Resource-dependent complexity of quantum channels

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Outline

What is complexity?

In general

Complexity is the quality or state of being complex. For example, if a math paper only involves simple calculus and linear algebra, we can say that the complexity of this paper is low(low complexity $\neq$ not good!). If it involves "fancy" tools, we say it is more complex.

In computer science

Complexity theory is a central topic in theoretical computer science. It helps determine the difficulty of a problem, often measured by how much time and space (memory) it takes to solve a particular problem. For example, some problems can be solved in polynomial amounts of time and others take exponential amounts of time, with respect to the input size.

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In quantum information theory

Quantum complexity is a fundamental concept in quantum information and quantum computation theory. It is a generalization of classical computational complexity and characterizes the inherent difficulty of various quantum information tasks.
Quantum information specialists primarily focus on the quantum circuit complexity, which emphasizes on the number of elementary gates required to simulate a complex unitary transformation.

Our work: mathematical framework of quantum complexity

- Quantum complexity is much more difficult to quantify and measure –**estimates instead.**
- Previous work mainly focus on the quantum circuit complexity – **propose a different measure that works for general quantum channels.**

Moreover, we build some connections to some existing complexity measures.

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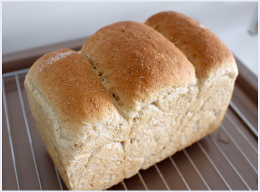
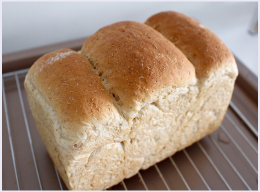
Resource-dependence of complexity

Different available resources produce different complexity.

Bread cooking: resource = flour or wheat



low complexity



high complexity

Preliminaries

Mathematical model of resources

Let $\mathcal{H}$ be a Hilbert space and let $\mathbb{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$. Suppose $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$ is a finite von Neumann algebra.

- (Definition 1)$\mathcal{N}$ is weakly closed *-algebra containing identity.
- (Definition 2, also known as double commutant theorem)$\mathcal{N}$ is a subalgebra closed under involution and its closure equal to its double commutant.
- (Definition 3)$\mathcal{N}$ is a C^*-algebra that has a predual.
- Finite means $\mathcal{N}$ is the direct integral of finite factors (implying the von Neumann algebra has a faithful normal tracial state).

A resource set S is any subset of $\mathcal{N}$.

Typical example of $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$

- $\mathcal{N} = L^\infty(X, \mu)$, with predual as integrable functions.
- $\mathcal{N} = (\mathbb{B}(\mathcal{H}), \mathrm{tr})$, with predual as trace class operators. In quantum information science, we mainly deal with the case when dimension of $\mathcal{H}$ is finite.
- $\mathcal{N}_* \cong L_1(\mathcal{N})$.

General framework can cover both classical setting(commutative) and quantum setting(non-commutative).

Mathematical model of quantum channels

A quantum channel Φ is a normal, trace-preserving, completely positive map:

$$\Phi : \mathcal{N}_* \to \mathcal{N}_*,$$

The dual map $\Phi^* : \mathcal{N} \to \mathcal{N}$ defined via $\mathrm{tr}(\Phi^*(\rho)x) = \mathrm{tr}(\rho\Phi(x))$ for any $\rho \in L_1(\mathcal{N}), x \in \mathcal{N}$, is a normal unital completely positive map.

Definition of resource-dependent complexity

Axiom

The ingredients are a given resource set $S \subset \mathcal{N}$ and quantum channels.
Mathematically, our goal is to find an appropriate complexity function C defined on quantum channels

$$C : CP(\mathcal{N}) \to \mathbb{R}_{\geq 0},$$

where $CP(\mathcal{N})$ is the set of all the quantum channels on $\mathcal{N}$, such that it satisfies some of the following axioms:

Axioms

1. $C(\Phi) = 0$ if and only if $\Phi = id$.
2. (*subadditivity under concatenation*) $C(\Phi\Psi) \leq C(\Phi) + C(\Psi)$.
3. (*Convexity*) For any probability distribution $\{p_i\}_{i\in I}$ and any channels $\{\Phi_i\}_{i\in I}$, we have
$$C(\sum_{i\in I} p_i\Phi_i) \leq \sum_{i\in I} p_i C(\Phi_i). \tag{1}$$
4. (*Tensor additivity*) For finitely many channels $\{\Phi_i\}_{1\leq i\leq m}$, we have
$$C(\bigotimes_{i=1}^{m} \Phi_i) = \sum_{i=1}^{m} C(\Phi_i). \tag{2}$$

(Only subadditive holds in fact.)

