# Lipschitz estimates in quasi-Banach Schatten ideals

F. Sukochev.

University of New South Wales

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# Introduction

This talk is a short survey concerning state-of-art in Lipschitz estimates for operator functions in Schatten ideals prompted by recent paper due to Edward McDonald and F.S. concerning Lipschitz estimates in the ideal $\mathcal{L}_p$ for $0 < p < 1$.

E. McDonald and F. S.: *Lipschitz estimates in quasi-Banach Schatten ideals*

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# Plan for this talk

1. The idea of operator Lipschitz, and some brief discussion of previous results
2. Wavelets and the wavelet description of Besov spaces
3. What we did
4. Some further comments

# Operator Lipschitz functions

Let $H$ be a (complex and separable) Hilbert space, and denote the operator norm by $\|\cdot\|_\infty$. A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *operator Lipschitz* if there exists a constant $C_f$ such that

$$\|f(A)-f(B)\|_\infty \leq C_f\|A-B\|_\infty, \quad A,B\in\mathcal{B}_{\mathrm{sa}}(H)$$

**Question (from Krein)**

Is every Lipschitz function operator Lipschitz? That is, does $|f(t)-f(s)|\lesssim|t-s|$ imply that $\|f(A)-f(B)\|_\infty\lesssim\|A-B\|_\infty$?

# Operator Lipschitz functions

## Answer

No.

Farforovskaya (1968): There exist Lipschitz functions that are not operator Lipschitz

Kato (1973): The absolute value function $f(t) = |t|$ is not operator Lipschitz

Johnson & Williams (1975): An operator Lipschitz function is differentiable.

# Operator Lipschitz functions

It is easy to check that sufficiently good functions are operator Lipschitz. The Duhamel formula states that

$$e^{i\xi A} - e^{i\xi B} = i\xi \int_0^1 e^{i\xi(1-\theta)A}(A-B)e^{i\xi\theta B}\,d\theta.$$

The integral converges in the Bochner sense. The triangle inequality implies

$$\|e^{i\xi A} - e^{i\xi B}\|_\infty \leq |\xi|\|A-B\|_\infty.$$

By Fourier inversion, if $f$ is Schwartz class on $\mathbb{R}$ then

$$\|f(A)-f(B)\|_\infty \leq \|A-B\|_\infty \cdot 2\pi\|\widehat{\partial f}\|_1.$$

But which functions are operator Lipschitz?

# Besov spaces

Denote by $\mathcal{S}(\mathbb{R})$ the algebra of all Schwartz class functions on $\mathbb{R}$, with its canonical Frechét topology, and denote by $\mathcal{S}'(\mathbb{R})$ its topological dual, the space of tempered distributions.
For $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ we consider the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$.

# Peller's theorem

The previous computation was based on Fourier inversion of $f$ and a description of $e^{i\xi A} - e^{i\xi B}$ as an integral (Duhamel's integral). Using a more subtle description of $e^{i\xi A} - e^{i\xi B}$, and handling the Littlewood-Paley components of $f$ individually, V. V. Peller has proved the following:

**Theorem (Peller (1990))**

*If $f$ is Lipschitz and belongs to the homogeneous Besov class $\dot{B}^1_{\infty,1}(\mathbb{R})$ then $f$ is operator Lipschitz.*

In other words, if

$$\int_0^\infty \sup_{t\in\mathbb{R}} \frac{|f(t-h)-2f(t)+f(t+h)|}{h^2}\,dh + \sup_{t\in\mathbb{R},h>0} \frac{|f(t+h)-f(t)|}{h} < \infty$$

then $f$ is operator Lipschitz.

# Peller's theorem

Peller's condition $\dot{B}^1_{\infty,1}$ is sufficient but is not necessary. In fact, an operator Lipschitz function does not need to be $C^1$.
Nonetheless the condition is "good enough" for virtually all purposes. For example, if $f' \in W^1_\infty(\mathbb{R})$ then $f$ is operator Lipschitz (see the preceding sliude with the explanation of Duhamel formula). Slightly better (i.e. weaker) sufficient conditions were found by Arazy, Barton and Friedman (1990). Interested listeners are referred to a substantial Aleksandrov-Peller 2016/7 survey in Russian Mathematical Surveys for the current state of art.

# Peller's Bernstein lemma

The core of Peller's proof is the following *operator Bernstein inequality*.

