The Novikov conjecture, groups of diffeomorphisms, and Hilbert-Hadamard spaces

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I. Motivations: rigidity phenomena of manifolds

Layers of structures on a Riemannian manifold M:

Rigidity phenomena $\Uparrow$?

- metric structure, e.g., curvatures

  $\Downarrow$
- smooth structure

  $\Downarrow$
- homeomorphism type

  $\Downarrow$
- homotopy type

Symmetry groups:

$\mathrm{Isom}(M, g)$

$\mathrm{Diff}(M)$

$\mathrm{Homeo}(M)$

The Borel Conjecture

For aspherical manifolds, homotopy equiv. implies homeomorphism.

- homeomorphism type

  $\Downarrow$
- homotopy type

Motivation: classification of manifolds

The Novikov Conjecture

The *higher signatures* of smooth orientable manifolds are invariant under oriented homotopy equivalences.

- metric structure, e.g., curvatures

$\Downarrow$

- smooth structure $\xleftarrow{\text{by Definition}}$

$\Downarrow$

- homeomorphism type $\xleftarrow{\text{Novikov Thm}}$

$\Downarrow$

- homotopy type $\xleftarrow{\text{Novikov Conj}}$

} Higher signature

The Gromov-Lawson Conjecture

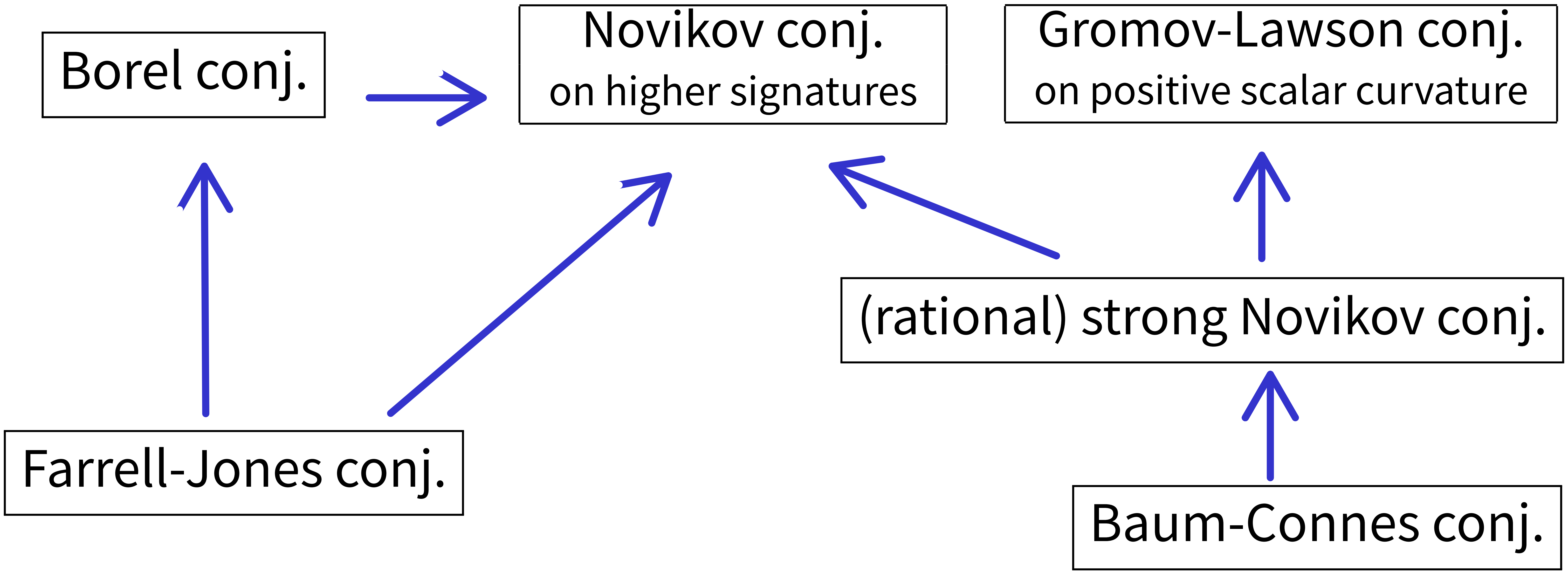
An $\underbrace{\text{aspherical}}_{\text{Homotopy structure}}$ manifold cannot have $\underbrace{\text{positive scalar curvature}}_{\text{Metric structure}}$.

