Semigroup C^*-Algebras arising from Graphs of monoids

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21 June 2023

0 Outline

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1 C*-algebras

A C*-algebra A is a Banach algebra endowed with an involution $*$ satisfying: For all $\lambda \in \mathbb{C}$ and all $x,\ y \in A$,

1. $(\lambda x + y)^* = \bar{\lambda} x^* + y^*$;
2. $x^{**} = x$;
3. $(xy)^* = y^* x^*$;
4. $\|x^* x\| = \|x\|^2$.

Examples: $M_n(\mathbb{C}),\ C_0(X),\ L_\infty(X),\ \mathcal{L}(H),\ \mathcal{K}(H)$; $L_p(X),\ 1 \leq p < \infty$.

Gelfand-Naimark Theorem: Every C*-algebra is isometrically $*$-isomorphic to a C*-subalgebra of $\mathcal{L}(H)$.

1 Group C*-algebras

Here we consider only discrete groups.

Given a group G, its left regular representation λ is the homomorphism of G into the unitary group of $\mathcal{L}(\ell_2(G))$, defined by $s \mapsto \lambda_s$ with $\lambda_s f(t) = f(s^{-1}t)$ for all $s \in G$ and all $f \in \ell_2(G)$. The left reduced group C*-algebra of G, denoted by $C_r^*(G)$, is defined to be the smallest C*-algebra in $\mathcal{L}(\ell_2(G))$, containg $\lambda_s, \ \forall s \in G$.

Theorem: If G is an abelian group, then $C_r^*(G) = C(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of G (consisting of all characters on G).

Example: $C_r^*(\mathbb{Z}) = C(\mathbb{T})$.

1 Semigroup C*-algebras

Let P be a left cancellative and discrete semigroup, its left regular representation is given as follows:

$$P \to \mathcal{L}(\ell_2(P)),\ p \ \mapsto V_p\ [\delta_x \mapsto \delta_{px}],\ \forall p \in P.$$

The left reduced semigroup C*-algebra of P, denoted by $C_r^*(P)$, is defined to be the smallest C*-algebra in $\mathcal{L}(\ell_2(P))$, containg $V_p,\ \forall p \in P$.

Example: $C_r^*(\mathbb{N})$ is the Toeplitz algebra.

A graph Γ is a pair $(V,\ E)$ with two maps $E \to V \times V,\ e \mapsto (o(e),\ t(e))$ and $E \to E,\ e \mapsto \bar{e}$, satisfying: $\bar{\bar{e}} = e$ and $\bar{e} \neq e$ and $o(e) = t(\bar{e})$. A tree is connected non-empty graph without circuits.

A graph of groups: $\Gamma = (V,\ E)$ connected, G_v and G_e.

$G_{\bar{e}} = G_e,\ G_e \to G_{t(e)}:\ x \mapsto x^e,\ G_e \to G_{o(e)}:\ x \mapsto x^{\bar{e}}$.

T maximal subtree, $E = T \cup A \cup \bar{A}$, The fundamental group $G = \pi_{1,\ T} :=< \{G_v\}_{v \in V} \cup A | g^e = g^{\bar{e}},\ \forall e \in T,\ g \in G_e, eg^e = g^{\bar{e}}e, \forall e \in A, g \in G_e >$.

Examples

$T = (V,\ E)$ a tree of groups $\Longrightarrow G$ is the amagamated free product of G_v along G_e

Assume $G_e = \{\epsilon\},\ \forall e \in E$: then G is the free product of G_v.

Γ a bouquet of circles, $V = \{v\}$, $G_v \cong \mathbb{Z}$ and $G_e \cong \mathbb{Z},\ \forall e \in E$.
$\Longrightarrow G$ is a one vertex generalised Baumslag-Solitar group.

$$G =< \{b\} \cup A|\ b^{n_e} e = e b^{\mathrm{sgn}(e) m_e},\ \forall e \in A >,$$

Here $n_e,\ m_e \in \mathbb{Z}_+$ and $\mathrm{sgn}(e) \in \{\pm 1\}$.
$\sharp A = 1 \Longrightarrow G$ is the classical Baumslag-Solitar group.