Definition of resource-dependent complexity

The induced Lipschitz (semi-)norm is defined by

$$|||f|||_S := \sup_{s \in S} \|[s, f]\|_\infty, \ \ f \in \mathcal{N}. \tag{3}$$

For a quantum channel $\Phi : \mathcal{N}_* \to \mathcal{N}_*$, the complexity is defined as

$$C_S(\Phi) := \|\Phi^* - id : (\mathcal{N}, |||\cdot|||_S) \to \mathbb{B}(\mathcal{H})\|. \tag{4}$$

- The complexity measure can be infinity, but if $\Phi_{S'} = id$, it will be finite.
- $|||\cdot|||_S$ is only a seminorm, but via standard quotient procedure, one can make it a form on the "mean" 0 space $\mathcal{A} \cong \{f \in \mathcal{N} : E_{S'}(f) = 0\}$. Here we assume S' is a von Neumann algebra which is the case in most settings.

Basic properties of resource-dependent complexity

For technical simplicity, we will assume the commutant S' is a von Neumann algebra and $E_{S'}$ is a conditional expectation, and we denote $E_{S'}$ as E_{fix} since the resource set is fixed.

1. $C_S(\Phi) = 0$ if and only if $\Phi = id$.
2. $C_S(\Phi\Psi) \leq C_S(\Phi) + C_S(\Psi)$ for any quantum channels Φ, Ψ.
3. For any probability tribution $\{p_i\}_{i\in I}$ and any channels $\{\Phi_i\}_{i\in I}$, we have
$$C_S(\sum_{i\in I} p_i\Phi_i) \leq \sum_{i\in I} p_i C_S(\Phi_i). \quad (5)$$
4. $C_S(\Phi) \leq C_S(E_{fix})\|\Phi^* - id : \mathcal{N} \to \mathcal{N}\|$.

Basic properties of resource-dependent complexity

Complete version

The complete(correlation-assisted) complexity of a quantum channel Φ is defined by

$$C_S^{cb}(\Phi) := \sup_{n\geq 1} \|id_n \otimes (\Phi^* - id) : (\mathbb{M}_n(\mathcal{A}), |||\cdot|||_S^n) \to \mathbb{M}_n(\mathbb{B}(\mathcal{H}))\|. \quad (6)$$

Standard argument shows that complete version of complexity is additive.

Return time trick

Denote $\mathcal{N}_{fix} = S'$ and suppose E_{fix} is the tracial conditional expectation onto $\mathcal{N}_{fix}$.
For any quantum channel Φ with $\Phi^* E_{fix} = E_{fix}$, we can define the *mixing time* of Φ as follows: for $\varepsilon > 0$,

$$t_{mix}(\varepsilon, \Phi) := \inf\{n \geq 1 : \|\Phi^{n*} - E_{fix} : \mathcal{N} \to \mathcal{N}\| = \|\Phi^n - E_{fix} : L_1(\mathcal{N}) \to L_1(\mathcal{N})\| \tag{7}$$

For $\varepsilon > 0$, the *return time* is given by

$$t_{ret}(\varepsilon, \Phi) := \inf\{n \geq 1 : (1-\varepsilon)E_{fix} \leq_{cp} \Phi^{n*} \leq_{cp} (1+\varepsilon)E_{fix}\}. \tag{8}$$

Note that similar definition for complete version is immediate.

