Compact orthogonally additive operators on Banach lattices

Marat Pliev

Southern Mathematical Institute of the Russian Academy of Sciences, Vladikavkaz, Russia, North-Ossetian State University, Vladikavkaz, Russia and North Caucasus Center for Mathematical Research, Vladikavkaz, Russia

Harbin — 2024

F. Riesz in 1928 in his talk *"On the decomposition of linear functionals into their positive and negative parts"*, at the International Congress of Mathematicians in Bologna, Italy, marked the beginnings of the study of vector lattices (Riesz spaces) and positive operators. The theory of vector lattices in first part of 20 century was developed Freudenthal, L. V. Kantorovich, G. Birkhoff. Later important contributions came from P. P. Korovkin, M. A. Krasnoselskii, P. P. Zabreiko, W. A. J. Luxemburg and A. C. Zaanen, Schaefer, and B. Z. Vulikh, P. G. Dodds, D. H. Fremlin, Abramovich, C. D. Aliprantis, A. V. Bukhvalov, O. Burkinshaw, D. I. Cartwright, J. J. Grobler, Luxemburg, M. Meyer, P. Meyer-Nieberg, R. J. Nagel, U. Schlotterbeck, H. H. Schaefer, A. R. Schep, C. T. Tucker, A. I. Veksler, A. W. Wickstead, M. Wolff, A. C. Zaanen and others.

- W. A. J. Luxemburg and A. C. Zaanen, Riesz Spaces, I, North-Holland, Amsterdam, 1971.
- L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker, Functional Analysis in Partially Ordered Spaces, Gosudarstv. Izdat. Tecn. and Teor. Lit., Moscow and Leningrad, 1950.
- D. H. Fremlin, Topological Riesz Spaces and Measure Theory, Cambridge University Press, London and New York, 1974.
- A. C. Zaanen, Riesz Spaces II , North-Holland, Amsterdam, 1983..
- P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin etc., 1991.
- C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, (2006).
- Y. A. Abramovich, C. D. Aliprantis, *An Invitation to Operator Theory*, AMS, 2002.
- A. G. Kusraev, *Dominated operators*, Kluwer Academic Publishers, (2000).

Let E be a real vector space, equipped with a partial order $\leq$, which satisfies the following conditions:

1) If $u, v \in E$, $u \leq v$, then $u + w \leq v + w$, for every $w \in E$;
2) If $u, v \in E$, $u \leq v$, then $\lambda u \leq \lambda v$, for every $\lambda \in \mathbb{R}_+$.

An element $u \in E$ is said to be positive, if $u \geq 0$.
An ordered vector space E is called *a vector lattice*, if every finite subset D of E has supremum and infimum.

We recall that the modulus of element e of a vector lattice E is defined by following $|e| = e \vee (-e)$. Two elements e, f of a vector lattice E are said to be *disjoint* (notation $e \perp f$) if $|e| \wedge |f| = 0$.

Definition

Let E be a vector lattice, and let F be a real linear space. A (not necessarily linear) function $T \colon E \to F$ is called *orthogonally additive operator* (OAO for brevity), if $T(e \sqcup f) = Te + Tf$ for every disjoint elements $e, f \in E$.

It is clear that $T(0) = 0$. The set of all orthogonally additive operators from E to F we denote by $\mathcal{OA}(E, F)$. Clearly, $\mathcal{OA}(E, F)$ is a real vector space with respect to the natural linear operations.