(V, E)
G_v totally ordered with positive cone P_v, $\forall v \in V$ and
$P_e := \{g \in G_e,\ g^e \in P_{t(e)}\}$, $\forall e \in E$.

Assume $P_e = P_{\bar{e}}$ for all $e \in T$ and either $P_e = P_{\bar{e}}$ or $P_e = P_{\bar{e}}^{-1}$ for all $e \in A$. Define $A_+ := \{e \in A |\ P_e = P_{\bar{e}}\}$ and $A_- := \{e \in A |\ P_e = P_{\bar{e}}^{-1}\}$.

Remark

The embedding $P_e \to P_{o(e)},\ g \mapsto g^{\bar{e}}$ is order preserving for all $e \in T \cup A_+$ and order reversing for all $e \in A_-$. For instance, in Example 2, A_+ consists exactly of those $e \in A$ with $\mathrm{sgn}(e) = 1$, and A_- consists exactly of those $e \in A$ with $\mathrm{sgn}(e) = -1$.

The fundamental monoid $P \subseteq G$: generated by $P_v,\ v \in V$ and A.
The submonoid $P_T \subseteq P$: generated by $P_v,\ v \in V$.

2 Right LCM Property

P right LCM: for all $p,\ q \in P$, $pP \cap qP = \emptyset$ or $pP \cap qP = rP$.

condition (LCM) for P: for all $e \in E$, $p \in P_{o(e)}$, either $p^{-1}P^{\bar{e}}_{\bar{e}} = \emptyset$ or $p^{-1}P^{\bar{e}}_{\bar{e}} = qP^{\bar{e}}_{\bar{e}}$ for some $q \in P_{o(e)}$, where

$$p^{-1}P^{\bar{e}}_{\bar{e}} := \{x \in P_{o(e)},\ px \in P^{\bar{e}}_{\bar{e}}\}.$$

Theorem (C. Chen, X. Li)

condition (LCM) for $P \implies P$ right LCM.

For convenience, we introduce the notation $\prec$ in P: $p,\ q \in P$, $p \prec q$ if $q \in pP$.

Theorem (C. Chen, X. Li)

Assume that condition (LCM) is satisfied. If $P_e = \{\epsilon\}$ for some $e \in T$ and there exists $v \in V$ and a sequence $x_n \in P_v \setminus \{\epsilon\}$ with $x_{n+1} \prec x_n$ such that, for every $p \in P_v \setminus \{\epsilon\}$, $x_n \prec p$ and $x_n \neq p$ for all sufficiently big n, then $C^*_\lambda(P)$ is purely infinite simple.

2 Closed invariant subsets

$V, \ E, \ G_v$ countable, condition (LCM) for P

The associated character space Ω: all nonzero filters (multiplicative) χ

$$\chi: \ \{pP, \ p \in P\} \cup \{\emptyset\} \to \{0, \ 1\},$$

with the pointwise convergence topology.

The partial action $G \curvearrowright \Omega$: $g: \ U_{g^{-1}} \to U_g, \ \chi \mapsto g \cdot \chi = \chi(g^{-1} \cdot)$,
$\chi \in U_{g^{-1}} \Leftrightarrow g = pq^{-1}$ for some $p, \ q \in P$ and $\chi(qP) = 1$.

Theorem [CELY17]

$C^*_\lambda(P) \cong C^*_r(G \ltimes \Omega)$.

For $w = x_1x_2x_3\cdots$, $x_* \in \{P_v \setminus \epsilon\}_{v\in V} \cup A$, define $\chi_w \in \Omega$:

$\chi_w(xP) = 1 \iff [w]_j := x_1x_2\cdots x_j \in xP$ for some j.

$\chi = \chi_w$ by [LOS18].

$\Omega_\infty : \chi \neq \chi_w$ with w finite word.

$\{\infty\} = \partial\Omega_{P_T}$.

$\Omega_{b,\ \infty} := \{\chi \in \Omega,\ (g\cdot\chi)(b^iP) = 1,\ \forall g \in G,\ \forall i \in \mathbb{N}\}$, where $b \in P_u$ for some $u \in V$ is fixed.

2 Additional assumption

Assume $G_v \subseteq (\mathbb{R}, +), \ v \in V$

Condition (D)

$P_e = \{\epsilon\}$ or $P_e \cong \mathbb{Z}_+$ for all $e \in T \cup A$.