Theorem: return time trick

For any channel Φ with $\Phi^* E_{fix} = E_{fix}$, we assume that for some $\varepsilon > 0$, the (complete) mixing time is finite. Then we have

$$(1-\varepsilon)C_S^{(cb)}(E_{fix}) \leq t_{mix}^{(cb)}(\varepsilon, \Phi) C_S^{(cb)}(\Phi) \leq t_{ret}^{(cb)}(\varepsilon, \Phi) C_S^{(cb)}(\Phi). \tag{9}$$

Perturbation argument

Perturbation argument helps us compare the complexity of different channels.
Suppose Φ_1, Φ_2 are two quantum channels such that

$$E_{fix}\Phi_i = E_{fix}. \tag{10}$$

Then we have

$$C_S(\Phi_2) \leq C_S(\Phi_1) + \|\Phi_1 - \Phi_2\|_\diamond C_S(E_{fix}). \tag{11}$$

The same argument holds for correlation assisted complexity C_S^{cb}.

Comparison to minimal length of quantum circuits

For quantum circuits, a natural definition of complexity is defined by the length. Suppose a gate set $S \subseteq U(d)$ is given.

Definition

For any $U \in U(d)$, the exact complexity(or length) of U is defined by

$$l_S(U) := \inf\{l \geq 1 : U = V_1 \cdots V_l,\ V_i \in S\}. \tag{12}$$

Given any $\delta > 0$, the approximate δ-complexity is defined by

$$l_{S,\delta}(U) := \inf\{l \geq 1 : \|U - V_1 \cdots V_l\|_\infty \leq \delta,\ V_i \in S\}. \tag{13}$$

Theorem

Suppose $\Phi = U \cdot U^\dagger$ is a unitary channel. Then the resource-dependent complexity $C_S(\Phi)$ of Φ is upper bounded by $l_S(U)$.

Comparison to Nielsen's geometric complexity

Lie group theory: basics

Suppose G is a locally compact Lie group, with Haar measure μ. For any unitary representation of G on a Hilbert space $\mathcal{K}$:

$$\pi : G \to U(\mathcal{K}),$$

define the induced representation of the Lie algebra $\mathfrak{g}$

$$d\pi : \mathfrak{g} \to \mathfrak{u}(\mathcal{K}),$$

which is connected by the exponential map: $\exp(d\pi(X)) = \pi(\exp(X))$, $X \in \mathfrak{g},\ \exp(X) \in G$.

Lie theory: continue

Let $H \subseteq \mathfrak{g}$ be a subspace of the Lie-algebra and $X_1, \cdots, X_m$ be a linearly independent set such that $span\{X_1, \cdots, X_m\} = H$. We consider $\{X_1, \cdots, X_m\}$ as the infinitesimal resource to construct the target unitary. For any element $h \in H$, define the induced norm of h by

$$\|h\|_H = \sum_j |\alpha_j|^2, \; h = \sum_j \alpha_j X_j. \tag{14}$$

Then the corresponding *Carnot-Caratheodory distance* is given by

$$d_H(g, h) := \inf\{\int_0^1 \|X_{\gamma(t)}\|_H dt : \gamma'(t) = X_{\gamma(t)}\gamma(t), X_{\gamma(t)} \in H\}, \tag{15}$$

where $\gamma(t)$ ranges over all the piecewise smooth curve such that $\gamma(0) = g, \gamma(1) = h, \; g, h \in G$. $\gamma'(t) = X_{\gamma(t)}\gamma(t), X_{\gamma(t)} \in H$ should be understood as the admissible directions of the curve are restricted within H.

Recovery of Geometric complexity

Suppose $G = SU(2^n)$ and H is taken to be the full Lie algebra $\mathfrak{su}(2^n)$. The orthonormal set is given by

$$\{\sigma, p\sigma' : \sigma \in \Sigma, \sigma' \in \Sigma'\}, \tag{16}$$

where $p > 0$ is a weight constant, Σ is the set of Pauli operators with weight less than or equal to 2, Σ' is the set of Pauli operators with weight greater than 2. For any $U \in SU(2^n)$, the geometric complexity is defined to be $d_H(g, I)$, which recovers the definition proposed by Nielsen and his collaborators.

Theorem

For any $g \in G$, we have

$$C_{H,\pi}(Ad_{\pi(g)}) \leq d_H(g, I). \tag{17}$$

Comparison to Wasserstein complexity of order 1

For n-qudit system $\mathcal{H}_n := \mathbb{C}_d^{\otimes n}$,:

- $\mathcal{O}_n^T$ is denoted as the traceless Hermitian operators on $\mathcal{H}_n$.
- $\mathcal{O}_n$ is denoted as the set of self-adjoint operators on $\mathcal{H}_n$.