In fact, these conjectures are statements about the fundamental groups $\pi_1(M)$.

Question

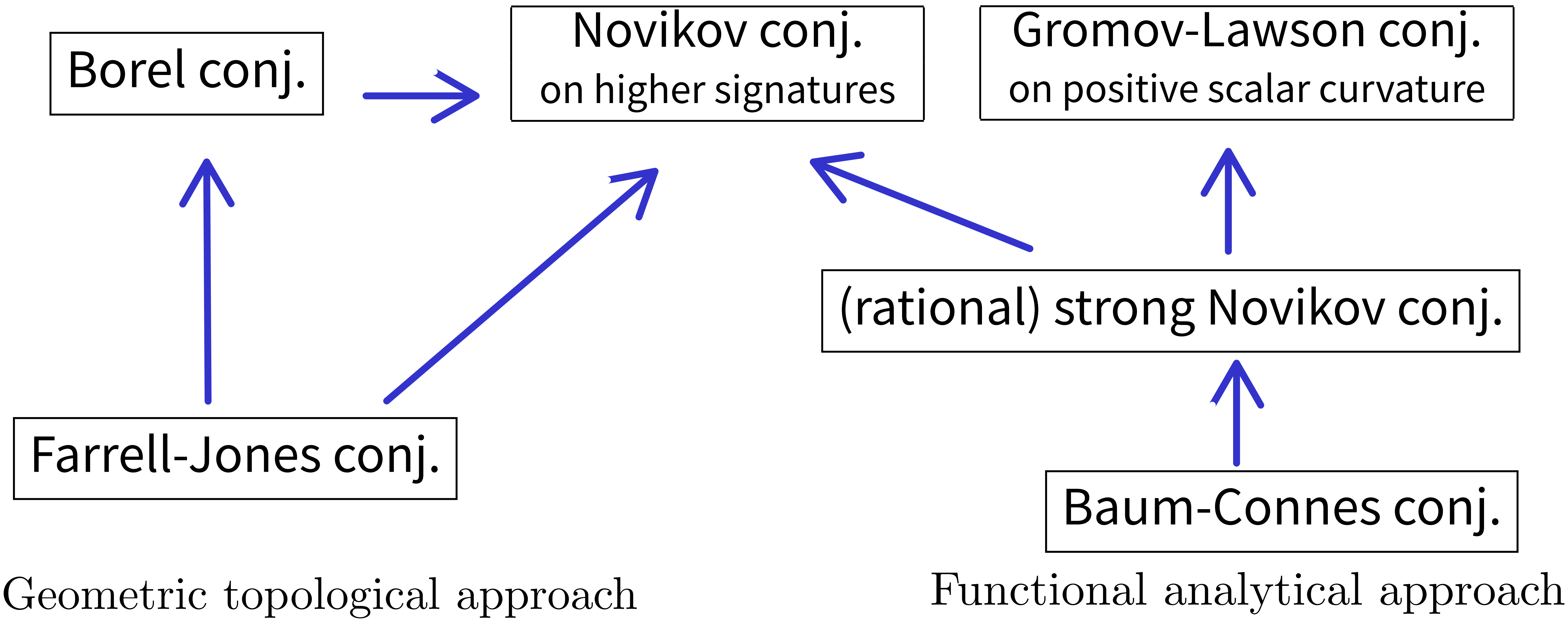
For what groups $\Gamma = \pi_1(M)$ are these conjectures true?

—— Many positive results and no counterexamples so far.



