Coverings of compact groups and automorphisms of semigroup C^*-algebras

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Our seminar and areas of interest

Seminar on Functional Analysis and Quantum Systems
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Organizers: Prof. Gumerov R.N.(KFU) and Prof. Lipacheva E.V.(KPEU, KFU).

Areas of interest:

- C^*-algebras, semigroups, representations and their applications to algebraic quantum field theory and non-commutative harmonic analysis;
- Topological groups,coverings and their applications to Weierstrass polynomials and algebraic equations over Banach algebras;
- Quantum channels and processes, entanglement, tensors and their ranks.

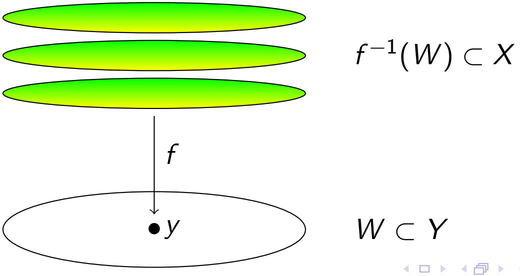
The subject and the plan of the report

- Covering mappings of the form $p : X \longrightarrow G$, where X is a connected compact space and G is a connected compact group.
- Weierstrass polynomials over the Banach algebra $C(G)$.
- Finite-sheeted covering mappings onto the P-adic solenoids.
- Semigroup C^*-algebras and their automorphisms.

Definition of a covering mapping (a covering)

Definition

For $k \in \mathbb{N}$, a surjective continuous mapping $f : X \to Y$ between topological spaces X and Y is called a *k-sheeted covering mapping* provided that for each $y \in Y$ there exists an open neighborhood $W \subset Y$ of y such that the inverse image $f^{-1}(W)$ can be written as the disjoint union of k open sets lying in X each of which is mapped homeomorphically onto W under f. The integer k is called a degree of f. If X and Y are connected spaces, then f is said to be connected.



Isomorphic (equivalent) covering mappings

Definition

A covering mapping $f_1 : X_1 \to Y$ is said to be *isomorphic* (or *equivalent*) to a covering $f_2 : X_2 \to Y$ provided that there exists a homeomorphism $\rho : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \rho\ } & X_2 \\ {\scriptstyle f_1}\searrow & & \swarrow{\scriptstyle f_2} \\ & Y & \end{array}$$

that is, the equality $f_1 = f_2 \circ \rho$ holds.

Examples of coverings

Example 1. Every homeomorphism $f : X \to X$ is a 1-sheeted covering mapping.

Example 2. Let $n \in \mathbb{N}$, $f : X \times \{1, 2, \ldots, n\} \longrightarrow X$,

$$f(x, m) = x, \quad m = 1, \ldots, n.$$

Definition

A covering mapping, which is isomorphic to f, is said to be trivial.

Example 3. Let $n \in \mathbb{N}$, $f_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 : z \longmapsto z^n$.

Example 4. The mapping $f : \mathbb{R} \longrightarrow \mathbb{S}^1$ given by

$$f(x) = (\cos(2\pi x), \sin(2\pi x)), x \in X,$$

is an infinite-sheeted covering mapping.

Motivations. Lifting of group structure.

$p : X \longrightarrow G$, where X is a space and G is a topological group

Does there exist **a group structure** on X such that: 1) this structure is consistent with the topology on X; 2) after equipping X with this structure, the mapping p becomes a continuous homomorphism between topological groups ?

In the case of positive answer, we say that the structure of a topological group lifts to a covering space, and the results of that kind are called the covering group theorems.

Motivations I. Pontryagin's covering group theorem

The following theorem is one of the main starting points of our research.

Pontryagin's Theorem [1].

Let $f : X \to G$ be a covering mapping from a path connected space X onto a connected locally path connected topological group G with identity e. Then for any point $\widetilde{e} \in f^{-1}(e)$ there exists a unique structure of topological group on X such that $\widetilde{e}$ is the identity and $f : X \to G$ is a morphism of topological groups. Furthermore, if G is abelian, then f is a morphism of abelian groups.

Since 2000: Grigorian S., G.R., Kazantsev A. [2, 3, 4, 5];
Eda K., Matijević V. [6, 7]; Clark A. [8]; Dydak J. [9], G.R. [10, 11].

Motivations II. A tool for studying mappings

A covering group theorem is a tool for studying structures of coverings and another mappings on topological groups.

It gives the possibility to jump from the topological category to the algebraic one and to transfer topological and analytic problems to algebraic and number-theoretic problems, to be bilingual.

For commutative topological groups, we can use the results of the Pontryagin duality theory.

Motivations III. Weierstrass polynomials and polynomial coverings.

The results on polynomials over Banach algebras of continuous functions and algebraic equations with functional coefficients and polynomial coverings contained in the following papers and the book.
Countryman R.S. (jr.) [12], Gorin E.A., Lin V.Ya.[13], Hansen V.L. [14, 15, 16], Kawamura K., Muira T. [17], Bardakov V.G., Vesnin A.Yu.[18].