**Theorem (Peller (1990))**

*Let* $f \in L_\infty(\mathbb{R})$ *have Fourier transform supported in the interval* $[-\sigma, \sigma]$*, where* $\sigma > 0$*. There is a constant* $c_{\mathrm{abs}}$ *such that*

$$\|f\|_{\mathrm{OL}} \leq c_{\mathrm{abs}} \sigma \|f\|_\infty.$$

Here,

$$\|f\|_{\mathrm{OL}} := \sup_{A,B \in \mathcal{B}_{\mathrm{sa}}(H)} \frac{\|f(A) - f(B)\|_\infty}{\|A - B\|_\infty}.$$

Peller's condition/result follows from this inequality via the Littlewood-Paley characterisation of Besov spaces. The books which I recommend are: Sawano, Y.: Theory of Besov spaces. Developments in Mathematics, vol. 56. Springer, Singapore (2018); Grafakos, L.: Modern Fourier Analysis, Volume 250 of Graduate Texts in Mathematics, 3rd edn. Springer, New York (2014).

# Besov spaces in terms LP-decomposition

Denote by $\mathcal{S}(\mathbb{R})$ the algebra of all Schwartz class functions on $\mathbb{R}$, with its canonical Frechét topology, and denote by $\mathcal{S}'(\mathbb{R})$ its topological dual, the space of tempered distributions. Let $\Phi$ be a smooth function on $\mathbb{R}$ supported in the set $[-2,-1+\frac{1}{7})\cup(1-\frac{1}{7},2]$, and identically equal to 1 in the set $[-2+\frac{2}{7},-1)\cup(1,2-\frac{2}{7}]$. We assume that

$$\sum_{n\in\mathbb{Z}}\Phi(2^{-n}\xi)=1 \quad \xi\neq 0.$$

We will use a homogeneous Littlewood-Paley decomposition $\{\Delta_n\}_{n\in\mathbb{Z}}$ where $\Delta_n$ is the operator on $\mathcal{S}'(\mathbb{R})$ of Fourier multiplication by the function $\xi\mapsto\Phi(2^{-n}\xi)$. For $s\in\mathbb{R}$ and $p,q\in(0,\infty]$ we consider the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$. In terms of the Littlewood-Paley decomposition $\{\Delta_j\}_{j\in\mathbb{Z}}$, a distribution $f\in\mathcal{S}'(\mathbb{R})$ is said to belong to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$, where $s\in\mathbb{R}$ and $p,q\in(0,\infty]$ if

$$\|f\|_{\dot{B}^s_{p,q}}:=\|\{2^{js}\|\Delta_j f\|_p\}_{j\in\mathbb{Z}}\|_{\ell_q(\mathbb{Z})}<\infty.$$

# Schatten ideals

If $T$ is a compact operator on $H$, the singular value sequence of $T$ is defined as

$$\mu(k,T) := \inf\{\|T-R\|_\infty \ :\ \mathrm{rank}(R)\le k\},\quad k\ge 0.$$

(Equivalently, $\mu(T)=\{\mu(k,T)\}_{k=0}^\infty$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities.) Note that $\|T\|_\infty=\mu(0,T)=\|\mu(T)\|_{\ell_\infty}$. For $1\le p<\infty$, the Schatten $\mathcal{L}_p$-norm of a compact operator $T$ is

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \left(\sum_{k=0}^\infty \mu(k,T)^p\right)^{\frac{1}{p}}.$$

Equivalently, $\|T\|_p=\mathrm{Tr}(|T|^p)^{1/p}$. It is not really obvious, but this is a norm (i.e. $\|T+S\|_p\le\|T\|_p+\|S\|_p$.)

# $\mathcal{L}_p$-operator Lipschitz functions

A function $f$ on $\mathbb{R}$ is said to be $\mathcal{L}_p$-operator Lipschitz if there exists a constant $C_f > 0$ such that

$$\|f(A) - f(B)\|_p \leq C_f \|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

By a duality argument, $\mathcal{L}_1$-operator Lipschitz is the same thing as operator Lipschitz.
What about $1 < p < \infty$?

**Theorem (D. Potapov and F. S., Acta Math (2011))**

*For $1 < p < \infty$, all Lipschitz functions are $\mathcal{L}_p$-operator Lipschitz.*

For $p = 2$ this is almost trivial and has been known for approx. 110 years.
For $p \neq 2$, this requires some deep harmonic analysis (Bourgain, Burkholder, UMD-spaces, vector-valued multipliers, transferrence, etc).

