# Schatten properties of commutators on twisted crossed product

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# Twisted crossed product

• Given a von Neumann algebra $\mathcal{M}$ and a locally compact group, suppose that there exists an action $\alpha$ of $G$ on $\mathcal{M}$, assume invariably that $\alpha$ is strong $*$-cotinuous, that is, for each fixed $x \in \mathcal{M}$, the map $s \mapsto \alpha_s(x)$ is strong $*$-continuous.

**Definition 1**

A *twisted dynamical system* is a quadruple $(\mathcal{M}, G, \alpha, \sigma)$ with a *twisted action* $(\alpha, \sigma)$ of $G$ on $\mathcal{M}$. Here the two functions $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ and $\sigma : G \times G \to \mathcal{U}(\mathcal{M})$ satisfy the following conditions: for any $s, t, r \in G$

1. $\alpha_s \circ \alpha_t = \mathrm{Ad}_{\sigma(s,t)} \circ \alpha_{st}$;
2. $\sigma(r, s)\sigma(rs, t) = \alpha_r(\sigma(s, t))\sigma(r, st)$;
3. $\sigma(e, s) = \sigma(s, e) = 1$.

# Twisted crossed product

## Definition 2

A *covariant homomorphism* of $(\mathcal{M}, G, \alpha, \sigma)$ is a pair $(\rho, u)$ of a normal representation $\rho$ of $\mathcal{M}$ on a Hilbert space $K$, and a function $u : G \to \mathcal{U}(K)$ such that

1. (i) $u(s)u(t) = \rho(\sigma(s,t))u(st)$, $s, t \in G$;
2. (ii) $\rho(\alpha_s(a)) = u(s)\rho(a)u(s)^*$, $a \in \mathcal{M}$, $s \in G$.

- $$(\pi_\alpha(a)\xi)(t) = \alpha_{t^{-1}}(a)\xi(t), \quad \xi \in L_2(G, H), t \in G,$$
$$(\lambda_\sigma(s)\xi)(t) = \sigma(t^{-1}, s)\xi(s^{-1}t), \quad \xi \in L_2(G, H), s, t \in G.$$

# Twisted crossed product

**Definition 3**

The von Neumann algebra generated by $\pi_\alpha(\mathcal{M})$ and $\lambda_\sigma(G)$ on $L_2(G,H)$ is called the twisted crossed product of $\mathcal{M}$ by $(\alpha,\sigma)$ and is denoted by $\mathcal{M} \rtimes_{\alpha,\sigma} G$.

- set

$$\mathcal{R} = \mathcal{M} \rtimes_{\alpha,\sigma} \mathbb{R}^d \quad \text{and} \quad \mathcal{N} = \mathcal{M}\overline{\otimes}B(L_2(\mathbb{R}^d)).$$

# Twisted crossed product

- For an element $f \in K(G, \mathcal{M})$, we put $\lambda_\sigma \times \pi_\alpha(f)$ to be

$$\lambda_\sigma \times \pi_\alpha(f) = \int_G \lambda(s)\pi_\alpha(f(s))ds.$$

**Proposition 4**

$$(\lambda_\sigma \times \pi_\alpha(K(G, \mathcal{M})))'' = span\{\lambda_\sigma(G) \cup \pi_\alpha(\mathcal{M})\}'' = \mathcal{M} \rtimes_{\alpha,\sigma} G$$

# Preliminaries: Dual trace

- For a given weight $\tau$ on $\mathcal{M}$, it is said to be semi-finite if

$$\mathfrak{p}_\tau = \{x \in \mathcal{M}_+ : \tau(x) < +\infty\}$$

generates $\mathcal{M}$; while

$$\mathfrak{n}_\tau = \{x \in \mathcal{M} : x^* x \in \mathfrak{p}_\tau\},$$

$$\mathfrak{m}_\tau = \{\sum_{i=1}^{n} y_i^* x_i : x_1, ..., x_n, y_1, ..., y_n \in \mathfrak{n}_\tau\}.$$

$\mathfrak{n}_\tau$ is a left ideal of $\mathcal{M}$, and $\mathfrak{m}_\tau \cap \mathcal{M}_+ = \mathfrak{p}_\tau$. For a fixed weight $\tau$ on $\mathcal{M}$. The set

$$N_\tau = \{x \in \mathcal{M} : \tau(x^* x) = 0\}$$

is a left ideal of $\mathcal{M}$ contained in $\mathfrak{n}_\tau$.