Weierstrass polynomials

Definition.

A Weierstrass polynomial of degree $n \in \mathbb{N}$ over a topological group G is a polynomial function $R : G \times \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$R(g,z) = z^n + f_1(g)z^{n-1} + f_2(g)z^{n-2} + \cdots + f_n(g), \tag{1}$$

where $g \in G$, $z \in \mathbb{C}$, and the coefficients $f_1, \ldots, f_n : G \longrightarrow \mathbb{C}$ are continuous functions. The Weierstrass polynomial (1) is said to be simple, if for every $g \in G$ the polynomial

$$R(g,z) = z^n + f_1(g)z^{n-1} + f_2(g)z^{n-2} + \cdots + f_n(g),$$

in the indeterminate z with complex coefficients $f_1(g), f_2(g), \ldots, f_n(g)$ has no multiple roots in the field $\mathbb{C}$.

Polynomial coverings. Definition.

Let R be a simple Weierstrass polynomial of degree n over G. Consider the set of its zeros $E(R) = \{(g,z) \in G \times \mathbb{C} \mid R(g,z) = 0\}$ as a subspace in $G \times \mathbb{C}$ and the projection

$$pr : E(R) \longrightarrow G : (g,z) \longmapsto g.$$

Proposition (Hansen V.L.,[15]).

The projection pr is an n-sheeted covering mapping.

Definition (Hansen V.L.,[15]).

The projection $pr : E(R) \longrightarrow G$ and the space $E(R)$ are called a polynomial covering and a polynomial covering space associated with the simple Weierstrass polynomial R respectively.

Polynomial coverings. Example 1.

Example 1. Consider the simple Weierstrass polynomial R given by

$$R(g, z) \ = \ (z-1)(z-2) \cdot \ldots \cdot (z-n),$$

where $n \geqslant 2$. It is easy to see that the n-sheeted polynomial covering associated with R is trivial. Indeed, it is the projection onto the first coordinate:

$$G \times \{1, 2, \ldots, n\} \longrightarrow G : (g, l) \longmapsto g, \quad l \in \{1, \ldots, n\}.$$

Example 2. For $n \in \mathbb{N}$, let us consider the simple Weierstrass polynomial $R : \mathbb{S}^1 \times \mathbb{C} \longrightarrow \mathbb{C}$ over $\mathbb{S}^1$ given by $R(g,z) = z^n - g$, where $g \in \mathbb{S}^1$, $z \in \mathbb{C}$.
We have the n-sheeted polynomial covering

$$pr : E(R) \longrightarrow \mathbb{S}^1 : (g,z) \longmapsto g,$$

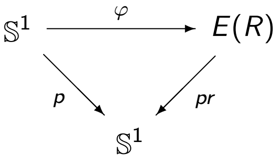
where $E(R) = \{(g,z) \in \mathbb{S}^1 \times \mathbb{C} \mid z^n - g = 0\}$.
We claim that the covering pr is equivalent to the n-sheeted covering

$$p : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 : z \longmapsto z^n,$$

that is, the nth potency mapping.

Polynomial coverings. Example 2.

Indeed, let us consider the diagram



Here φ is the homeomorphism defined by $\varphi(z) = (z^n, z)$.
It is straightforward to check that the diagram commutes, that is,

$$pr \circ \varphi = p.$$

Thus the coverings p and pr are equivalent.

Problem. Find topological groups G satisfying the following property: all covering mappings onto G are equivalent to polynomial coverings.

Theorem 1.

Let $p : X \longrightarrow G$ be a finite-sheeted covering mapping from a connected space X onto a compact group G with the identity e. Then for any point $\widetilde{e} \in p^{-1}(e)$ there exists a unique structure of topological group on X such that $\widetilde{e}$ is the identity and $p : X \longrightarrow G$ is a continuous homomorphism between compact groups. Furthermore, if G is abelian, then p is a continuous homomorphism of abelian groups.

Idea of the proof: approximation of the covering

We consider an inverse system $\{ G_\lambda, \pi_\lambda^\mu, \Lambda \}$ consisting of compact connected Lie groups G_λ and open surjective homomorphisms π_λ^μ such that

$$(G, \{\pi_\lambda\}) = \varprojlim \{ G_\lambda, \pi_\lambda^\mu, \Lambda \}.$$

If G is abelian, then every group G_λ is also abelian.
For every $\lambda \in \Lambda$, we construct a finite-sheeted covering $p_\lambda : X_\lambda \longrightarrow G_\lambda$.
Informally speaking, the family of coverings $\{p_\lambda \mid \lambda \in \Lambda\}$ approximates the given covering $p : X \longrightarrow G$.

Proposition.