Definition

Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces. By $(A \times B, \mu \otimes \nu)$ we denote the completion of their product measure space. The union $\Gamma \cup \Theta$ of two disjoint measurable sets $\Gamma, \Theta \in A$ we denote by $\Gamma \sqcup \Theta$. A map $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \nu$-measurable for all $r \in \mathbb{R}$;

(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

We say that a Carathéodory function K is *normalized* if $K(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

Let E be an order ideal of $L_0(\nu)$, let $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ be a normalized Carathéodory function and for every $f \in E$ the function $K(s, \cdot, f(\cdot)) \in L_1(\nu)$ for almost all $s \in A$. Suppose that the function $s \mapsto \int_B K(s, t, f(t))\, d\nu(t)$ belongs to F. Then is defined orthogonally additive operator $T\colon E \to F$ by setting

$$Tf(s) = \int_B K(s, t, f(t))\, d\nu(t).$$

- M. A. Krasnosel'skii, P. P. Zabrejko, E. I. Pustil'nikov, P. E. Sobolevski, *Integral operators in spaces of summable functions*, Noordhoff, Leiden (1976)..

Let (A, Σ, μ) be a finite measure space. We say that $N : A \times \mathbb{R} \to \mathbb{R}$ is a *superpositionally measurable function*, or *sup-measurable* for brevity, if $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in L_0(\mu)$. A sup-measurable function N is called *normalized* if $N(s, 0) = 0$ for μ-almost all $s \in A$.
With every normalized sup-measurable function N is associated an orthogonally additive operator $\mathcal{N} \colon L_0(\mu) \to L_0(\mu)$ defined by

$$\mathcal{N}(f)(s) = N(s, f(s)), \quad f \in L_0(\mu).$$

We note that the operator $\mathcal{N}$ is known in literature as the nonlinear superposition operator or Nemytskij operator.

- J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.
- T. Runst, W. Sickel,*Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series of Nonlinear Analysis and Applications, 1996.
- J. Appell, J. Banas, N. Merentes *Bounded variation and around*, De Gruyter, 2014.

Now, we are going to show how the theory of OAOs relates with the classical theory of linear regular operators. Let E be a vector lattice and E be a Dedekind complete vector lattice. Consider a map $J\colon L_r(E,F) \to \mathcal{OA}_r(E,F)$ defined by

$$J(T)x = T|x|, \quad x \in E.$$

Then the following statement holds.

Lemma

The map J is a linear operator from $L_r(E,F)$ to $\mathcal{OA}_r(E,F)$ and it possesses the following properties.

1. *J is an injective positive linear operator;*
2. *if, in addition, E has the principal projection property then for every $T \in L_r(E,F)$ one has $J(|T|) = |J(T)|$.*

- M. Pliev, M. Popov, *Representation theorems for regular operators*, Math. Nach., **297** (2024), 1, 126–143.

We say that $J\colon L_r(E,F) \to \mathcal{U}(E,F)$ is the *canonical embedding* of $L_r(E,F)$ to $\mathcal{OA}_r(E,F)$. The image $J(L_r(E,F))$ of $L_r(E,F)$ in $\mathcal{OA}_r(E,F)$ we denote by $\mathcal{OAL}(E,F)$.

- M. Popov, *Banach lattices of orthogonally additive operators*, J. Math. Anal. Appl., **514** (2022), 1, 126279.
- W. A. Feldman, *Radon-Nikodym theorem for nonlinear functionals on Banach lattices*, Proc. Amer. Math. Soc., Series B, **9** (2022), 9, 150–158.
- N. Erkursun Ozcan, M. Pliev, *On orthogonally additive operators in C-complete vector lattices*, Banach J. Math. Anal., **16** (2022), 1, article number 6.
- N. A. Dzhusoeva, S. Y. Itarova, *On orthogonally additive operators in lattice-normed spaces*, Math. Notes, **113** (2023), 1, 59–71.
- O. Fotiy, V. Kadets, M. Popov, *Some remarks on orthogonally additive operators*, Positivity, **27** (2023), article number 53.
- B. Turan, D. Tulu, *On orthogonally additive band operators and orthogonally additive disjointness preserving operators*, Turkish J. Math., **47** (2023), 4, article number 15.