# Preliminaries: Dual trace

• Define a canonical quotient map $\eta_\tau : \mathfrak{n}_\tau \to \mathfrak{n}_\tau / N_\varphi$ by:

$$\eta_\tau(x) = x + N_\tau \in \mathfrak{n}_\tau / N_\tau.$$

Define a sesquilinear functional:

$$\langle \eta_\tau(x), \eta_\tau(y) \rangle = \tau(y^* x)$$

on $\mathfrak{n}_\tau / N_\tau$.

• Take the completion of $\mathfrak{n}_\tau / N_\tau$ with respect to this sesquilinear functional and denote it by $\mathfrak{H}_\tau$.

# Preliminaries: Dual trace

• Define a representation $\pi_\tau$ of $\mathcal{M}$ on $\mathfrak{H}_\tau$ by

$$\pi_\tau(a)\eta_\tau(x) = \eta_\tau(ax).$$

• The triplet $\{\pi_\tau, \mathfrak{H}_\tau, \eta_\tau\}$ is called the semi-cyclic representation of $\mathcal{M}$.

• Let $K(G, \mathcal{M})$ be the space of all $\sigma$-strongly-$*$ continuous $\mathcal{M}$ valued functions on $G$ with compact support.

# Preliminaries: Dual trace

- For $x, y \in K(G, \mathcal{M})$ define

$$x *_\sigma y(s) = \int_G \sigma(s^{-1}, s)^* \sigma(s^{-1}, st) \alpha_t(x(st)) \sigma(t, t^{-1}) y(t^{-1}) dt,$$
$$x^{\#}(s) = \delta_G(s)^{-1} \sigma(s^{-1}, s)^* \alpha_{s^{-1}}(x(s^{-1}))^*.$$

and

$$\langle x, y \rangle_{\mathcal{M}} = \int_G y(t)^* x(t) dt.$$

Where $\delta_G$ is the modular function of $G$.

# Preliminaries: Dual trace

- We define

$$(x \cdot a)(s) = x(s)a,$$
$$(a \cdot x)(s) = \alpha_s^{-1}(a)x(s),$$

for $x \in K(G, \mathcal{M})$, $a \in \mathcal{M}$, then $K(G, \mathcal{M})$ is a right module over $\mathcal{M}$.

- Set

$$\mathfrak{b}_\tau = K(G, \mathcal{M}) \cdot \mathfrak{n}_\tau = \operatorname{span}\{x \cdot a : x \in K(G, \mathcal{M}), a \in \mathfrak{n}_\tau\}.$$

# Preliminaries: Dual trace

• Define the map $\tilde{\eta}_\tau : x \in \mathfrak{b}_\tau \mapsto \tilde{\eta}_\tau(x) \in L_2(G, \mathfrak{H}_\tau)$ by

$$\tilde{\eta}_\tau(x)(s) = \eta_\tau(\sigma(s^{-1}, s)x(s))$$

for $x \in \mathfrak{b}_\tau$ and $s \in G$.

• $\tilde{\mathfrak{A}}_\tau = \tilde{\eta}_\tau(\mathfrak{b}_\tau \cap \mathfrak{b}_\tau^{\#})$ is a left Hilbert algebra with respect to the following operations:

$$\tilde{\eta}_\tau(x)\tilde{\eta}_\tau(y) = \tilde{\eta}_\tau(x *_\sigma y),\ \ x, y \in \mathfrak{b}_\tau \cap \mathfrak{b}_\tau^{\#},$$

$$\tilde{\eta}_\tau(x)^{\#} = \tilde{\eta}_\tau(x^{\#}).$$

# Preliminaries: Dual trace

• Given a normal, semi-finite and faithful weight $\tau$ on $\mathcal{M}$, the normal, semi-finite and faithful weight $\tilde{\tau}$ associated with the left Hilbert algebra $\tilde{\mathfrak{A}}_\tau$ is called the dual weight of $\tau$, namely, the weight is in the following form for $x \in \mathcal{R}_\ell(\tilde{\mathfrak{A}}_\tau)_+$:

$$\tilde{\tau}(x) = \begin{cases} \|\xi\|^2 & \text{if } \ x = \pi_\ell(\xi)^* \pi_\ell(\xi), \ \xi \in \tilde{\mathfrak{A}}_\tau \\ +\infty & \text{otherwise} \ . \end{cases}$$