For any $X \in \mathcal{O}_n^T$,

$$\|X\|_{W_1} := \frac{1}{2} \min \left\{ \sum_{i=1}^{n} \|X^{(i)}\|_1 : \ X^{(i)} \in \mathcal{O}_n^T, \mathrm{tr}_i(X^{(i)}) = 0, X = \sum_{i=1}^{n} X^{(i)} \right\}. \tag{18}$$

For any quantum channel $\Phi : \mathbb{B}(\mathcal{H}_n) \to \mathbb{B}(\mathcal{H}_n)$, the Wasserstein complexity $C_{W_1}(\Phi)$ is defined by

$$C_{W_1}(\Phi) := \sup_{\rho \in \mathcal{D}(\mathcal{H}_n)} \|\rho - \Phi(\rho)\|_{W_1} = \|id - \Phi : \mathbb{B}(\mathcal{H}_n)_* \to (\mathcal{O}_n^T, \|\cdot\|_{W_1})\|, \tag{19}$$

Theorem

If S is given by Pauli gate, C_{W_1} is equivalent to C_S.

Linear growth conjecture of complexity

Brown-Susskind conjecture: a generic random quantum circuit keeps growing linearly with time before exponential time.
We can show that given any random quantum circuit produced by the resource set, the resourse-dependent complexity grows linearly with time before return time.

Hamiltonian simulation

The starting point is a Hamiltonian given by

$$H = \sum_{j=1}^{L} h_j H_j$$

where H_j are some 'more elementary' components such that $\|H\|_\infty \leq 1$ and h_j are some positive weights. Denote

$$\lambda := \sum_{j=1}^{L} h_j. \tag{20}$$

In this setting, we consider

$$S = \{H_j | 1 \leq j \leq L\} \tag{21}$$

as a set of resources. The goal is to approximate the unitary $U(t) = \exp(itH)$ using a product of $U_j(\tau) = \exp\left(i\tau H_j\right), 1 \leq j \leq L$ up to some desired precision.

Bound on the cost of quantum computation

The number of elementary gates $U_j(\tau) = \exp(i\tau H_j)$ used is called the cost of this quantum computation.
Qdrift protocol: Suppose $N \geq 1$ is a fixed large integer and denote $\tau = \frac{t\lambda}{N}$. Assume $j_1, j_2, \cdots, j_N$ are i.i.d. random variables distributed as

$$\mathbb{P}(j_1 = j) = \frac{h_j}{\sum_{l=1}^{L} h_l}, 1 \leq j \leq L. \tag{22}$$

Then the random unitary of length N defined by

$$V_{j_1,\cdots,j_N} = \prod_{k=1}^{N} \exp(i\tau H_{j_k}) \tag{23}$$

is a candidate to approximate $U(t)$ up to precision ε.

In fact, The mean of $V_{j_1,\cdots,j_N}$ is given as follows:

$$\mathbb{E}\, V_{j_1,\cdots,j_N}\,\rho\, V^*_{j_1,\cdots,j_N} = \Phi_\tau^N(\rho),$$

where for any $\tau > 0$, Φ_τ is defined by

$$\Phi_\tau(\rho) := \sum_{j=1}^{L} p_j \exp\big(i\tau H_j\big)\,\rho\,\exp\big(-i\tau H_j\big), \qquad (24)$$

where $p_j := \frac{h_j}{\lambda} = \frac{h_j}{\sum_{j=1}^{L} h_j}$.
It is shown in (Campbell 2019) that an upper bound of the gate count N is given by $O(\lambda^2 t^2/\varepsilon)$.

Theorem

$C_S(Ad_{U(t)}) \sim t$ for small t and it provides a lower bound on the cost of computation.

Summary

- Our flexible notion for quantum channels falls into the axiomatic regime from Wasserstein complexity.
- By choosing the resource set suitably, we can show upper and lower bounds for the complexity of random circuits and continuous time evolution.
- The linear aspect of the Brown-Süsskind conjecture is confirmed in our context. The new Brown-Süsskind threshold is determined by the geometric properties of the resource set.

Thank you!