- V. Mykhaylyuk, M. Pliev, M. Popov, *The lateral order on Riesz spaces and orthogonally additive operators-ii*, Positivity, **28** (2024), 1, article number 8.
- M. Pliev, M. Popov, *Representation theorems for regular operators*, Math. Nach., **297** (2024), 1, 126–143.
- N. Dzhusoeva, E. Grishenko, M. Pliev, F. Sukochev, *Narrow operators on C-complete lattice-normed spaces*, Nonlinear Anal., **243** (2024), 113524.
- N. M. Abasov, N. A. Dzhusoeva, M. A. Pliev, *Diffuse orthogonally additive operators*, Sbornik Math., **215** (2024), 1, 3–32.
- M. Pliev, M. Popov, *Orthogonally additive operators on vector lattices*, In: Aron, R.M., Moslehian, M.S., Spitkovsky, I.M., Woerdeman, H.J. (eds) Operator and Norm Inequalities and Related Topics. Trends in Mathematics. Birkhäuser, Cham, 2022, pp. 321–351.

The next definition is motivated by problems of the convex geometry.

Definition

A valuation is a map $\mathcal{V}$, defined on a given class of sets $\mathcal{A}$, such that the equality

$$\mathcal{V}(A \cup B) + \mathcal{V}(A \cap B) = \mathcal{V}(A) + \mathcal{V}(B),$$

holds for all $A, B \in \mathcal{A}$ whenever $A \cup B$ and $A \cap B$ also belong to $\mathcal{A}$. Valuations are a generalization of the notion of measure, and play an important role in the modern convex geometry.

- M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuation*, Annals Math., **172** (2010), 2, 1219–1267.
- S. Alesker, *Continuous rotation invariant valuations on convex sets*, Annals Math., **149** (1999), 3, 977–1005.
- P. Tradacete, and I. Villanueva, *Valuations on Banach lattices*,Int. Math. Res. Not., **2020** (2020), 1, 287–319.

Due to the natural identification between convex bodies and their support functions, the notion of valuation can be extended to the setting of maps defined on function spaces or more generally on Banach lattices.

Definition

Let E be a Banach lattice. A function $T\colon E \to \mathbb{R}$ is called a *valuation*, if the following conditions hold:

1. $T(f \vee g) + T(f \wedge g) = Tf + Tg, \ f, g \in E$;
2. $T(0) = 0$.

The next statement shows how OAOs related with the theory of valuations.

Theorem

Let E be a σ-Dedekind complete Banach lattice and $V : E \to \mathbb{R}$ be a function from E to $\mathbb{R}$. Then the following statements are equivalent:

1. *V is a valuation;*
2. *V is an orthogonally additive functional.*

- P. Tradacete, and I. Villanueva, *Valuations on Banach lattices*, Int. Math. Res. Not., **2020** (2020), 1, 287–319.

Hence, we expect that the methods and the technique of OAOs in Banach lattices can be useful for the theory of valuations.

Definition

Let E, F be vector lattices. An OAO $T \colon E \to F$ is said to be:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *regular* if $T = S_1 - S_2$, where S_1, S_2 are positive OAOs from E to F;
- *order bounded*, or an *abstract Uryson operator*, if it maps order bounded sets in E to order bounded sets in F;
- *C-bounded*, or a *Popov operator*, if the set $T(\mathfrak{F}_x)$ is order bounded in F for every $x \in E$.

The sets of positive, regular, order bounded and C-bounded orthogonally additive operators from E to F we denote by $\mathcal{OA}_+(E,F)$, $\mathcal{OA}_r(E,F)$, $\mathcal{OA}_b(E,F)$ and $\mathcal{P}(E,F)$ respectively. There is a natural partial order on $\mathcal{OA}_r(E,F)$, namely $S \leq T \Leftrightarrow (T-S) \in \mathcal{OA}_+(E,F)$. The next assertion is the Riesz-Kantorovich type theorem for orthogonally additive operators.