• By the Plancherel formula, the map $f \mapsto \lambda_\sigma \times \pi_\alpha(f)$ establishes an isometry from $L_2(\mathbb{R}^d, L_2(\mathcal{M}))$ onto $L_2(\mathcal{R})$.

# Preliminaries: Dual trace

**Theorem 5**

*For $x \in \mathfrak{b}_\tau$,*

$$\tilde{\tau}((\lambda_\sigma \times \pi_\alpha(x))^*(\lambda_\sigma \times \pi_\alpha(x))) = \tau((x^{\#} * x)(e)).$$

*In addition, there exists uniquely an operator valued weight $T$ from $\mathcal{M} \rtimes_{\alpha,\sigma} G$ onto $\pi_\alpha(\mathcal{M})$ such that for $x \in (\mathcal{M} \rtimes_{\alpha,\sigma} G)_+$,*

$$\tilde{\tau}(x) = \tau \circ \pi_\alpha^{-1}(T(x))$$

*for any faithful semi-finite normal weight $\tau$ on $\mathcal{M}$.*

# Preliminaries: Takesaki Duality

• Suppose the group $G$ is abelian, the action $\alpha$ admits a dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on the twisted crossed product $\mathcal{M} \rtimes_{\alpha,\sigma} G$ as follows, let $\omega$ be the unitary representation of $\widehat{G}$ on $L_2(G, H)$ in the following form:

$$\big(w(\gamma)\xi\big)(h) = \overline{\gamma(h)}\xi(h), \quad \xi \in L_2(G,H),\ h \in G,\ \gamma \in \widehat{G}.$$

Then the dual action $\widehat{\alpha}$ is implemented by $w$:

$$\widehat{\alpha}_\gamma(x) = w(\gamma)xw(\gamma)^*, \quad x \in \mathcal{M} \rtimes_{\alpha,\sigma} G,\ \gamma \in \widehat{G}. \tag{1}$$

•

$$\widehat{\alpha}_\gamma(\pi_\alpha(x)) = \pi_\alpha(x), \quad \widehat{\alpha}_\gamma(\lambda_\sigma(g)) = \overline{\gamma(g)}\lambda_\sigma(g), \quad x \in \mathcal{M},\ g \in G,\ \gamma \in \widehat{G}. \tag{2}$$

# Preliminaries: Takesaki Duality

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**Definition 6**

The action $\widehat{\alpha}$ defined in (1) and (2) is called the dual action of $\widehat{G}$ on $\mathcal{M} \rtimes_{\alpha,\sigma} G$ and $\{\mathcal{M} \rtimes_{\alpha,\sigma} G, \widehat{G}, \alpha\}$ is called the dual twisted covariant system.

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**Theorem 7**

*The dual action $\widehat{\alpha}$ of $\widehat{G}$ on $\mathcal{M} \rtimes_{\alpha,\sigma} G$ has the following properties:*

1. *A faithful weight $\tilde{\tau}$ on $\mathcal{M} \rtimes_{\alpha,\sigma} G$ is dual to a faithful weight $\tau$ on $\mathcal{M}$ if and only if $\tilde{\tau}$ is $\widehat{\alpha}$ invariant.*
2. *Considering the second crossed product $\mathcal{M} \rtimes_{\alpha,\sigma} G \rtimes_{\widehat{\alpha}} \widehat{G}$, there exists a unique isomorphism $\Phi$ of $\mathcal{M} \rtimes_{\alpha,\sigma} G \rtimes_{\widehat{\alpha}} \widehat{G}$ onto $\mathcal{M}\overline{\otimes}B(L_2(G))$.*

• $\widehat{\alpha}_\gamma$ is $\tilde{\tau}$ invariant, $\widehat{\alpha}_\gamma$ extends to an isometric action $\widehat{\alpha}_\gamma^{(p)}$ on $L_p(\mathcal{M} \rtimes_{\alpha,\sigma} G)$.

