Exponential Ergodicity in Certain Quantum Markov Semigroups

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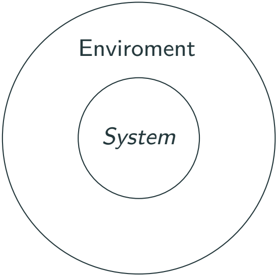
Background

An ideal quantum system is not realistic because it should be perfectly isolated; however, in practice, it is influenced by coupling to an environment.



Time evolution is governed by the global Hamiltonian

$$H = H_{\mathrm{S}} + H_{\mathrm{E}} + H_{\mathrm{int}}.$$

By taking the partial trace and assuming the Markov property we have the following Lindblad equation:

$$\frac{\mathrm{d}\rho_{\mathrm{S}}(t)}{\mathrm{d}t} = \mathcal{L}_*(\rho_{\mathrm{S}}(t)).$$

Mathematically, from a closed quantum system to an open quantum system, the Hamiltonian is replaced by a Lindblad operator

$$H_{\mathrm{S}} \rightsquigarrow \mathcal{L}_*.$$

Meanwhile, the time evolution is no longer described by means of one-parameter groups of unitary maps $\mathrm{e}^{\mathrm{i}tH_{\mathrm{S}}}$, but one needs to introduce semigroups of completely positive maps $\mathrm{e}^{t\mathcal{L}_*}$, thus leading to the concept of quantum Markov semigroups.

Background: Quantum Markov Semigroups

Definition 1 (Quantum Markov semigroup)

Let $\mathcal{A}$ be a von Neumann algebra. A quantum dynamical semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{A}$ is a family of bounded operators on $\mathcal{A}$ with the following properties:

- $\mathcal{T}_0(a) = a$ *for all* $a \in \mathcal{A}$ *and* $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s$ *for all* $t, s \geq 0$,
- $\mathcal{T}_t$ *is completely positive for all* $t \geq 0$,
- $\mathcal{T}_t$ *is* σ*-weakly continuous on* $\mathcal{A}$ *for all* $t \geq 0$,
- *for each* $a \in \mathcal{A}$, *the map* $t \mapsto \mathcal{T}_t(a)$ *is continuous w.r.t. the* σ*-weak topology.*

If $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ *in addition, we call* $(\mathcal{T}_t)_{t\geq 0}$ *a quantum Markov semigroup.*

Definition 2 (Predual semigroup)

The predual semigroup of $(\mathcal{T}_t)_{t\geq 0}$, $(\mathcal{T}_{*t})_{t\geq 0}$, *is a semigroup on* $\mathcal{A}_*$ *defined by*

$$\mathcal{T}_{*t}(\omega)(a) := \omega(\mathcal{T}_t(a)), \quad \forall a \in \mathcal{A}, \quad \omega \in \mathcal{A}_*.$$

Background: GKSL Forms

The characterization of the generator of a quantum dynamical semigroup due to Lindblad [Lin76] in the case of an arbitrary Hilbert space and to Gorini, Kossakowski and Sudarshan [GKS76] in the case of a finite-dimensional Hilbert space.

Theorem 3

A bounded operator $\mathcal{L}$ on $\mathcal{B}(\mathcal{H})$ is the generator of a ***uniformly continuous*** *quantum dynamical semigroup if and only if*

$$\mathcal{L}(a) = \mathrm{i}[H, a] - \frac{1}{2}\sum_j \left(V_j^\dagger V_j a - V_j^\dagger a V_j + a V_j^\dagger V_j\right),$$

where $V_j \in \mathcal{B}(\mathcal{H})$, $\sum_j V_j \in \mathcal{B}(\mathcal{H})$ and $H \in \mathcal{B}(\mathcal{H})$ self-adjoint. In this case, the predual generator is of the form

$$\mathcal{L}_*(\rho) = -\mathrm{i}[H, \rho] - \frac{1}{2}\sum_j \left(V_j^\dagger V_j \rho - V_j \rho V_j^\dagger + \rho V_j^\dagger V_j\right).$$

Exponential Ergodicity

Let $(\mathcal{T}_t)_{t\geq 0}$ be a quantum Markov semigroup on the von Neumann algebra $\mathcal{A}$. Assume that $(\mathcal{T}_t)_{t\geq 0}$ possesses a faithful normal invariant state ρ, i.e.

$$\rho(a) > 0, \quad a \in \mathcal{A}_+ \setminus \{0\}; \quad \rho(\mathcal{T}_t(a)) = \rho(a), \quad \forall t \geq 0, \quad \forall a \in \mathcal{A}.$$

Induced semigroup

Let $(\mathcal{H}, \pi, \xi)$ be the Gelfand-Naimark-Segal representation associated to the faithful normal state ρ. Then, we can construct a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on the Hilbert space $\mathcal{H}$ by

$$T_t(\pi(a)\xi) := \pi(\mathcal{T}_t(a))\xi, \quad a \in \mathcal{A}.$$

$(T_t)_{t\geq 0}$ is referred to be the induced semigroup of $(\mathcal{T}_t)_{t\geq 0}$. By L we denote the generator of the induced semigroup $(T_t)_{t\geq 0}$.