# What about $0 < p < 1$?

For $0 < p < 1$, we can still define

$$\|T\|_p := \|\mu(T)\|_{\ell_p} = \mathrm{Tr}(|T|^p)^{\frac{1}{p}}.$$

This is not a norm, merely a quasi-norm. There is no triangle inequality, merely a quasi-triangle inequality

$$\|T+S\|_p \leq 2^{\frac{1}{p}-1}(\|T\|_p + \|S\|_p).$$

For $0 < p < 1$, we have the $p$-triangle inequality

$$\|T+S\|_p^p \leq \|T\|_p^p + \|S\|_p^p, \quad T, S \in \mathcal{L}_p.$$

While it still has a complete translation-invariant metric, $\mathcal{L}_p$ for $0 < p < 1$ can seem quite pathological:

- There is *no* Hahn-Banach theorem for $\mathcal{L}_p$;
- The unit ball of $\mathcal{L}_p$, $0 < p < 1$ is not convex;
- There is *no* Bochner integration theory for $\mathcal{L}_p$-valued functions.

# $\mathcal{L}_p$-Lipschitz functions via Schur multiplication

The common method of the study of Lipschitz continuity for all $p \in (0, \infty]$ is the estimates for the norm of Schur multipliers for the function of two variables, $f^{[1]}$, the divided difference for the function $f$, in the corresponding Schatten ideal. We denote its norm in the Schatten ideal $S_p$ by $\|f^{[1]}\|_{\mathrm{m}_p}$.

# A simple new (actually, old) observation

Let $f$ be a Lipschitz function on $\mathbb{R}$, and let $0 < p \leq \infty$. The following assertions are equivalent:

1. There is a constant $c_f$ such that $\|f(A) - f(B)\|_p \leq c_f \|A - B\|_p$ for all bounded self-adjoint $A$ and $B$ with $A - B \in \mathcal{L}_p$,
2. The matrix of divided differences $\{f^{[1]}(t,s)\}_{t,s\in\mathbb{R}}$, where $f^{[1]}$ is defined as
$$f^{[1]}(t,s) := \frac{f(t) - f(s)}{t - s}, \quad t \neq s \in \mathbb{R}.$$
is a Schur multiplier of $\mathcal{L}_p$ in the sense that
$$\sup_{\lambda,\mu} \|\{f^{[1]}(\lambda_j, \mu_k)\}_{j,k=0}^n\|_{\mathrm{m}_p} < \infty$$
where the supremum ranges over all *disjoint* sequences $\lambda, \mu \subset \mathbb{R}$ and all $n \geq 1$.

# $\mathcal{L}_p$-Lipschitz functions for $0 < p < 1$.

Which functions are Lipschitz in $\mathcal{L}_p$ when $0 < p < 1$? Alternatively, for which fucntions we have $\|f^{[1]}\|_{\mathrm{m}_p} < \infty$?

At least some functions are, for example $f(t) = (t+\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

What about $f(t) = \exp(it\xi)$ for $\xi \in \mathbb{R}$?

# Periodic functions are not $\mathcal{L}_p$-Lipschitz for $0 < p < 1$.

A first hint that the $0 < p < 1$ case is interesting and somewhat unusual comes from the following:

**Lemma (E. McDonald and F.S. (2022))**

*Let $0 < p < 1$, and let $f$ be a periodic function on $\mathbb{R}$. Then $f$ is $\mathcal{L}_p$-Lipschitz if and only if it is constant.*

What does this imply?

- Even $C^\infty$ functions with all derivatives bounded may not be $\mathcal{L}_p$-Lipschitz;
- In particular $f(t) = \exp(it\xi)$, $\xi \neq 0$ is not $\mathcal{L}_p$-Lipschitz for any $0 < p < 1$. This is a strong indication that methods based on a Fourier decomposition are unlikely to work.

What will work?

# Wavelets (compactly supported)

Nonetheless, it is possible to prove that if $f$ is a compactly supported $C^k$ function where $k > \frac{1}{p}$ then $f$ is $\mathcal{L}_p$-Lipschitz. To see this, let us firstly ask ourselves:" it What is a good way of approximating a general function from compactly supported $C^k$-functions?