• We can define the convolution between a function $f \in L_1(\mathbb{R}^d)$ and an element $x \in L_p(\mathcal{R})$.

$$f * x = \int_{\mathbb{R}^d} f(s)\widehat{\alpha}_{-s}^{(p)}(x)ds. \tag{3}$$

# Preliminaries: Distribution space

• $\mathcal{M}^\infty$ is the smooth subalgebra with $x \in \mathcal{M}$ such that the map $s \mapsto \alpha_s(x)$ is smooth.

• The class of Schwartz functions on $\mathcal{R}$ is defined as the image of the Schwartz class $\mathcal{S}(\mathbb{R}^d, \mathcal{M}^\infty)$ under $\lambda_\sigma \times \pi_\alpha$. That is,

$$\mathcal{S}(\mathcal{R}) = \{\lambda_\sigma \times \pi_\alpha(f) : f \in \mathcal{S}(\mathbb{R}^d, \mathcal{M}^\infty)\}. \tag{4}$$

• The space of *tempered distributions* on $\mathcal{R}$ is the topological dual space $\mathcal{S}'(\mathcal{R})$ of $\mathcal{S}(\mathcal{R})$, i.e., the space of continuous linear functionals on $\mathcal{S}(\mathcal{R})$.

• For $x = \lambda_\sigma \times \pi_\alpha(f) \in \mathcal{S}(\mathcal{R})$, $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$, we set

$$\partial^\alpha x = \int_{\mathbb{R}^d} s^\alpha \lambda_\sigma(s) \pi_\alpha(f(s)) ds,$$

where $s^\alpha = s_1^{\alpha_1} \cdots s_d^{\alpha_d}$.

• $\partial^\alpha x$ belongs to $\mathcal{S}(\mathcal{R})$ too. By duality, these partial derivations extend to all distributions.

# Preliminaries: Derivatives on twisted crossed product

• Let $\Delta = \partial_1^2 + \cdots + \partial_d^2$ be the Laplacian. We will frequently use the Bessel and Riesz operators $(1+\Delta)^{\frac{1}{2}}$ and $\Delta^{\frac{1}{2}}$ which will be abbreviated as $J$ and $I$ respectively. More generally, for $a \in \mathbb{R}$, define $J^a = (1+\Delta)^{\frac{a}{2}}$ and $I^a = \Delta^{\frac{a}{2}}$.

• The Bessel potential $J^a$ operates on $\mathcal{S}'(\mathcal{R})$. While for the Riesz potential $I^a$. Let

$$\mathcal{S}_0(\mathbb{R}^d, \mathcal{M}^\infty) = \{x : \widehat{\partial^\alpha x}(0) = 0 \quad \forall\, \alpha \in \mathbb{N}_0^d\}.$$

Then $I^a$ operates on $\mathcal{S}_0(\mathcal{R}) = \lambda_\sigma \times \pi_\alpha\big(\mathcal{S}_0(\mathbb{R}^d, \mathcal{M}^\infty)\big)$, and by duality, on the dual space $\mathcal{S}_0'(\mathbb{R}_\theta^d)$.

# Preliminaries: Fourier multipliers and convolution

• We denote $\check{\phi}$ as the inverse Fourier transform of $\phi$. Now assume that $\check{\phi} \in L_1(\mathbb{R}^d)$. Define

$$\check{\phi} * x = \int_{\mathbb{R}^d} \check{\phi}(t) \widehat{\alpha}_{-t}(x) dt. \tag{5}$$

• For $x = \lambda_\sigma \times \pi_\alpha(f)$ with $f \in \mathcal{S}(\mathbb{R}^d, \mathcal{M}^\infty)$, we have for the Fourier multiplier $T_\phi$,

$$T_\phi(x) = \lambda_\sigma \times \pi_\alpha(\phi f) = \check{\phi} * x.$$

# Preliminaries: Commutators

• Given $x \in \mathcal{R}$, denote by $M_x : y \mapsto xy$ the left multiplication on $L_2(\mathcal{R})$. Then $M_x$ is a bounded linear operator on $L_2(\mathcal{R})$. We now define the commutator

$$\mathbf{C}_{\phi,x} = [T_\phi, M_x].$$

This is a so-called Calderón-Zygmund transform on $\mathcal{R}$, it is bounded on $L_2(\mathcal{R})$.