We use $\mathcal{S}(\mathcal{A})$ to denote all normal states on the von Neumann algebra $\mathcal{A}$.

Special states

By $\mathcal{S}_\rho(\mathcal{A})$ we denote the set of all normal states on $\mathcal{A}$ which are majorized by a scalar multiple of ρ. That is,

$$\mathcal{S}_\rho(\mathcal{A}) := \{\phi \in \mathcal{S}(\mathcal{A}) : \exists \lambda \geq 0 \text{ s.t. } \phi \leq \lambda\rho\}.$$

$\mathcal{S}_\rho(\mathcal{A})$ is dense in $\mathcal{S}(\mathcal{A})$, and the linear span of $\mathcal{S}_\rho(\mathcal{A})$ is dense in $\mathcal{A}_*$.

Lemma 4

ϕ is a positive σ-weakly continuous functional on $\mathcal{A}$ that is majorized by $\lambda\rho$ for some $\lambda \geq 0$ if and only if there exists a unique $x_\phi \in \pi(\mathcal{A})'$ with $0 \leq x_\phi \leq \lambda\mathbb{1}$ such that

$$\phi(a) = \langle x_\phi \xi, \pi(a)\xi \rangle, \quad \forall a \in \mathcal{A},$$

where $\pi(\mathcal{A})'$ denotes the commutant of $\pi(\mathcal{A})$.

Spectral gap

The spectral gap of the induced generator L is the non-negative number α defined as follows:

$$\alpha := \inf \left\{ -\operatorname{Re}\langle x, Lx \rangle : x \in \operatorname{Dom} L \subset \mathcal{H},\ \|x\| = 1,\ x \in \ker L^{\perp} \right\}.$$

Notice that $\ker L$ characterizes invariant vectors of $(T_t)_{t \geq 0}$.

This term is referred to as the “spectral gap” because in the case where the generator L is **self-adjoint**, α represents the maximum value for which there is no part of the spectrum of L within the interval $(-\alpha, 0)$.

Remark

Roughly speaking, the self-adjointness of L is equivalent to the reversibility (detailed balance) of $(T_t)_{t \geq 0}$. We do not assume it in our work.

Theorem 5 (Exponential Ergodicity)

Assume that the infinitesimal generator L of the induced semigroup $(T_t)_{t\geq 0}$ has a spectral gap $\alpha > 0$, and there exists a common core for L and L^*. Then, there exists a projection $\mathcal{P}$ onto σ-weakly continuous functionals that are invariant under $(\mathcal{T}_{*t})_{t\geq 0}$, and for all $\psi \in \mathcal{A}_*$, $\mathcal{T}_{*t}(\psi) \to \mathcal{P}(\psi)$ in the norm topology as $t \to +\infty$. In particular, if $\psi \in \mathrm{Span}\{\mathcal{S}_\rho(\mathcal{A})\}$, then

$$\|\mathcal{T}_{*t}(\psi) - \mathcal{P}(\psi)\|_{\mathcal{A}_*} \leq \mathrm{e}^{-\alpha t} \left\| \sum_{k=1}^{n} \overline{c}_k x_{\psi_k} \xi - P\left(\sum_{k=1}^{n} \overline{c}_k x_{\psi_k} \xi \right) \right\|_{\mathcal{H}},$$

where P is the projection onto invariant vectors of $(T_t)_{t\geq 0}$.

In the following we will answer the following two questions: In general, can we observe uniform exponential convergence for normal states in $\mathcal{S}_\rho(\mathcal{A})$? Do σ-weakly continuous functionals in $\mathcal{A}_* \setminus \mathrm{Span}\{\mathcal{S}_\rho(\mathcal{A})\}$ demonstrate exponential convergence?

A quantum Ornstein-Uhlenbeck semigroup models the evolution of an open quantum system that is coupled to a reservoir with inverse temperature $\beta > 0$.