Every n-sheeted covering mapping $p : X \longrightarrow G$ from a connected topological space X onto a compact group G is up to isomorphism the limit morphism induced by the morphism

$$\{p_\lambda : \lambda \in \Lambda\} : \{X_\lambda, h_\lambda^\mu, \Lambda\} \longrightarrow \{G_\lambda, \pi_\lambda^\mu, \Lambda\}$$

of inverse systems in the category of compact spaces and their continuous mappings, where p_λ is an n-sheeted covering mapping for every $\lambda \in \Lambda$.

$$\begin{array}{ccc} X_\lambda & \xleftarrow{\ h_\lambda^\mu\ } & X_\mu \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle p_\mu} \\ G_\lambda & \xleftarrow{\ \pi_\lambda^\mu\ } & G_\mu, \end{array}$$

Applications of the covering group theorem.

- Polynomial coverings.
- Structures of coverings.
- Criterion of triviality.
- Generalized means.
- A covering space of the P-adic solenoid is only the P-adic solenoid itself.

Applications to coverings of abelian groups.

Theorem 2.

Every finite-sheeted covering mapping onto a connected compact abelian group is equivalent to a polynomial covering.

Coverings of prime degree

In what follows, for a compact abelian group G we denote by $\widehat{G}$ its character group consisting of continuous homomorphisms from G to $\mathbb{S}^1$.

Theorem 3.

Let $p : X \longrightarrow G$ be an n-sheeted covering mapping from a connected topological space X onto a compact connected abelian G, where n is a prime number. Then there exists a simple Weierstrass polynomial

$$R(g, z) = z^n - \chi(g)$$

of degree n over G, where $\chi \in \widehat{G}$, such that the covering p is equivalent to the polynomial covering associated with the polynomial R.

Non-existence of coverings.

Definition.

Let $k \in \mathbb{N}$. An additive abelian group $\widehat{G}$ is said to be *k-divisible*, or we say that the group $\widehat{G}$ *admits division by* k, if for every $g \in \widehat{G}$ there exists $h \in \widehat{G}$ such that $kh = g$.

Corollary 1.

Let $k \geqslant 2$. If the character group $\widehat{G}$ is k-divisible, then there is NO a k-sheeted covering mapping from a connected topological space onto G.

The converse statement is not true.

1st criterion of triviality.

Theorem (Gorin E.A., Lin V.Ya.[13]).

All polynomial coverings associated with Weierstrass polynomials of degree $k \in \mathbb{N}$ over a compact connected abelian group G are trivial if and only if the character group $\widehat{G}$ admits division by $k!$.

Theorem 4.

All finite-sheeted coverings onto a compact connected abelian group G of degree $k \in \mathbb{N}$ are trivial if and only if the character group $\widehat{G}$ admits division by $k!$.

Keesling's criterion for the divisibility of $\widehat{G}$.

There are distinct necessary and sufficient conditions for the k-divisibility in the character group $\widehat{G}$. In particular, J. Keesling proved in 1972 [19] the following criterion.

Keesling's criterion.

Let G be a compact connected abelian group and $k \geqslant 2$ be an integer. Then the character group $\widehat{G}$ is k-divisible if and only if the group G admits a k-mean.

A generalized mean on a topological space.

Definition.

Let X be a topological space and $k \in \mathbb{N}, k \geqslant 2$. A continuous mapping

$$\mu : X \times X \times \ldots \times X \longrightarrow X$$

from the Cartesian product of k copies of X is called a (generalized) k-mean on X if it satisfies the following properties:

1. $\mu(x, x, \ldots, x) = x$;
2. $\mu(x_1, x_2, \ldots, x_k) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)})$

for all $x, x_1, x_2, \ldots, x_k \in X$ and for every permutation σ of the set $\{1, 2, \ldots, k\}$.

Kolmogoroff A.N.[20], Aumann G. [21], Eckmann B. [22].

Non-existence of means. 2nd criterion of triviality.

Corollary 2.

Let $k \geqslant 2$ be an integer. If there is a k-sheeted covering mapping from a connected topological space onto a compact connected abelian group G, then G does NOT admit a k-mean.

The converse statement is not true.

As an immediate consequence of Theorem 4 and Keesling's criterion, we obtain the following criterion of triviality.

Theorem 5.

All finite-sheeted covering mappings of degree $k \in \mathbb{N}$ onto a compact connected abelian group G are trivial if and only if G admits a $k!$-mean.

The P-adic solenoids

Let $P = (p_1, p_2, \ldots)$ be an arbitrary sequence of prime numbers.

The *P-adic solenoid* Σ_P is the inverse limit of the inverse sequence

$$\mathbb{S}^1 \xleftarrow{f_1} \mathbb{S}^1 \xleftarrow{f_2} \mathbb{S}^1 \xleftarrow{f_3} \cdots,$$

where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$, and $f_n(z) = z^{p_n}$, $z \in \mathbb{S}^1$, $n \in \mathbb{N}$.

