^*(X) \to M_\rho \subset \mathbb{B}(l^2(X))$ is a bounded derivation then there exists $b \in \mathbb{B}(l^2))$ such that $d(a) = [a, b]$ for any $a \in C_u^*(X)$.
Let $b = b_0 + b_1$, where b_0 (resp., b_1) is the diagonal (resp., off-diagonal) part of b with respect to the standard basis of $l^2(X)$. Then $d = d_0 + d_1$, where $d_0(\cdot) = [\cdot, b_0]$, $d_1(\cdot) = [\cdot, b_1]$.
To deal with the derivation d_1, we can use the argument from [M. Lorentz, R. Willett. *Bounded Derivations on Uniform Roe Algebras.* Rocky Mountain J. Math. **50** (2020), 1747–1758] to show that if $[a, b_1] \in M_\rho$ for any $a \in C_b(X)$ then $b_1 \in M_\rho$, i.e. d_1 is inner.

Proof (continued)

Thus, any bounded derivation d is of the form

$$d(\cdot) = [\cdot, f] + [\cdot, m],$$

$m \in M_\rho$. Since $[\cdot, f]$ is a derivation, $f \in C_h(X, \mathcal{D})$. Define a map

$$j : \mathrm{Der}(C_u^*(X), M_\rho) \to C_h(X, \mathcal{D})/C_0(X, \mathcal{D})^\sim$$

by $j(d) = f + C_0(X, \mathcal{D})^\sim$.

If $d(\cdot) = [\cdot, f] + [\cdot, m] = [\cdot, f'] + [\cdot, m']$ with $f, f' \in C_h(X, \mathcal{D})$ and $m, m' \in M_\rho$ then $[a, f - f' - m + m'] = 0$ for any $a \in C_u^*(X)$. As $C_u^*(X)$ contains all compact operators, its commutant consists only of scalars, hence, $f - f' - m + m'$ is scalar. Then $g = m - m'$ is a diagonal operator, and $g \in M_\rho$, hence $g \in C_0(X, \mathcal{D})^\sim$, i.e. $f - f' \in C_0(X, \mathcal{D})^\sim$, so the map j is well defined. Clearly, j is surjective, and its kernel is $C_0(X, \mathcal{D})^\sim$. $\square$

Higher Hochschild cohomology

A similar argument can show that the odd Hochschild cohomology is non-trivial.

Theorem

If $M_\rho \neq C_u^*(X)$ then $HH^{2k+1}(C_u^*(X), M_\rho)$ contains a copy of $C_h(X,\mathcal{D})/C_0(X,\mathcal{D})^\sim$ for any $k \in \mathbb{N}$.

Proof. Let $\psi \in C^{2k}(C_u^*(X), \mathbb{B}(l^2(X)))$ be given by $\psi(a_1,\ldots,a_{2k}) = a_1 \cdots a_{2k} f$, where $f \in C_h(X,\mathcal{D})$, $a_1,\ldots,a_{2k+1} \in C_u^*(X)$. Then we have

$$\phi(a_1,\ldots,a_{2k+1}) = \delta_{2k}(\psi)(a_1,\ldots,a_{2k+1}) = \pm a_1 \cdots a_{2k}[a_{2k+1}, f].$$

Clearly, $\phi \in C^{2k+1}(C_u^*(X), M_\rho)$, and $\delta_{2k+1}(\phi) = 0$, hence $[\phi] \in HH^{2k+1}(C_u^*(X), M_\rho)$. It is not zero unless $f \in C_0(X,\mathcal{D})^\sim$, as ψ takes values outside M_ρ. □

The even case is more difficult, and we cannot evaluate even the second cohomology, but we shall give some calculations for the case of the minimal uniform Roe bimodule, which is $\mathbb{K}(l^2(X))$. For shortness' sake we write $\mathbb{K}$ for $\mathbb{K}(l^2(X))$.
Recall that if we complete the algebraic tensor product $\mathbb{K} \odot_{\mathbb{K}} \mathbb{K}$ with respect to an appropriate norm then the formula $a \otimes b \mapsto ab$, $a, b \in \mathbb{K}$, defines an isometric isomorphism $\mathbb{K} \odot_{\mathbb{K}} \mathbb{K} \to \mathbb{K}$. We write $\mathbb{K} \otimes_{\mathbb{K}} \mathbb{K}$ for this completion, and identify it with $\mathbb{K}$ via the map $a \otimes b \mapsto ab$, $a, b \in \mathbb{K}$.
Recall that, given two A-bimodules over a Banach algebra A, the cup product for the Hochschild cohomology is defined as $[\alpha] \smile [\beta] \in HH^{p+q}(A, M \otimes_A N)$ for $[\alpha] \in HH^p(A, M)$, $[\beta] \in HH^q(A, N)$. When $p = q = 1$, $\alpha : A \to M$, $\beta : A \to N$, then $(\alpha \smile \beta)(a, b) = \alpha(a) \otimes \beta(b)$, $a, b \in A$. If the Künneth theorem would hold in this situation, non-triviality of the first Hochschild cohomology would imply non-trivility of the second one. We shall show that this is not the case.

No Künneth formula for Hochshild cohomology

Lemma

The map

$$\smile : HH^1(C^*_u(X), \mathbb{K}) \times HH^1(C^*_u(X), \mathbb{K}) \to HH^2(C^*_u(X), \mathbb{K})$$

is zero.

Proof. If α, β are derivations of $C^*_u(X)$ with coefficients in $\mathbb{K}$ then $\alpha(\cdot) = [\cdot, f]$, $\beta(\cdot) = [\cdot, g]$ for some $f, g \in C_h(X)$. Then $(\alpha \smile \beta)(a, b) = [a, f][b, g]$. But $\alpha \smile \beta = \delta_1(\psi)$, where $\psi(a) = [a, f]g$. As $[a, f] \in \mathbb{K}$, $\psi(a) \in \mathbb{K}$ for any $a \in C^*_u(X)$, thus $[\alpha] \smile [\beta] = 0$. □