Let $\mathfrak{H}$ be a complex separable Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The quantum Ornstein-Uhlenbeck semigroup $(\mathcal{T}_t^\beta)_{t\geq 0}$ associated with the inverse temperature β is given by the generator

$$\mathcal{L}^\beta(x) = \frac{\mathrm{e}^\beta}{\mathrm{e}^\beta - 1}\left(-\frac{1}{2}a^\dagger ax + a^\dagger xa - \frac{1}{2}xa^\dagger a\right) + \frac{1}{\mathrm{e}^\beta - 1}\left(-\frac{1}{2}aa^\dagger x + axa^\dagger - \frac{1}{2}xaa^\dagger\right),$$

where $x \in \operatorname{Dom}\mathcal{L}^\beta \subset \mathcal{B}(\mathfrak{H})$, a is the annihilation operator, and $a^\dagger$ is the creation operator.

$$ae_n = \sqrt{n}\, e_{n-1}, \quad n \geq 1, \quad ae_0 = 0; \quad a^\dagger e_n = \sqrt{n+1}\, e_{n+1}, \quad n \geq 0.$$

. Moreover, the position operator q, the momentum operator p and the number operator N are given by

$$qe_n = \frac{a + a^\dagger}{\sqrt{2}} e_n, \quad pe_n = \frac{\mathrm{i}(a^\dagger - a)}{\sqrt{2}} e_n, \quad Ne_n = ne_n.$$

Remarks

1. a and $a^\dagger$ are unbounded operators.
2. $(\mathcal{T}_t^\beta)_{t\geq 0}$ is indeed self-adjoint due to the lack of a Hamiltonian part in its generator.

It was proved in [CFL00] that $(\mathcal{T}_t^\beta)_{t\geq 0}$ has a unique faithful normal invariant state

$$\rho^\beta := (1-\mathrm{e}^{-\beta})\mathrm{e}^{-\beta N} = (1-\mathrm{e}^{-\beta})\sum_{k=0}^{\infty} \mathrm{e}^{-\beta k} |e_k\rangle\langle e_k| \,,$$

and its induced semigroup admits a spectral gap.

The restriction of the quantum Ornstein-Uhlenbeck semigroup $(\mathcal{T}_t^{\beta})_{t\geq 0}$ to the subalgbera of the position operator corresponds to a classical Ornstein-Uhlenbeck process. The restriction of its predual semigroup $(\mathcal{T}_{*t}^{\beta})_{t\geq 0}$ to the subalgebra of the number operator is a classical birth-and-death process.

Lemma 6

$(\mathcal{T}_{*t}^{\beta} \upharpoonright l^1(\mathbb{N}))_{t\geq 0}$ is the classical birth-and-death process with birth rates $((n+1)/(e^{\beta}-1))_{n\in\mathbb{N}}$ and death rates $(n\mathrm{e}^{\beta}/(\mathrm{e}^{\beta}-1))_{n\in\mathbb{N}}$. In addition, $(\mathcal{T}_t^{\beta} \upharpoonright l^{\infty}(\mathbb{N}))_{t\geq 0}$ has a spectral gap $\alpha = 1$.

Just notice that

$$\mathcal{L}_*^{\beta}(|e_n\rangle\langle e_n|) = \frac{n\mathrm{e}^{\beta}}{\mathrm{e}^{\beta}-1}|e_{n-1}\rangle\langle e_{n-1}| - \frac{n\mathrm{e}^{\beta}+n+1}{\mathrm{e}^{\beta}-1}|e_n\rangle\langle e_n| + \frac{n+1}{\mathrm{e}^{\beta}-1}|e_{n+1}\rangle\langle e_{n+1}|,$$

and we can define the following transition probabilities

$$p_{ij}^{\beta}(t) := \mathrm{Tr}\left(\mathcal{T}_{*t}^{\beta}(|e_i\rangle\langle e_i|)\,|e_j\rangle\langle e_j|\right), \quad i,j\in\mathbb{N}.$$

Uniformly exponentially convergence for CTMCs

Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with state space $I = \mathbb{N}$ and transition probabilities $P(t) = (p_{ij}(t))_{i,j\in I}$. Suppose there exists a unique invariant density $(\pi_i)_{i\in I}$ for $(X_t)_{t\geq 0}$. Similar to the case of quantum Markov semigroups, we say that $P(t)$ is uniformly exponentially convergent if there exists $M > 0$ and $\alpha > 0$ such that $|p_{ij}(t) - \pi_j| < M\mathrm{e}^{-\alpha t}$ for all $i, j \in I$.

The following theorem shows that uniform exponential convergence can be characterized by the mean hitting times to state 0. Recall that, starting from state $n \in \mathbb{N}$, the mean time taken for $(X_t)_{t\geq 0}$ to reach state 0 is given by $k_n := \mathbb{E}[T|X_0 = n]$, where $T := \inf\{t \geq 0 : X_t = 0\}$.