•The *homogeneous Sobolev space* $\dot{W}_p^m(\mathcal{R})$ consists of those $x \in \mathcal{S}'(\mathcal{R})$ such that every partial derivative of order $m$ is in $L_p(\mathcal{R})$, equipped with the seminorm:

$$\|x\|_{\dot{W}_p^m} = \Big(\sum_{|\alpha|=m} \|\partial^\alpha x\|_p\Big)^{\frac{1}{p}}.$$

# Function spaces on twisted crossed product

• Besov spaces are defined by using a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\begin{cases} \operatorname{supp} \varphi \subset \{\xi : 2^{-1} \le |\xi| \le 2\}, \\ \varphi > 0 \text{ on } \{\xi : 2^{-1} < |\xi| < 2\}, \\ \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \ \xi \neq 0. \end{cases} \tag{6}$$

The sequence $\{\varphi(2^{-k}\cdot)\}_{k\in\mathbb{Z}}$ is a Littlewood-Paley decomposition of $\mathbb{R}^d$, modulo constant functions. Denote by $\varphi_k$ the inverse Fourier transform of $\varphi(2^{-k}\cdot)$.

## Definition 8

Let $1 \leq p, q \leq \infty$ and $a \in \mathbb{R}$. The *homogeneous Besov space* on $\mathbb{R}_\theta^d$ is defined by

$$B_{p,q}^a(\mathcal{R}) = \{x \in L_p(\mathcal{R}) : \|x\|_{B_{p,q}^a} < \infty\},$$

where

$$\|x\|_{B_{p,q}^a} = \Big(\sum_{k\in\mathbb{Z}} 2^{qka}\|\varphi_k * x\|_p^q\Big)^{\frac{1}{q}}.$$

Let $B_{p,c_0}^a(\mathcal{R})$ be the subspace of $B_{p,\infty}^a(\mathcal{R})$ consisting of all $x$ such that $2^{kr}\|\varphi_k * x\|_p \to 0$ as $|k| \to \infty$.

# Function spaces on twisted crossed product

- Denote by $A(\widehat{G})$ the Fourier algebra of $\widehat{G}$ which is the image of $L_1(G)$ under the Fourier transform.
- For an action $\beta$ of $G$ on $\mathcal{M}$, with a function $f \in A(\widehat{G})$, define

$$\beta_f(x) = \int_G \check{f}(t)\beta_-t(x)dt.$$

- For each $x \in \mathcal{M}$, putting

$$I(x) = \{f \in A(\widehat{G}) : \beta_f(x) = 0\}$$

- The Arveson's $\beta$- spectrum $\sigma_\beta(x)$ is defined by

$$\sigma_\beta(x) = \{p \in \widehat{G} : f(p) = 0, f \in I(x)\}.$$

# Function spaces on twisted crossed product

- Define $\mathcal{A}(\mathcal{R}) = \{x \in \mathcal{R} \cap L_1(\mathcal{R}) : \ \sigma_{\widehat{\alpha}}(x) \text{ is compact}\}$.
- $\mathcal{A}(\mathcal{R})$ is a $*$-algebra.
- $\mathcal{A}(\mathcal{R})$ is dense in $B^a_{p,q}(\mathcal{R})$ for $1 \leq p < \infty$ and $1 \leq q < \infty$.

# Function spaces on twisted crossed product

• $\mathcal{A}(\mathcal{R})$ is norm-dense in $W_p^m(\mathcal{R})$ when $m \geq 0$ and $1 \leq p < \infty$; the density of $\mathcal{A}(\mathcal{R})$ in $\dot{W}_p^m(\mathcal{R})$ holds only when $m \geq 0$ and $1 < p < \infty$

• The dual space of $B_{p,q}^a(\mathcal{R})$ coincides isomorphically with $B_{p',q'}^{-a}(\mathcal{R})$ for $1 \leq p < \infty$ and $1 \leq q < \infty$

• $J^b$ and $I^b$ are isomorphisms between $B_{p,q}^a(\mathcal{R})$ and $B_{p,q}^{a-b}(\mathcal{R})$.