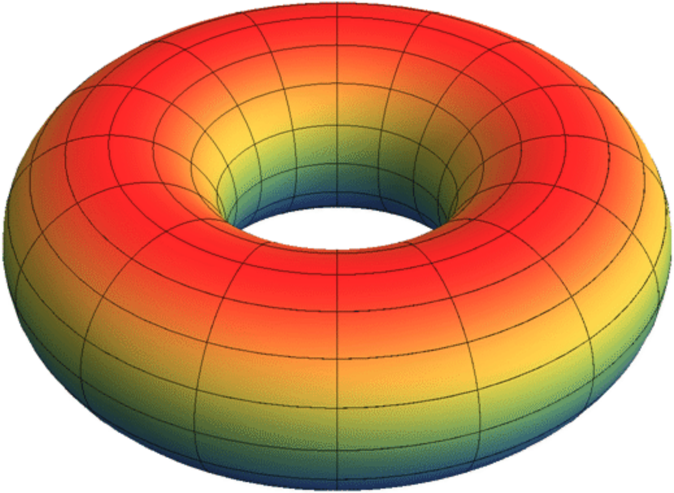
Thus, Σ_P is a subspace of the Cartesian product $\prod_{n\in\mathbb{N}} \mathbb{S}^1$:

$$\Sigma_P = \varprojlim\{\mathbb{S}^1, f_n\} = \{(z_1, z_2, \ldots) : z_n \in \mathbb{S}^1, z_{n+1}^{p_n} = z_n, n \in \mathbb{N}\}.$$

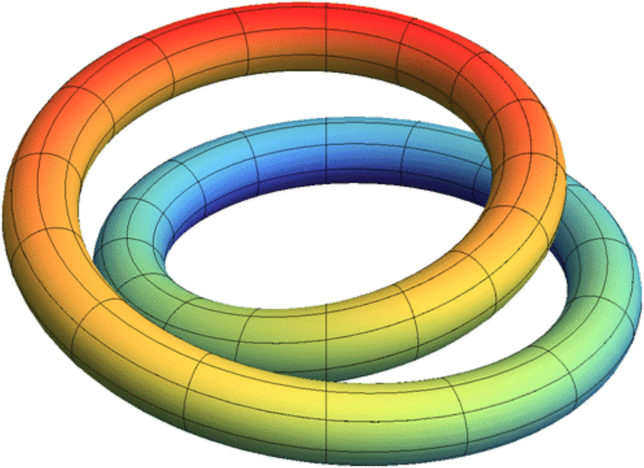
If $P = (2, 2, \ldots)$, then the solenoid Σ_P is said to be *dyadic*.

Vietoris L. in 1927 [23].
Van Dantzig D. and Van der Waerden B. (1928) [24]. (1930) [25].
The solenoid provides a topological model for the Smale attractor.

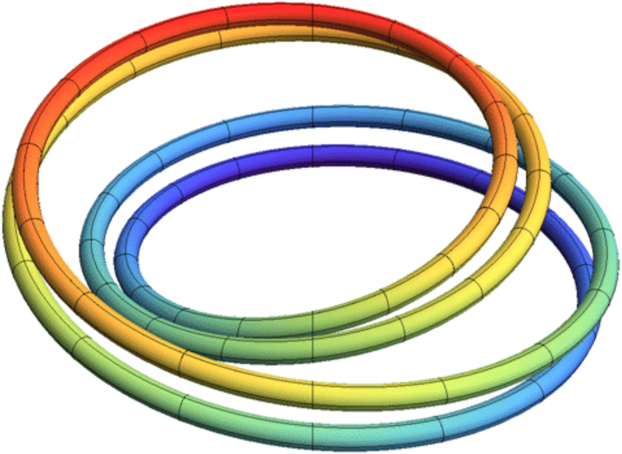
Embedding of the dyadic solenoid. Torus $T_0 \subset \mathbb{R}^3$



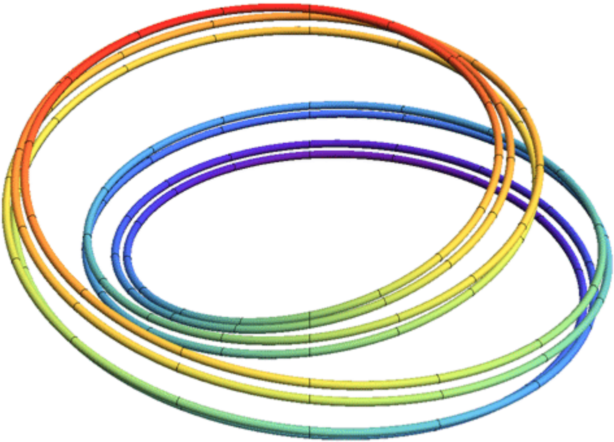
Embedding of the dyadic solenoid. Torus $T_1 \subset T_0$