Theorem (C. Chen, X. Li)

Suppose that $G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$,and that $\sharp V > 1$ or $V = \{v\},\ G_v \subseteq (\mathbb{R}, +)$ dense and $A \neq \emptyset$. Further assume that conditions (LCM) and (D) are satisfied. The lists of all closed invariant subspaces of Ω are as follows:

(i) Assume that $P_e = \{\epsilon\}$ for some $e \in T$.

(i_1)G_v dense in $\mathbb{R}$ for some v: $\partial\Omega = \Omega$.

(i_2)$P_v \cong \mathbb{Z}_{\geq 0}$ for all v: $\partial\Omega = \overline{\Omega_\infty} \subseteq \Omega$.

(ii) Assume that $P_e \neq \{\epsilon\}$ for all $e \in T$.

(ii_1)$\sharp A=0$:$\partial\Omega=\{\infty\}\subsetneq\overline{\Omega_\infty} \subseteq \Omega$.

(ii_2)G_v dense in $\mathbb{R}$ for some v and $\sharp A \geq 1$: $\partial\Omega = \Omega_{b,\ \infty} \subsetneq \Omega$.

(ii_3)$P_v \cong \mathbb{Z}_{\geq 0}$ for all v, $\sharp A \geq 1$ and $\sharp V > 1$: $\partial\Omega = \Omega_{b,\ \infty} \subsetneq \overline{\Omega_\infty} \subseteq \Omega$.

$G \curvearrowright X$ topologically free

$\Longleftrightarrow \exists X' \subseteq X$ dense s.t. $g \cdot x = x,\ g \in G,\ x \in X'$ implies $g = \epsilon$.

Assume that $G_v \subseteq (\mathbb{R},\ +)$ for all $v \in V$, and $\sharp V > 1$ or $A \neq \emptyset$, and that conditions (LCM) and (D) are satisfied.

Assume in addition $P_e \neq \{\epsilon\}$ for all $e \in T$. Given $e \in A$, let $v = o(e)$ and $w = t(e)$. Let b_v be the generator of P_v and b_w the generator of P_w. Let $m_e,\ n_e \in \mathbb{Z}_+$ be such that $()^{\bar{e}}:\ P_{\bar{e}} \to P_v$ is given by $z \mapsto n_e z$ and $()^e:\ P_{\bar{e}} \to P_w$ is given by $z \mapsto \pm m_e z$. Then we have $b_v^{n_e} e = e b_w^{\pm m_e}$ in G. Moreover, as $P_e \neq \{\epsilon\}$ for all $e \in T$,
$< b_v^{n_e} > \cap < b_w^{m_e} > = < b_v^{l_e n_e} > = < b_w^{k_e m_e} >$ for some $k_e,\ l_e \in \mathbb{Z}_+$.

Theorem

$G \curvearrowright X$ is topologically free for every closed invariant subspace $X \subseteq \Omega$ if and only if one of the following holds:
(i) $P_e = \{\epsilon\}$ for some $e \in T$.

(ii) $P_e \neq \{\epsilon\}$ for all $e \in T$, $\sharp V > 1$, $A \neq \emptyset$, and one of the following holds:
(ii_1) $k_e \nmid l_e$ for some $e \in A$,
(ii_2) $k_e \mid l_e$ for all $e \in A$ and $(\cap_{e \in A} < b_v^{k_e n_e}) \cap (\cap_{v \in V} G_v) = \{\epsilon\}$.

(iii) $\sharp V = 1$, $\sharp A = \infty$, $\sharp A_+ \in \{0, \infty\}$, and (ii_1) or (ii_2) holds.

(iv) $\sharp V = 1$, $\sharp A = \infty$, $\sharp A_+ < \infty$, (ii_1) or (ii_2) holds, and either $\sharp A_+ \geq 2$ or $\sharp A_+ = 1, \ A_+ = \{e\}$ and $m_e \neq 1$.

Corollary

If one of (i)-(iv) in the above theorem is satisfied, the the map $X \mapsto C_r^*(G \ltimes (\Omega \setminus X))$ is a one-to-one correspondence between closed invariant subspaces of Ω and ideals of $C_\lambda^*(P) \cong C_r^*(G \ltimes \Omega)$.