Theorem

Let E, F be vector lattices with F being Dedekind complete. Then $\mathcal{OA}_r(E,F) = \mathcal{P}(E,F)$, and $\mathcal{OA}_r(E,F)$ is a Dedekind complete vector lattice. Moreover, for every S, $T \in \mathcal{OA}_r(E,F)$ and every $x \in E$ the following hold:

1. $(T \vee S)x = \sup\{Ty + Sz : \; x = y \sqcup z\}$;
2. $(T \wedge S)x = \inf\{Ty + Sz : \; x = y \sqcup z\}$;
3. $T^+x = \sup\{Ty : \; y \sqsubseteq x\}$;
4. $T^-x = -\inf\{Ty : \; y \sqsubseteq x\}$;
5. $|Tx| \leq |T|x$.

- M. Pliev, K. Ramdane, *Order unbounded orthogonally additive operators in vector lattices*, Mediter. J. Math., **15** (2018), 2, article number 55.

Compact linear operators on Banach spaces is one of the most important subclasses of linear operators studied in Analysis. This subject in the setting of Banach lattices have been investigated numerous authors. In contrast with the general theory of bounded operators in Banach spaces, the theory of regular operators in Banach lattices contains some new effects by the order structure of Banach lattices. Of greatest interest here are regular questions of the relationship between the order and topological properties of a linear operator.

One of the most famous results in this direction concerning the compactness property of a positive linear operator was obtained by Dodds and Fremlin in their groundbreaking paper

- P. G. Dodds, D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math., **34** (1979), 287–320.

Theorem

Let E, F be Banach lattices, such that E' and F have order continuous norms and $T\colon E \to F$ be a positive compact operator. Then every linear operator $S \in [0, T] \subset L_+(E, F)$ is compact.

This theorem influenced a number of papers on positive compact operators.

- C. D. Aliprantis, O. Burkinshaw,*Positive compact operators on Banach lattices*, Math. Z., **174** (1980), 289–298.
- C. D. Aliprantis, O. Burkinshaw, *Dunford-Pettis operators on Banach lattices*, Trans. Amer. Math. Soc., **274** (1982), 227–238.
- C. D. Aliprantis, O. Burkinshaw, *Factoring compact and weakly compact operators through reflexive Banach lattices*, Trans. Amer. Math. Soc., **283** (1984), 369–381.
- N. J. Kalton, P. Saab, *Ideal properties of regular operators between Banach lattices*, Illinois J. Math., **29** (1985), 3, 382–400.
- B. de Pagter, *Irreducible compact operators on Banach lattices*, Math. Z., **192** (1986), 149–153.
- A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems*, Math. Proc. Camb. Philosop. Soc., **120** (1996), 1, 175–179.

Definition

Suppose that E be a vector lattice and F be a normed lattice. An orthogonally additive operator $T : E \to F$ is said to be:

1. *AM-compact*, if T maps order bounded subsets of E to relatively compact subsets of F;
2. *C-compact*, if $T(\mathfrak{F}_e)$ is a relatively compact subset of F for every $e \in E$;
3. *compact*, if E is a normed lattice and T maps norm bounded sets in E to relatively compact sets of F;
4. *almost compact*, if for every norm bounded subset D of E and $\varepsilon > 0$, there exists $f \in E_+$ such that $T(D) \subset T([-f, f]) + \varepsilon B_F$;
5. *narrow*, if for any $e \in E$ and $\varepsilon > 0$ there exists a pair e_1, e_2 of mutually complemented fragments of e, such that $\|Te_1 - Te_2\| < \varepsilon$;

- B. Aqzzouz, R. Nouira, L. Zraoula, *Compactness properties of operators dominated by AM-compact operators*, Proc. Amer. Math. Soc., **135** (2007), 1151–1157.
- Z. L. Chen, A. W. Wickstead, *The order properties of r-compact operators on Banach lattices*, Acta Mathematica Sinica, **23** (2007), 3, 457–466.
- B. Aqzzouz, A. Elbour, A. W. Wickstead, *Positive almost Dunford-Pettis operators and their duality*, Positivity, **15** (2011), 185–197.