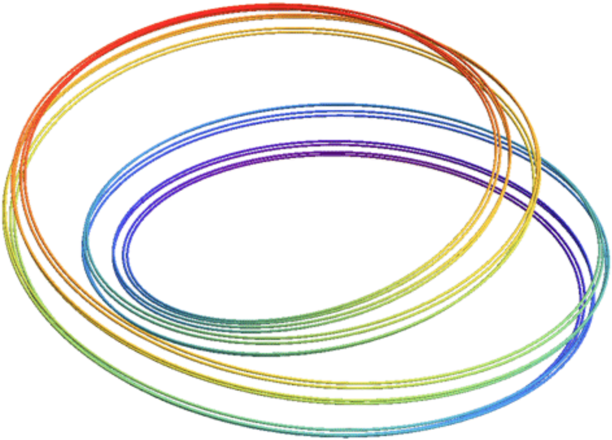
Embedding of the dyadic solenoid. Torus $T_2 \subset T_1$



Embedding of the dyadic solenoid. Torus $T_3 \subset T_2$



Embedding of the dyadic solenoid into $\mathbb{R}^3$.



Solenoids and their coverings

The P-adic solenoid is a compact connected abelian group under the coordinatewise multiplication with the identity $(1, 1, \ldots)$.

It is not locally connected at any point.

The study of coverings of solenoids: Fox R.([26, 27], 1972); Moore T.([28], 1978) — applied the theory of overlays.
Youcheng Z. ([29], 2000), Kwapisz J. ([30], 2001), Charatonik J., Covarrubias P. ([31], 2002), Aarts J.M., Fokkink R.J. ([32], 2003), Matijević V. ([33], 2003), Bogatyi S., Frolkina O. ([34], 2005), Jiang B., Wang S., Zheng H. ([35], 2008) and others.

Limit coverings

Fix $k \in \mathbb{N}$. Consider the morphism $\{h_n^k : n \in \mathbb{N}\}$ between two copies of the same inverse sequence and the limit mapping h_P^k induced by this morphism:

$$\begin{array}{ccccccccc}
\mathbb{S}^1 & \xleftarrow{f_1} & \mathbb{S}^1 & \xleftarrow{f_2} & \mathbb{S}^1 & \xleftarrow{f_3} & \cdots & \Sigma_P \\
{\scriptstyle h_1^k}\downarrow & & \downarrow{\scriptstyle h_2^k} & & \downarrow{\scriptstyle h_3^k} & & & \downarrow{\scriptstyle h_P^k} \\
\mathbb{S}^1 & \xleftarrow{f_1} & \mathbb{S}^1 & \xleftarrow{f_2} & \mathbb{S}^1 & \xleftarrow{f_3} & \cdots & \Sigma_P,
\end{array}$$

where $h_n^k(z) = z^k \quad z \in \mathbb{S}^1$. Thus, we have $h_P^k(g) = g^k, \; g \in \Sigma_P$.

Youcheng Z. ([29], 2000), Charatonik J., Covarrubias P. ([31], 2002) — for the constant sequence $P = (2, 2, \ldots)$.

G. ([36], 2003; [37], 2005; [38], 2018) — for all sequences of primes $P = (p_1, p_2, \ldots)$.

The dichotomy

Theorem 6.

Let P be a sequence of prime numbers and $k \in \mathbb{N}$. If k is a multiple of some prime number which occurs infinitely many times in P, then there is no a k-fold connected covering of the P-adic solenoid Σ_P. Otherwise, the limit mapping h_P^k is a k-fold connected covering and, moreover, each k-fold connected covering of Σ_P is isomorphic to h_P^k.

Brownlowe N., Raeburn I.([39], 2013) have used this result in studying the crossed product C^*-algebras.

The character group of solenoid

Consider the character group of the P-adic solenoid:

$$\widehat{\Sigma_P} \simeq \mathbb{Q}_P := \left\{ \frac{m}{p_1 p_2 \dots p_n} | m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

Let Γ denote either the group $\mathbb{Z}$ or $\mathbb{Q}_P$ and let $\Gamma^+ = \Gamma \cap [0; +\infty)$

$l^2(\Gamma^+)$ is the Hilbert space of square summable complex-valued functions $\{e_a | a \in \Gamma^+\}$ its basis, here $e_a(b) = \delta_{a,b}$, $\delta_{a,b} = 1$ if $a = b$ and 0 if $a \neq b$.

Let $B(l^2(\Gamma^+))$ be the C^*-algebra of bounded linear operators on $l^2(\Gamma^+)$.

For $a \in \Gamma^+$, define the isometry $V_a \in B(l^2(\Gamma^+))$ by setting

$$V_a e_b = e_{a+b}, b \in \Gamma^+.$$

The sub-C^*-algebra of $B(l^2(\Gamma^+))$ generated by the set $\{V_a | a \in \Gamma^+\}$ is called *the reduced semigroup C^*-algebra of* Γ^+, or *the Toeplitz algebra generated by* Γ^+. It is denoted by $C_r^*(\Gamma^+)$.