**Theorem (Daubechies (1988))**

*For all $k > 0$, there exists a compactly supported $C^k$ function $\psi$ such that the system of translations and dilations*

$$\psi_{j,k}(t) := 2^{\frac{j}{2}}\psi(2^j t - k), \quad j, k \in \mathbb{Z}$$

*forms an orthonormal basis of $L_2(\mathbb{R})$.*

We will fix a compactly supported wavelet $\psi$ of regularity $C^r$ for some $r > 1$

# Wavelet method

Every $f \in L_2(\mathbb{R})$ admits an $L_2$-convergent wavelet decomposition

$$f = \sum_{j,k\in\mathbb{Z}} \psi_{j,k}\langle f, \psi_{j,k}\rangle.$$

This is called a wavelet series. For brevity, denote

$$f_j = \sum_{k\in\mathbb{Z}} \psi_{j,k}\langle f, \psi_{j,k}\rangle \in L_2(\mathbb{R}), \quad j \in \mathbb{Z}$$

That is, we have the $L_2$-convergent series

$$f = \sum_{j\in\mathbb{Z}} f_j, \quad f_j(t) = \sum_{k\in\mathbb{Z}} 2^{\frac{j}{2}}\psi(2^j t - k)\langle f, \psi_{j,k}\rangle, \quad t \in \mathbb{R}.$$

Roughly speaking, our strategy will be to bound $\|f^{[1]}\|_{\mathrm{m}_p}$ using the wavelet decomposition as follows

$$\|f^{[1]}\|_{\mathrm{m}_p}^p \le \sum_{j\in\mathbb{Z}} \|f_j^{[1]}\|_{\mathrm{m}_p}^p.$$

# Wavelets and Besov spaces

The mysterious Besov spaces have a very simple characterisation in terms of wavelet coefficients.

## Theorem (Meyer (1986))

*Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let $\psi$ be a compactly supported $C^k$ wavelet where $k > -s$. Then a distribution $f \in \mathcal{D}'(\mathbb{R})$ belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ if and only if*

$$\|f\|_{B^s_{p,q}} \approx \sum_{j \in \mathbb{Z}} 2^{jq(s+\frac{1}{2}-\frac{1}{p})} \left( \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^p \right)^{\frac{q}{p}} < \infty.$$

# A new result

Using wavelet methods we can get the following:

**Theorem (E. McDonald and F. S. (2022))**

*Let $0 < p < 1$. Let $f \in \dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ be Lipschitz continuous. Then $f$ is $\mathcal{L}_p$-Lipschitz and*

$$\|f(A) - f(B)\|_p \leq C_p(\|f'\|_\infty + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})})\|A - B\|_p, \quad A, B \in \mathcal{B}_{\mathrm{sa}}(H).$$

In other words, we require that $f$ be Lipschitz and for some $n > \frac{1}{p}$ that

$$\int_0^\infty \left(\int_{-\infty}^\infty \left|\sum_{k=0}^n \binom{n}{k}(-1)^{n-k} f(t+kh)\right|^{\frac{p}{1-p}} dt\right)^{1-p} \frac{dh}{h^2} < \infty.$$

# What else can we do?

Wavelets are not new, but their application to this theory is!

Some other things we can achieve:

- A new proof of Peller's Theorem (1990) asserting that any $f \in \dot{B}^1_{\infty 1}$ is operator Lipschitz.
- For all $n \geq 0$, the inequality

$$\sum_{k=0}^{n} \mu(k, f(A) - f(B))^p \lesssim (\|f'\|_\infty + \|f\|_{\dot{B}^{\frac{1}{p}}_{\frac{p}{p-1},p}}) \sum_{k=0}^{n} \mu(k, A - B)^p$$

  (this recovers the previous result with $n = \infty$.)
- Hölder-type estimates of the form

$$\|f(A) - f(B)\|_p \lesssim_f \||A - B|^\alpha\|_p$$

  for $f$ in some Besov space.

# Quasi-Banach Bernstein Lemma

The core of the proof of the $B^{\frac{1}{p}}_{\frac{p}{1-p},p}(\mathbb{R})$ sufficient condition is the following "Bernstein" inequality. Here, $\frac{p}{p-1} = p^\sharp$.

**Theorem**

*Let $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ be a scalar sequence, and let $\phi \in C_c^N(-\frac{1}{2}, \frac{1}{2})$ where $N$ is large enough ($N > \frac{2}{p}$ will do). Define*

$$\phi_\alpha(t) = \sum_{k\in\mathbb{Z}} \alpha_k \phi(t-k).$$

*Then*

$$\|\phi_\alpha\|_{\mathrm{OL}_p} \leq c_{\mathrm{abs}} \|\alpha\|_{\ell_{p^\sharp}}.$$

The $B^{\frac{1}{p}}_{\frac{p}{1-p},p}$ sufficient condition follows from this inequality via the wavelet characterisation of Besov spaces.

Thank you for listening!