Some unexpected connections of Dodds-Fremlin theorem with the Mathematical Physics was observed in

- J. Avron, I. Herbst, B. Simon, *Schrodinger operators with magnetic fields. I. General interactions*, Duke Math. J., **45** (1978), 4, 847–883.

We generalize the Dodds-Fremlin's theorem to the nonlinear setting. This required the introduction of new concepts and constructions. In particular, we introduce *normal and* $\mathfrak{C}$*-full vector lattices of order bounded orthogonally additive operators. We also generalize to the nonlinear setting the property of a Banach lattice* E *to have order continuous norm on* E'*, saying that "a Banach lattice* E *has property* $\mathcal{E}$*". We show that the vector lattice* $\mathcal{OAL}(E,F)$*, that is the canonical embedding of vector lattice of all regular linear operators* $L_r(E,F)$ *from* E *to* F *into the vector lattice* $\mathcal{OA}_r(E,F)$ *of all regular orthogonally additive operators, is normal and* $\mathfrak{C}$*-full.*

Theorem

Let E be a Banach lattice, F be a Banach lattice with order continuous norm, $\mathfrak{C}(E)$ be a subset of $\mathfrak{I}(E)$ the set of all lateral ideals of E, and $\mathcal{E}(E,F)$ be a normal $\mathfrak{C}$-full vector sublattice of $\mathcal{U}(E,F)$. Suppose that E has the property $\mathcal{E}$ and $T \in \mathcal{E}_+(E,F)$ is a compact OAO. Then every orthogonally additive operator $S \in [0,T] \subset \mathcal{E}_+(E,F)$ is compact.

Suppose that E is a vector lattice and $e \in E$. We say that $f \in E$ is a *fragment* (a *component*) of e, and use the notation $f \sqsubseteq e$, if $f \perp (e - f)$. We characterize the Boolean algebra $\mathfrak{F}_T$ of fragments of a positive orthogonally additive operator $T \colon E \to F$ from a vector lattice E to a Dedekind complete vector lattice F.

Let E be a vector lattice and $e \in E_+$. The order ideal generated by e we denote by E_e. An *e-step function* is an element $f \in E$ for which there exist pairwise disjoint fragments $e_1, \dots, e_n$ of e with $e = \bigsqcup_{i=1}^{n} e_i$ and real numbers $\lambda_1, \dots, \lambda_n$ satisfying $f = \sum_{i=1}^{n} \lambda_i e_i$.

Theorem

(Freudenthal Spectral Theorem) Let E be a vector lattice with the principal projection property and let $e \in E_+$. Then for every $f \in E_e$ there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of e-step functions satisfying $0 \leq f - v_n \leq \frac{1}{n}e$ for each n and $v_n \uparrow f$.

Definition

Let E be a vector lattice. We say that a subset $\mathcal{I} \subseteq E$ is a *lateral ideal of E if the following hold*

1. *if $e \in \mathcal{I}$ then $\mathfrak{F}_e \subseteq \mathcal{I}$;*
2. *$e + f \in \mathcal{I}$ for every disjoint $e, f \in \mathcal{I}$.*

The next example will be important for the further exposition.

Example

Suppose E is a vector lattice and $e \in E$. Then the Boolean algebra $\mathfrak{F}_e$ of fragments of e is a lateral ideal of E.

Suppose F is a Dedekind complete vector lattice. Given a lateral ideal $\mathcal{I}$ of E, we define the *associated map* $\mathfrak{p}_{\mathcal{I}} \colon \mathcal{OA}_r(E,F) \to \mathcal{OA}_r(E,F)$ by setting

$$
\begin{aligned}
(\mathfrak{p}_{\mathcal{I}} T)e := \sup\{Tf : f \in \mathcal{I} \cap \mathfrak{F}_e\}, \quad e \in E, \ \ T \in \mathcal{OA}_+(E,F); \\
\mathfrak{p}_{\mathcal{I}} T = \mathfrak{p}_{\mathcal{I}} T_+ - \mathfrak{p}_{\mathcal{I}} T_-, \quad T \in \mathcal{OA}_r(E,F).
\end{aligned}
$$

For the partial case where $\mathcal{I} = \mathfrak{F}_u$ for some $u \in E$, we write $\mathfrak{p}_u T$ instead of $\mathfrak{p}_{\mathcal{I}} T$.