If $\Gamma = \mathbb{Z}$, we denote $C_r^*(\mathbb{Z}^+)$ by $\mathcal{T}$ and use the symbols $T := V_1$ and $T^n := V_n$, where $n \in \mathbb{Z}^+$. Thus, T is the right-shift operator

$$T : l^2(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+) : (\lambda_0, \lambda_1, \lambda_2, \ldots) \longmapsto (0, \lambda_0, \lambda_1, \lambda_2, \ldots),$$

T^* is the left-shift operator

$$T^* : l^2(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+) : (\lambda_0, \lambda_1, \lambda_2, \ldots) \longmapsto (\lambda_1, \lambda_2, \ldots).$$

$\mathcal{T}$ is the C^*-subalgebra in $B(l^2(\mathbb{Z}^+))$ generated by the operator T.

Coburn L.([40], 1967; [41], 1969) — the case of $\mathcal{T}$.
Douglas R.([42], 1972) — for subsemigroups in $\mathbb{R}$.
Murphy G. ([43], [44], [45], [46], 1987–1994) — a generalization to the ordered semigroups and to the left cancellative semigroups.
From the middle of 1990s to the present days: Adji S., Cuntz J., Laca M., Li X., Nica A., Nilsen M., Norling M., Raeburn I. and others. For the history and the results we refer to [47, 48, 49] and the references therein.

Li X. ([47], 2012) posed a question: does a given homomorphism $\tau : S \longrightarrow L$ between semigroups with the left cancellation property induce a morphism of corresponding semigroup C^*-algebras by the rule $V_a \longmapsto V_{\tau(a)}$, $a \in S$?
Lipacheva E. ([50], 2021; [51, 52], 2022) studied it and its applications for the reduced semigroup C^*-algebras. In particular, the positive answer formulated in terms of the normal extensions of semigroups was given in [50].

Isometric representation

An isometric homomorphism(or representation) from Γ^+ into a unital C^*-algebra B is a mapping

$$\rho : \Gamma^+ \to B : a \mapsto W_a$$

for which $W_a^* W_a = 1$ and $W_{a+b} = W_a W_b$ for all $a, b \in \Gamma^+$.

Clearly, the mapping

$$\pi : \Gamma^+ \to C_r^*(\Gamma^+) : a \mapsto V_a$$

is an isometric homomorphism.

Furthermore, it has the following universal property.

The universal property of the isometric homomorphism π

Theorem(Coburn [53, Theorem 3.5.18], Murphy[43]).

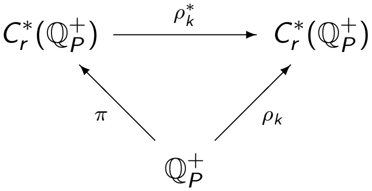
Let Γ be $\mathbb{Z}$ or $\mathbb{Q}_P$, B be a unital C^*-algebra and $\rho : \Gamma^+ \to B$ be an isometric homomorphism. Then there exists a unique $*$-homomorphism $\rho^* : C_r^*(\Gamma^+) \to B$ such that the diagram

$$\begin{array}{ccc} C_r^*(\Gamma^+) & \xrightarrow{\rho^*} & B \\ & \nwarrow^{\pi} \quad \nearrow_{\rho} & \\ & \Gamma^+ & \end{array}$$

commutes,i.e., $\rho^* \circ \pi = \rho$. Moreover, if $\rho(a)$ is non-unitary for every $a > 0$ then the $*$-homomorphism ρ^* is injective.

Definition of the $*$-endomorphisms ρ_k^*.

For $k \in \mathbb{N}$, the $*$-homomorphism $\rho_k^* : C_r^*(\mathbb{Q}_P^+) \to C_r^*(\mathbb{Q}_P^+)$defined by the universal property of isometric homomorphisms:



Here,

$$\rho_k : \mathbb{Q}_P^+ \longrightarrow C_r^*(\mathbb{Q}_P^+) : a \longmapsto V_{ka};$$
$$\pi : \mathbb{Q}_P^+ \longrightarrow C_r^*(\mathbb{Q}_P^+) : a \longmapsto V_a.$$

The idea: to represent ρ_k^* as a limit endomorphism.

Direct(inductive) sequences of C^*-algebras

Next, we recall the definitions of the direct sequences and the direct limits in the category of C^*-algebras and their $*$-homomorphisms (see details, for example, in [53, Ch. 6]).