# Backgrounds and motivations

• The first results [Mcdonald, Sukochev and Xiong, Commun. Math. Phys. 2019] concerning quantum differentiability in the noncommutative euclidean space are the characterizations of the Schatten $S_{d,\infty}$ properties of

$$đx := \sum_{j=1}^{d} \gamma_j \otimes đx_j \tag{7}$$

on noncommutative euclidean space $\mathbb{R}^d_\theta$.

• $\gamma_j$'s denote the $d$-dimensional euclidean gamma matrices, and $đx_j := \mathrm{i}[R_j, M_x]$, where for $1 \leq j \leq d$, $R_j = T_\phi$ for $\phi(s) = \frac{s_j}{|s|}$ denote the quantum counterpart of Riesz transforms on $\mathbb{R}^d_\theta$.

- Our research in the second part is motivated by the following:

**Theorem 9 (Mcdonald, Sukochev and Xiong, 2019)**

*$\bar{d}x_i$ has bounded extension in $S_{d,\infty}$ for every $1 \le i \le d$ iff $x$ belongs to the homogeneous Sobolev space $\dot{W}^1_d(\mathbb{R}^d_\theta)$.*

- One related result is the formula on Dixmier Trace. For any continuous normalised trace $\mathrm{tr}$ on $S_{1,\infty}$ we have

$$\mathrm{Tr}_\omega(|\bar{d}x|^d) = c_d \left\| \sum_{j=1}^{d} \gamma_j \otimes \left(\partial_j x - s_j \sum_{k=1}^{d} s_k \partial_k x\right) \right\|_d^d. \tag{8}$$

## Main results

• We aim to extend the aforementioned results to a more general setting. Here are our results.

**Theorem 10**

*Let $d < p < \infty$. If $x \in B^{\frac{d}{p}}_{p,p}(\mathcal{R})$, then $\mathbf{C}_{\phi,x}$ has a bounded extension in $S_p$ and*

$$\left\|\mathbf{C}_{\phi,x}\right\|_{S_p} \lesssim_{d,p} \Big[ \sup_{s\in\mathbb{S}^{d-1}} |\phi(s)| + \sup_{s\in\mathbb{S}=^{d-1}} |\nabla\phi(s)| \Big] \left\|x\right\|_{B^{\frac{d}{p}}_{p,p}} .$$

*Conversely, assume additionally that $\phi$ is not constant. If $x \in \mathcal{R}$ and $\mathbf{C}_{\phi,x} \in S_p$, then $x \in B^{\frac{d}{p}}_{p,p}(\mathcal{R})$ and*

$$\left\|x\right\|_{B^{\frac{d}{p}}_{p,p}} \lesssim_{d,p} \Big[ \sup_{s\in\mathbb{S}^{d-1}} |\phi(s)| + \sup_{s\in\mathbb{S}^{d-1}} | \nabla\phi(s)| \Big] \left\|\mathbf{C}_{\phi,x}\right\|_{S_p} .$$

# Main results: Application to noncommutative Euclidean space

- For the critical case, i.e., the $S_{d,\infty}$ properties of $\mathbf{C}_{\phi,x}$ for $p \leq d$.

**Theorem 11**

*If $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$, then $\mathbf{C}_{\phi,x}$ has bounded extension in $S_{d,\infty}$.*

# Main results: Applications to noncommutative Euclidean space

- The following trace formula is new even for classical setting.

**Theorem 12**

*Let $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$. Then for every continuous normalised trace $\mathrm{Tr}_\omega$ on $S_{1,\infty}$, we have*

$$\mathrm{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d) = C_d \int_{\mathbb{S}^{d-1}} \tau_\theta\Big(\Big|\sum_{1\le k\le d} \partial_{s_k}\phi \, \partial_k x\Big|^d\Big) ds.$$

*Here the integral over $\mathbb{S}^{d-1}$ is taken with respect to the rotation-invariant measure $ds$ on $\mathbb{S}^{d-1}$.*