Suppose that $\mathfrak{C}(E)$ is a subset of the set $\mathfrak{I}(E)$ of all lateral ideals of E; $\mathcal{I}_1, \dots, \mathcal{I}_n$ are elements of $\mathfrak{C}(E)$ and $\rho_1, \dots, \rho_n$, $n \in \mathbb{N}$ are pairwise disjoint order projections on F. We note that all operators $\rho_1 \mathfrak{p}_{\mathcal{I}_1} T, \dots, \rho_n \mathfrak{p}_{\mathcal{I}_n} T$ are pairwise disjoint and therefore the operator

$$S := \sum_{i=1}^{n} \rho_i \mathfrak{p}_{\mathcal{I}_i} T = \bigvee_{i=1}^{n} \rho_i \mathfrak{p}_{\mathcal{I}_i} T$$

is well defined and $S \in \mathfrak{F}_T$. We say that S is an $\mathfrak{C}$-*elementary* fragment of T. The set of all $\mathfrak{C}$-elementary fragments of T we denote by $\mathcal{A}_T^{\mathfrak{C}}$.

For a subset $\mathcal{A}$ of a vector lattice W we employ the following notation:

$$\mathcal{A}^{\uparrow} = \{x \in W : \exists \text{ a net } (x_\alpha) \subset \mathcal{A} \text{ with } x_\alpha \uparrow x\}.$$

The meanings of $\mathcal{A}^{\downarrow}$ are analogous. As usual, we also write

$$\mathcal{A}^{\downarrow\uparrow} = (\mathcal{A}^{\downarrow})^{\uparrow}; \mathcal{A}^{\uparrow\downarrow\uparrow} = ((\mathcal{A}^{\uparrow})^{\downarrow})^{\uparrow}.$$

It is clear that $\mathcal{A}^{\downarrow\downarrow} = \mathcal{A}^{\downarrow}$, $\mathcal{A}^{\uparrow\uparrow} = \mathcal{A}^{\uparrow}$. Consider a positive OAO $T : E \to F$, where F is Dedekind complete. Under the partial ordering induced by $\mathcal{OA}_r(E, F)$ the set $\mathcal{F}_T$ is a Boolean algebra with smallest element 0 and largest element T and $\neg S = T - S$ for every $S \in \mathcal{F}_T$.
Since $\mathcal{F}_T$ is a Boolean algebra, it is closed under finite suprema and infima. In particular, all "ups and downs" of $\mathcal{F}_T$ are likewise closed under finite suprema and infima, and therefore they are also directed upward and, respectively, downward.

Theorem

Let E, F be vector lattices with F Dedekind complete and $T \in \mathcal{OA}_+(E, F)$. Then $\mathfrak{F}_T = (\mathcal{A}_T^{\mathcal{F}})^{\uparrow\downarrow\uparrow}$.

To prove the main result, we apply Freudenthal Spectral Theorem and propose the following procedure:

1. fix an compact OAO $T \in \mathcal{E}_+(E, F)$ and show that every $\mathfrak{C}$-elementary fragment of T is also compact;
2. prove that, if $(S_\alpha)_{\alpha \in A}$ is an increasing (decreasing) net of compact OAOs in $\mathfrak{F}_T$ and $S = \sup\limits_{\alpha \in A} S_\alpha (\inf\limits_{\alpha \in A} S_\alpha)$, then S is a compact OAO;
3. if a sequence $(S_n)_{n \in \mathbb{N}}$ of compact OAOs in $[0, T] := \{S \in \mathcal{E}(E, F) : 0 \leq S \leq T\}$ converges relatively uniformly to $S \in [0, T]$, then S is also a compact OAO.