Definition

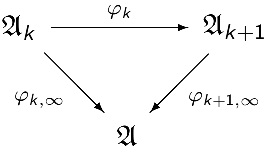
A direct, or *inductive, sequence* is a collection $\{\mathfrak{A}_k, \varphi_k\}_{k=1}^{+\infty}$ consisting of C^*-algebras $\mathfrak{A}_k$ and $*$-homomorphisms $\varphi_k : \mathfrak{A}_k \longrightarrow \mathfrak{A}_{k+1}$. To represent this sequence, one often uses the diagram

$$\mathfrak{A}_1 \xrightarrow{\varphi_1} \mathfrak{A}_2 \xrightarrow{\varphi_2} \mathfrak{A}_3 \xrightarrow{\varphi_3} \dots \tag{2}$$

Direct(inductive) limit i)

A direct, or *inductive, limit* of the direct sequence (2) is a pair $(\mathfrak{A}, \{\varphi_{k,\infty}\}_{k=1}^{+\infty})$ consisting of a C^*-algebra $\mathfrak{A}$ and a sequence of $*$-homomorphisms $\{\varphi_{k,\infty} : \mathfrak{A}_k \longrightarrow \mathfrak{A}\}_{k=1}^{+\infty}$ with the following two properties:

i) for every $k \in \mathbb{N}$ the diagram

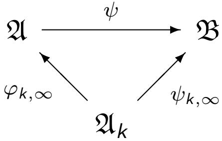


is commutative, i.e.,

$$\varphi_{k,\infty} = \varphi_{k+1,\infty} \circ \varphi_k;$$

Direct(inductive) limit ii)

ii) (*the universal property of the direct limit*) for any C^*-algebra $\mathfrak{B}$ and any sequence of $*$-homomorphisms $\{\psi_{k,\infty} : \mathfrak{A}_k \longrightarrow \mathfrak{B}\}_{k=1}^{+\infty}$ satisfying the condition $\psi_{k,\infty} = \psi_{k+1,\infty} \circ \varphi_k$ for every $k \in \mathbb{N}$, there exists a unique $*$-homomorphism $\psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ making the diagram



commute for each $k \in \mathbb{N}$, i.e., $\psi_{k,\infty} = \psi \circ \varphi_{k,\infty}$.

A direct limit $(\mathfrak{A}, \{\varphi_{k,\infty}\}_{k=1}^{+\infty})$ exists for any direct sequence $\{\mathfrak{A}_k, \varphi_k\}_{k=1}^{+\infty}$. Often, the C^*-algebra $\mathfrak{A}$ itself is called a direct limit. It is denoted by $\varinjlim\{\mathfrak{A}_k, \varphi_k\}$. Moreover, two direct limits of the same direct sequence are isomorphic.

Definition of a limit $*$-homomorphism.

A sequence $\{\theta_k : \mathfrak{A}_k \longrightarrow \mathfrak{B}_k \mid k \in \mathbb{N}\}$ of $*$-homomorphisms of C^*-algebras is called a morphism from a direct sequence $\{\mathfrak{A}_k, \varphi_k\}_{k=1}^{+\infty}$ of C^*-algebras to a direct sequence $\{\mathfrak{B}_k, \psi_k\}_{k=1}^{+\infty}$ of C^*-algebras provided that the diagram

$$\begin{array}{ccc} \mathfrak{A}_k & \xrightarrow{\varphi_k} & \mathfrak{A}_{k+1} \\ \theta_k \downarrow & & \downarrow \theta_{k+1} \\ \mathfrak{B}_k & \xrightarrow{\psi_k} & \mathfrak{B}_{k+1} \end{array}$$

commutes for each $k \in \mathbb{N}$, i.e., $\psi_k \circ \theta_k = \theta_{k+1} \circ \varphi_k$. In this case, using the universal property of the direct limit $\varinjlim\{\mathfrak{A}_k, \varphi_k\}$, one has *the limit $*$-homomorphism*

$$\varinjlim\{\theta_k\} : \varinjlim\{\mathfrak{A}_k, \varphi_k\} \longrightarrow \varinjlim\{\mathfrak{B}_k, \psi_k\}$$

induced by the morphism $\{\theta_k : \mathfrak{A}_k \longrightarrow \mathfrak{B}_k \mid k \in \mathbb{N}\}$.

Inductive sequence of Toeplitz algebras

Lemma

For each positive integer n there is a unique unital $*$-homomorphism of C^*-algebras $\varphi : \mathcal{T} \longrightarrow \mathcal{T} : T \mapsto T^n$. Furthermore, the $*$-homomorphism φ is isometric.

Definition ([54, 55]).

For an arbitrary sequence of prime numbers $P = \{p_1, p_2, \dots\}$ and the sequence of isometric $*$-homomorphisms $\{\varphi_n : \mathcal{T} \longrightarrow \mathcal{T} : T \mapsto T^{p_n}\}_{n=1}^{\infty}$ the inductive sequence

$$\mathcal{T} \xrightarrow{\varphi_1} \mathcal{T} \xrightarrow{\varphi_2} \mathcal{T} \xrightarrow{\varphi_3} \dots$$

is called *the inductive sequence of Toeplitz algebras associated with the sequence* P.

Inductive limit

Theorem 7.

Let $P = \{p_1, p_2, \dots\}$ be a sequence of prime numbers and let $\{\mathcal{T}, \varphi_n\}_{n=1}^{+\infty}$ be the inductive sequence of Toeplitz algebras associated with P. Then there exists an isomorphism of C^*-algebras:

$$\varinjlim\{\mathcal{T}, \varphi_n\} \cong C_r^*(\mathbb{Q}_P^+).$$

Limit endomorphism φ_P^k.