# Proof of Theorem 10: Basic ingredients

• We view $\mathcal{M} \rtimes_{\alpha,\sigma} G$ as a right Hilbert w* module on $\mathcal{M}$ with the inner product

$$\langle x, y\rangle = T(x^* y).$$

• $\mathcal{M} \rtimes_{\alpha,\sigma} G$ can be embedded as a submodule of $\mathcal{C}_I(\mathcal{M}) = \bigoplus_{i\in I} \mathcal{M}$ for an index sets $I$, i.e., there exist right module map $u = (u_i)_{i\in I}$ such that for $x, y \in \mathcal{M} \rtimes_{\alpha,\sigma} G$, we have

$$\begin{aligned} \langle x, y\rangle &= \langle u(x), u(y)\rangle \\ &= \sum_{i\in I} u_i(x)^* u_i(y) \end{aligned} \tag{9}$$

# Proof of Theorem 10: Basic ingredients

• For a element $x \in L_p(\mathcal{R})$, we define the Fourier transform of $x$ by

$$\widehat{x}(s) = T(\lambda_\sigma(s)^* x).$$

• With this Fourier coefficient, we can write $x$ formally as

$$x = \int_{\mathbb{R}^d} \lambda_\sigma(s) \pi_\alpha(\widehat{x}(s)) ds.$$

• For instance, if we have $f \in L_1(G, \mathcal{M}) + L_\infty(G, \mathcal{M})$, then we can calculate

$$\lambda_\sigma \widehat{\times \pi_\alpha}(f)(s) = f(s).$$

# Proof of Theorem 10: Upper bounds estimate

We use the complex interpolation to obtain the desired estimate. Indeed, we have the following three endpoint cases.

• Let $a > 0, b > 0$ and $a + b < 1$. If $x \in B^{a+b}_{\infty,\infty}(\mathcal{R})$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_\infty(L_2(\mathcal{R}))$ and

$$\|I^a \mathbf{C}_{\phi,x} I^b\|_{S_\infty} \lesssim_{d,a,b} \|x\|_{B^{a+b}_{\infty,\infty}}.$$

• Let $a > -\frac{d}{2}$, $b > -\frac{d}{2}$ and $a + b + d < 1$. If $x \in B^{a+b+d}_{1,1}(\mathcal{R})$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_1$ and

$$\left\|I^a \mathbf{C}_{\phi,x} I^b\right\|_{S_1} \lesssim_{d,a,b} \left\|x\right\|_{B^{a+b+d}_{1,1}}. \tag{10}$$

# Proof of Theorem 10: Upper bounds estimate

- Let $a, b > -\frac{d}{2}$ and $a + b + \frac{d}{2} < 1$. If $x \in B_{2,2}^{a+b+\frac{d}{2}}(\mathcal{R})$, then $I^a \mathbf{C}_{\phi,x} I^b \in S_2$ and

$$\left\| I^a \mathbf{C}_{\phi,x} I^b \right\|_{S_2} \lesssim_{d,a,b} \left\| x \right\|_{B_{2,2}^{a+b+\frac{d}{2}}}.$$

## Theorem 13

*Let $1 \leq p \leq \infty$, $a + b + \frac{d}{p} < 1$ and $a, b > \max(-\frac{d}{p}, -\frac{d}{2})$. If $x \in B_{p,p}^{a+b+\frac{d}{p}}(\mathcal{R})$, then $I^a \mathbf{C}_{\phi,x} I^b$ belongs to $B_{p,p}^{a+b+\frac{d}{p}}(\mathcal{R})$ and*

$$\left\| I^a \mathbf{C}_{\phi,x} I^b \right\|_{S_p} \lesssim_{d,p,a,b} \left\| x \right\|_{B_{p,p}^{a+b+\frac{d}{p}}}.$$

# Proof of Theorem 10: Upper bounds estimate

• We end this part with a generalization to higher commutators. Namely, let $\phi_1, \cdots, \phi_N \in C^\infty(\mathbb{S}^{d-1})$ be $N$ non-constant functions. Define

$$\mathbf{C}_{\phi_1,\cdots,\phi_N,x} = [T_{\phi_N}, ..., [T_{\phi_1}, M_x]...] \tag{11}$$

• Theorem 13 extends to higher commutators.