Theorem (C. Chen, X. Li)

If condition (LCM) for P is satisfied, then $C^*_\lambda(P)$ is nuclear iff $C^*_\lambda(P_T)$ nuclear.

Assume, in addition, $G_v \subseteq (\mathbb{R}, +)$, $\sharp V > 1$ or $\sharp A > 0$, $C^*_\lambda(P)$ is nuclear iff

For all $T' \subseteq T$ with $P_e \neq \{\epsilon\}$ for all $e \in T'$, either T' consists of a single vertex or T' consists of two vertices v, w and a pair of edges $e, \bar{e}$ with $o(e) = v, t(e) = w$, such that $P_v \cong \mathbb{Z}_+, P_w \cong \mathbb{Z}_+$, and the embeddings $()^e, ()^{\bar{e}}$ are both given by $\mathbb{Z}_+ \to \mathbb{Z}_+, z \mapsto 2z$.

Theorem (C. Chen, X. Li)

Suppose that $G_v \subseteq (\mathbb{R}, +)$ for all $v \in V$,and that $\sharp V > 1$ or $V = \{v\}$, $G_v \subseteq (\mathbb{R}, +)$ dense and $A \neq \emptyset$. Further assume that conditions (LCM) and (D) are satisfied.
(i)

$$K_0(C_r^*(G \ltimes \Omega)) \cong \mathbb{Z} \text{ and } K_1(C_r^*(G \ltimes \Omega)) \cong 0;$$

$$K_*(C_r^*(G \ltimes \Omega_{b,\ \infty})) \cong K_*(C(\Omega_{b,\ \infty}) \rtimes_r G) \cong K_*(C_\lambda^*(G_T));$$

$$K_*(C_r^*(G \ltimes \{\infty\})) \cong K_*(C_\lambda^*(G_T))$$

if $\{\infty\}$ is closed in Ω.

Theorem (C. Chen, X. Li)

(ii) When Ω_∞ is closed in Ω,

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z} \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}$$

if $P_e \neq \{\epsilon\}$ for all $e \in T$ and

$$K_0\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong \mathbb{Z}_n \text{ and } K_1\big(C_r^*(G \ltimes \Omega_\infty)\big) \cong 0$$

if $P_e = \{\epsilon\}$ for some $e \in T$. Here $n := 1/2\sharp\{e \in T : \ P_e = \{\epsilon\}\}$.

Theorem (C. Chen, X. Li)

$\partial C^*_\lambda(P) = C^*_r(G \ltimes \partial\Omega)$ is a UCT kirchberg algebra and is completely classified by K-theory, if the following two conditions are satisfied:

(TF) One of the following holds.
(i) There exists $e \in T$ with $P_e = \{\epsilon\}$.
(ii) For all $e \in T$, $P_e \neq \{\epsilon\}$, $\sharp A > 0$ and there exists $e \in A$ with $k_e \nmid l_e$.
(iii) For all $e \in T$, $P_e \neq \{\epsilon\}$, $\sharp A > 0$, for all $e \in A$, $k_e \mid l_e$ and $(\cap_{e \in A} < b_{t(e)}^{k_e n_e} >) \cap (\cap_{v \in V} G_v) = \{\epsilon\}$.

(N) For all $T' \subseteq T$ with $P_e \neq \{\epsilon\}$ for all $e \in T'$, either T' consists of a single vertex or T' consists of exactly two vertices v, w and one pair of edges e, $\bar{e}$ with $o(e) = v$, $t(e) = w$ such that $P_v \cong \mathbb{Z}_+$, $P_w \cong \mathbb{Z}_+$, and the embeddings $(\cdot)^e$, $(\cdot)^{\bar{e}}$ are both given by $\mathbb{Z}_+ \to \mathbb{Z}_+$, $z \mapsto 2z$.

The discussion of closed invariant subsets and K-theory above is incomplete. Also, we have some work on the topological freeness of the group action on closed invariant subsets, ideal structures of the semigroup C^*-algebras and an application to Cartan pair in UCT kirchberg algebras. We refer interested readers to [CL22].

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3 References

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Thank you for listening!