For $k \in \mathbb{N}$, we consider the limit endomorphism $\varphi_P^k : C_r^*(\mathbb{Q}_P^+) \to C_r^*(\mathbb{Q}_P^+)$ defined by the morphism $\{\varphi_n^k : n \in \mathbb{N}\}$:

$$\begin{array}{ccccccccc}
\mathcal{T} & \xrightarrow{\varphi_1} & \mathcal{T} & \xrightarrow{\varphi_2} & \mathcal{T} & \xrightarrow{\varphi_3} & \cdots & C_r^*(\mathbb{Q}_P^+) \\
\varphi_1^k \downarrow & & \downarrow \varphi_2^k & & \downarrow \varphi_3^k & & & \downarrow \varphi_P^k \\
\mathcal{T} & \xrightarrow{\varphi_1} & \mathcal{T} & \xrightarrow{\varphi_2} & \mathcal{T} & \xrightarrow{\varphi_3} & \cdots & C_r^*(\mathbb{Q}_P^+),
\end{array}$$

where $\varphi_n^k : \mathcal{T} \longrightarrow \mathcal{T} : T \longmapsto T^k$.

It is clear that $\rho_k^* = \varphi_P^k$.

Criteria for endomorphisms to be automorphisms.

Theorem 8.

Let P be a sequence of prime numbers and let $k \geqslant 2$ be an integer. Then the following conditions are equivalent:

1) the endomorphism $\rho_k^* = \varphi_P^k : C_r^*(\mathbb{Q}_P^+) \longrightarrow C_r^*(\mathbb{Q}_P^+)$ is an automorphism of C^*-algebras;
2) each prime divisor of k occurs infinitely many times in the sequence P;
3) the group of rationals $\mathbb{Q}_P$ is k-divisible;
4) the P-adic solenoid Σ_P admits a k-mean;
5) for every prime divisor q of k there is no a q-sheeted covering mapping from a connected topological space onto the P-adic solenoid Σ_P.

Example 1.

Example 1. Let $P = (2, 2, \dots)$, $k \in \mathbb{N}$. The group of the dyadic rationals

$$\mathbb{Q}_{(2,2,\dots)} := \left\{ \frac{m}{2^n} | m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

$\varphi^k_{(2,2,\dots)} : C^*_r(\mathbb{Q}^+_{(2,2,\dots)}) \longrightarrow C^*_r(\mathbb{Q}^+_{(2,2,\dots)})$ is an automorphism if and only if k is a power of 2.
For the dyadic solenoid $\Sigma_{(2,2,\dots)}$, we have a mean (2-mean)

$$\mu : \Sigma_{(2,2,\dots)} \times \Sigma_{(2,2,\dots)} \longrightarrow \Sigma_{(2,2,\dots)} : (x, y) \longmapsto (x_2 y_2, x_3 y_3, \dots),$$

where $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \Sigma_{(2,2,\dots)}$.

The mapping $\nu : \Sigma_{(2,2,\dots)} \times \Sigma_{(2,2,\dots)} \times \Sigma_{(2,2,\dots)} \times \Sigma_{(2,2,\dots)} \longrightarrow \Sigma_{(2,2,\dots)}$

such that $\nu(x, y, z, w) = \mu(\mu(x, y), \mu(z, w))$ for $x, y, z, w \in \Sigma_{(2,2,\dots)}$ is a 4-mean.

Examples 2,3,4.

Example 2. Let $n \in \mathbb{N}$. Take n different primes $p_1, \ldots, p_n$. Consider the periodic sequence $P = (p_1, \ldots, p_n, p_1, \ldots, p_n, \ldots)$. The endomorphism $\varphi^k_{(2,2,\ldots)} : C^*_r(\mathbb{Q}^+_{(2,2,\ldots)}) \longrightarrow C^*_r(\mathbb{Q}^+_{(2,2,\ldots)})$ is an automorphism if and only if there exist nonnegative integers $m_1, \ldots, m_n$ such that $k = p_1^{m_1} \cdot \ldots \cdot p_n^{m_n}$. For instance, if $P = (2, 3, 2, 3, \ldots)$, then $\varphi^k_{(2,3,\ldots)}$ is an automorphism if and only if $k = 2^s 3^t$, where $s, t \in \mathbb{Z}^+$.

Example 3. Let $P = (2, 2, 3, 2, 3, 5, 2, 3, 5, 7, \ldots)$. Then $\mathbb{Q}_P$ coincides with the group of all rationals $\mathbb{Q}$. In this case, each endomorphism φ^k_P is an automorphism.

Example 4. Let $P = (2, 3, 5, 7, \ldots)$ be the sequence of all primes. Then, for each $k \geqslant 2$, the endomorphism φ^k_P is not an automorphism.

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Thank you!