Theorem 7 (Characterizations of uniform exponential convergence)

The following statements are equivalent:

1. $P(t)$ *is uniformly exponentially convergent.*
2. $\lim_{t\to+\infty} \sup_{i\in I} |p_{il}(t) - \pi_l| = 0$ *for some* $l \in I$ *with* $\pi_l > 0$.
3. $\lim_{t\to+\infty} \sup_{i\in I} \sum_{j\in I} |p_{ij}(t) - \pi_j| = 0$.
4. $\delta(P(t)) < 1$ *for some* $t > 0$, *where* $\delta(P(t)) := \frac{1}{2} \sup_{i,j\in I} \sum_{h\in I} |p_{ih}(t) - p_{jh}(t)|$.
5. *the sequence of mean hitting times* $(k_n)_{n\geq 0}$ *is uniformly bounded.*

Let $(X_t^\beta)_{t\geq 0}$ be the birth-and-death process associated to the quantum Ornstein-Uhlenbeck semigroup $(\mathcal{T}_t^\beta)_{t\geq 0}$. We have the following results:

Proposition 8

For the process $(X_t^\beta)_{t\geq 0}$, starting from state n, the mean hitting time of state 0 equals $\sum_{m=1}^{n} 1/m$.

The following result is immediate:

Theorem 9

For the quantum Ornstein-Uhlenbeck semigroup $(\mathcal{T}_t^\beta)_{t\geq 0}$, there does not exist $M > 0$ and $\alpha > 0$ such that

$$\left\|\mathcal{T}_{*t}^\beta(\phi) - \rho^\beta\right\| \leq M\mathrm{e}^{-\alpha t}, \quad \forall \phi \in \mathcal{S}_\rho(\mathcal{A}).$$

In fact, we can conduct more detailed computations and analysis. The transition probabilities of a birth-and-death process have what is known as the Kendall representation. Let $(X_t^\beta)_{t\geq 0}$ be the birth-and-death process associated to $(\mathcal{T}_t^\beta)_{t\geq 0}$, and let $(\pi_j^\beta)_{j\in\mathbb{N}}$ denote its unique invariant distribution. According to [KM58], the Kendall representation of $(p_{ij}^\beta(t))_{i,j\in\mathbb{N}}$ is

$$p_{ij}^\beta(t) = \pi_j^\beta \sum_{n=0}^{\infty} \mathrm{e}^{-tn} Q_i^\beta(n) Q_j^\beta(n) \mathrm{e}^{-\beta n}, \quad i,j \in \mathbb{N}. \tag{1}$$

The above $(Q_i^\beta)_{i\in\mathbb{N}}$ are Meixner polynomials defined by

$$Q_i^\beta(x) = \sum_{k=0}^{\infty} \frac{(-i)_k(-x)_k(1-\mathrm{e}^\beta)^k}{(k!)^2}, \quad x \in \mathbb{R},$$

where

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}, \quad a \in \mathbb{R}, \quad k \in \mathbb{N}.$$

For a non-positive integer a, the above $(a)_k$ is defined by continuation.

The proposition below demonstrates that the convergence speed towards the unique faithful normal invariant state ρ^{β} for normal states in the form of $|e_i\rangle\langle e_i|$ cannot have an exponential rate with parameter $1+\epsilon$, where $\epsilon > 0$.

Proposition 10

There does not exists $\epsilon > 0$ and $M_i > 0$ such that

$$\left\| \mathcal{T}^{\beta}_{*t}(|e_i\rangle\langle e_i|) - \rho^{\beta} \right\| \leq M_i \mathrm{e}^{-(1+\epsilon)t}.$$

Notice that

$$\left\| \mathcal{T}^{\beta}_{*t}(|e_i\rangle\langle e_i|) - \rho^{\beta} \right\| = \sum_{j=0}^{\infty} \left| p^{\beta}_{ij}(t) - (1-\mathrm{e}^{-\beta})\mathrm{e}^{-\beta j} \right|.$$

Let $\hat{\omega} := \kappa \sum_{k=1}^{\infty} k^{-2} |e_n\rangle \langle e_n|$ with $\kappa := 6/\pi^2$.

Proposition 11

When $\beta > -\log 1/2$, there does not exist an $M > 0$ such that

$$\left\| \mathcal{T}_{*t}^{\beta}(\hat{\omega}) - \rho^{\beta} \right\| \leq M\mathrm{e}^{-t}, \quad \forall t \geq 0.$$

Therefore, we have discovered a normal state outside of $\mathcal{S}_{\rho}(\mathcal{A})$ that is not α-exponentially convergent.

References

References

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