Geometric topological approach

Functional analytical approach



The Rational Strong Novikov Conjecture

For any countable group Γ, the following *higher index map* (a variant of the *Baum-Connes assembly*) is injective:

$$\mu_\Gamma : \underbrace{K_*(B\Gamma)}\otimes_{\mathbb{Z}} \mathbb{Q} \to \underbrace{K_*(C^*_r(\Gamma))}\otimes_{\mathbb{Z}} \mathbb{Q}.$$

K-homology of the classifying space of Γ	*K-theory* of the *reduced group C^*-algebra* of Γ
"topological"	"analytical"
easier to calculate	difficult to calculate

—— Verified for subgroups of $\mathrm{GL}(n, R)$ and connected Lie groups, groups that coarsely embed into Hilbert spaces, etc

II. Our focus: groups of diffeomorphisms

Let $M =$ closed smooth manifold

Theorem (Connes, 1986)

The Novikov conjecture holds for the Gel'fand-Fuchs classes of a group of diffeomorphisms of M.

Theorem (Gong-W-Yu, 2021) [Fix a volume form ω on M]

The rational strong Novikov conjecture holds for ω-preserving (ω-)discrete countable subgroups of $\mathrm{Diff}(M)$.

Comparing the results: pros (+) and cons (-)

Connes:	$+$ No restrictions on the subgroups of $\mathrm{Diff}(M)$.
G-W-Y:	$+$ No restrictions on the characteristic classes. $+$ Rational strong Novikov $\Rightarrow$ Novikov + Gromov-Lawson.

Theorem (Gong-W-Yu, 2021) [Fix a volume form ω on M]

The rational strong Novikov conjecture holds for ω-preserving (ω-)discrete countable subgroups of $\mathrm{Diff}(M)$.

Assumptions on the subgroups Γ of $\mathrm{Diff}(M)$:

- volume-form-preserving
- ω-discrete, more precisely:

$$\Gamma \ni \gamma \mapsto \lambda_+(\gamma) := \Big(\int_{y \in N} (\log \| D_y \gamma \|)^2 \, d\omega(y) \Big)^{1/2} \xrightarrow{\gamma \to \infty} \infty$$

where $D_y\gamma \colon T_yN \to T_{\gamma \cdot y}N$ is the derivative and $\| \cdot \|$ is the operator norm (w.r.t. some fixed Riemannian metric g).

Note:

– The integral measures how far γ is from being isometric (w.r.t. g).

– The definition of ω-discreteness is independent of g.

Theorem (Gong-W-Yu, 2021) [Fix a volume form ω on M]

The rational strong Novikov conjecture holds for ω-preserving (ω-)discrete countable subgroups of $\mathrm{Diff}(M)$.

Theorem (Gong-W-Xie-Yu, 2023+) [Fix a prob. measure μ on M]

The rational strong Novikov conjecture holds for μ-discrete countable subgroups of $\mathrm{Diff}(M)$.

This removes the "volume-form-preserving" condition above.

In this more general case, "μ-discreteness" means:

$$\inf_{\gamma' \in \Gamma} d_{\mu,g}(\gamma'\gamma, \gamma') \xrightarrow{\gamma \to \infty} \infty ,$$

where we use a pseudometric on $\mathrm{Diff}(N)$:

$$d_{\mu,g}(\varphi, \psi) := \left(\int_{x \in N} \left(\log \left(\left\| D_{\varphi(x)} \left(\psi \varphi^{-1} \right) \right\|_g \vee \left\| D_{\psi(x)} \left(\varphi \psi^{-1} \right) \right\|_g \right) \right)^2 \mathrm{d}\,\mu(x) \right)^{\frac{1}{2}}$$

III. Outlook: merging two extreme cases for $\Gamma < \mathrm{Diff}(M)$?

"tame" Γ fixes a Riem. metric $\Longrightarrow$ $\Gamma <$ Lie group

$\Downarrow$ [Guentner-Higson-Weinberger]

"wild" Γ is μ-discrete $\overset{\text{[GWYX]}}{\Longrightarrow}$ rational strong Novikov conj.