# Proof of Theorem 10: Lower bounds estimate

- This part is devoted to the converse results of those in the previous part.

- We need the following nondegeneracy condition:

$$\forall\, s \in \mathbb{R}^d \setminus \{0\}\ \exists\, t \in \mathbb{R}^d \setminus \{0\}\ \text{ such that } \prod_{i=1}^{N} (\phi_i(s) - \phi_i(t)) \neq 0. \quad (12)$$

For $N = 1$, this condition means that $\phi_1$ is not a constant function.

# Proof of Theorem 10: Lower bounds estimate

• Denote $\gamma = -(a + a_1 + b + b_1 + d)$ and set

$$\omega(s) = |s|^{\gamma} \int_{\mathbb{R}^d} \prod_{i=1}^{N} |\phi_i(s+t) - \phi_i(t)|^{2k} |s+t|^{a+a_1} |t|^{b+b_1} dt. \qquad (13)$$

• Suppose that $\phi_1, ..., \phi_N$ satisfy condition 12, we can show that $\omega$ is a homogeneous function of order 0 and never vanishes for $s \neq 0$.

• $\omega$ is a Fourier multiplier on $B^r_{1,1}(\mathcal{R}))$ for some $r$. By a Tauberian result, we see that $\omega^{-1}$ is a Fourier multiplier on $B^a_{p,p}(\mathcal{R})$ for any $a \in \mathbb{R}$.

# Proof of Theorem 10: Lower bounds estimate

- For $k \geq 1$ set

$$\mathbf{C}_{N,k,y} = \mathbf{C}_{\underbrace{\phi_1, \dots, \phi_N}_{k \text{ tuple}}, \underbrace{\bar{\phi}_1, \dots \bar{\phi}_N}_{k-1 \text{ tuple}}, y},$$

- By the duality, we have

$$\langle I^a C_{\phi_1,\dots,\phi_N,x} I^b, \; I^{a_1} \mathbf{C}_{N,k,y} I^{b_1} \rangle = \langle I^{-\gamma} T_\omega(x), y \rangle.$$

Thus,

$$\| T_\omega(x) \|_{B_{p,p}^{a+b+\frac{d}{p}}} \leq C \| I^a \mathbf{C}_{\phi_1,\dots,\phi_N,x} I^b \|_{S_p}.$$

# The trace formula: Pseudodifferential operator

- Given $f \in \mathcal{S}(\mathbb{R}^d)$ and $\rho \in S^m(\mathbb{R}^d; \mathcal{S}(\mathbb{R}^d_\theta))$, we set

$$P_\rho(\lambda_\theta(f)) = \int_{\mathbb{R}^d} f(\xi)\rho(\xi)\lambda_\theta(\xi)d\xi.$$

The operator $P_\rho$ is called the pseudo-differential operator of symbol $\rho$.

# The trace formula

• We replace $T_\phi$ by another Fourier multiplier $T_{\widetilde{\phi}}$ whose symbol is smooth on the whole $\mathbb{R}^d$.

• We put

$$A = \frac{1}{2\pi \mathrm{i}} \sum_{1 \leq k \leq d} T_{|\xi| \partial_{\xi_k} \widetilde{\phi}} M_{\partial_k x}. \tag{14}$$

We are going to reduce the computation of $\mathrm{Tr}_\omega(|\mathbf{C}_{\phi,x}|^d)$ to that of $\mathrm{Tr}_\omega(|A|^d (1+\Delta)^{-\frac{d}{2}})$.

# The trace formula

- Compute the symbol of $\mathbf{C}_{\widetilde{\phi},x} - AJ^{-1}$ is of order $-2$. We see that

$$M_y \mathbf{C}_{\widetilde{\phi},x} - M_y AJ^{-1} \in S_{\frac{d}{2},\infty}.$$

Then we have

$$|M_y \mathbf{C}_{\phi,x}|^d - |M_y A|^d J^{-d} \in S_1.$$

- We have

$$\mathrm{Tr}_\omega(|M_y \mathbf{C}_{\phi,x}|^d) = \mathrm{Tr}_\omega(|M_y A|^d J^{-d}).$$

So we can apply the trace formula in [McDonald, Sukochev and Zanin, Math. Ann. 2018] to deduce our trace formula.