IV. Strategy: exploit geometric properties of groups

"Abstract" Theorem (Gong-W-Yu, 2021)

The rational strong Novikov conjecture holds for groups acting isometrically and properly on *admissible Hilbert-Hadamard spaces*.

isometrically and properly: $\forall/\exists\, x \in \mathcal{H},\, d(x, \gamma \cdot x) \xrightarrow{\gamma\to\infty} \infty$

admissible Hilbert-Hadamard spaces: nonpositively curved "manifold-like" metric spaces; common generalization of Hilbert spaces and Hadamard manifolds

A Hilbert-Hadamard space constructed from M [Fix $\mu \in \mathrm{Prob}(M)$]

Obs: $\{$inner products on $\mathbb{R}^n\} \cong \{$positive definite $n\times n$-matrices$\}$
$\cong GL(n,\mathbb{R})/O(n)$ (= a nonpositively curved symmetric space)
$\Rightarrow \{$Riemannian metrics on $M\} \cong \{$smooth sections of $\underbrace{\mathrm{Riem}(M)}_{\text{a } GL(n,\mathbb{R})/O(n)\text{-bundle over } M}\}$

with an "L^2-metric" $d(\xi,\eta) := \left(\int_M d_{GL/O}(\xi(x),\eta(x))^2\, \mathrm{d}\mu(x)\right)^{1/2}$

$\xrightarrow{L^2\text{-completion}} \mathrm{Riem}(M)_\mu$ = the "space of L^2-Riemannian metrics".

Obs: $\Gamma < \mathrm{Diff}(M)$ is μ-preserving and discrete
$\Rightarrow \Gamma \curvearrowright \mathrm{Riem}(M)_\mu$ isometrically & properly. $\Longrightarrow$ ("Abstract" Theorem $\Rightarrow$ Theorem GWY 2021).

V. Technical innovation: continuous fields of Hilbert-Hadamard spaces

Variation of measures [Fix $Z \subseteq \mathrm{Prob}(M)$]

$$\mathrm{Riem}(M)|_Z := \underbrace{\{\mathrm{Riem}(M)_\mu\}_{\mu \in Z}}_{}$$

a continuous field of Hilbert-Hadamard spaces over Z

Obs: $\Gamma < \mathrm{Diff}(M)$ is μ-discrete
$\Rightarrow \Gamma \curvearrowright \mathrm{Riem}(M)|_Z$ isometrically & properly where $Z = \overline{\Gamma \cdot \mu} \subseteq \mathrm{Prob}(M)$.

Hence Theorem GWXY 2023+ follows from:

"Abstract" Theorem (Gong-W-Xie-Yu, 2023+)

The rational strong Novikov conjecture holds for groups acting isometrically and properly on *admissible continuous fields of Hilbert-Hadamard spaces*.

Key proof ingredients:

- The construction of a C^*-algebra associated to a continuous field of Hilbert-Hadamard spaces.
- New deformation & trivilization techniques to aid K-theory computations.

VI. Related research: ∞-dimensional symmetric spaces

• Point of departure: $\mathrm{Riem}(M)_\mu$ is a symmetric Hilbert-Hadamard space, i.e., $\exists$ inversion symmetry at every point.

• Goal: Extend the classification and structure theory of classical symmetric spaces (à la Cartan; using Riemannian geometry & Lie theory) to this setting. $\rightsquigarrow$ Functional analysis on nonlinear spaces!

• They appear to be related to von Neumann algebras.

Construction (essentially [Bowen-Hayes-Lin] "A multiplicative ergodic theorem..."; think: ∞-dim'l analog of $GL(n,\mathbb{C})/U(n)$)

- (M,τ): a von Neumann algebra with a semifinite trace.
- $L^0(M,\tau)$: the algebra of operators on $L^2(M,\tau)$ that are affiliated with (M,τ) and have essentially dense domains.
- $\mathcal{P}(M,\tau) \subseteq L^0(M,\tau)$ consisting of operators x satisfying:
  1. x is positive,
  2. x is invertible, i.e., $x^{-1} \in L^0(M,\tau)$, and
  3. $\log|x| \in L^2(M,\tau)$.
- Equip $\mathcal{P}(M,\tau)$ with a metric $d(x,y) = \left\|\log\left(x^{-1/2}yx^{-1/2}\right)\right\|_2$

谢谢！