Khintchine type inequalities and Fourier multipliers on HNN extensions

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Fourier Multipliers

Definition

Let $f : \mathbb{R} \to \mathbb{C}$ be a sufficiently regular function and let $\widehat{f}$ be its Fourier transform. Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function, which we will call the **symbol**. Then the **Fourier multiplier** with symbol m is an operator T_m defined by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Boundedness Problem

When is T_m well-defined and bounded on $L_p(\mathbb{R})$ ($p \neq 2$ and $1 < p < \infty$)?

The Hilbert transform on $\mathbb{R}$

Definition

For $f \in C_c^\infty(\mathbb{R})$,

$$(Hf)(x) = \mathrm{p.v.} \int_{\mathbb{R}} f(y) \frac{1}{x-y} dy.$$

As a Fourier multiplier:

$$\widehat{(Hf)}(\xi) = -i\, \mathrm{sgn}(\xi) \cdot \widehat{f}(\xi), \quad \xi \in \mathbb{R}.$$

Question

Is H well-defined and bounded on $L_p(\mathbb{R})$ ($p \neq 2$)?

Motivation I:

$$\begin{array}{ccc} C^\infty(\mathbf{R}) & \xrightarrow{P_r} & \mathcal{H}(\mathbf{H}^2) \\ \downarrow H & & \downarrow J \\ C^\infty(\mathbf{R}) & \xleftarrow{|_{\partial \mathbf{D}}} & \mathcal{H}(\mathbf{H}^2) \end{array}$$

$u + iJ(u)$ is holomorphic.

Motivation II:

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k\theta} \longrightarrow f$$

in L_p-norm, for any $f \in L_p(\mathbb{T})$?

Mikhlin Multipliers

Theorem (Mikhlin '1956)

Let m be a smooth function on $\mathbb{R}^d \setminus \{0\}$ such that

$$\sup_{0 \le |j| \le \frac{d}{2}+1} |\xi^j \nabla^j m(\xi)| \le C \quad \textit{for any } x \in \mathbb{R}^d \setminus \{0\}.$$

The Fourier multiplier $T_m : e^{2\pi i \xi x} \mapsto m(\xi) e^{2\pi i \xi x}$ extends to a bounded map on $L_p(\mathbb{R}^d)$ for any $1 < p < \infty$.

Theorem (Discrete case)

Let m be a function on $\mathbb{Z}$ such that

$$\sup_{k \in \mathbb{Z}} \{|m(k)|, |k(m(k) - m(k-1))|\} \le C.$$

The Fourier multiplier $T_m : z^k \mapsto m(k) z^k$ extends to a bounded map on $L_p(\mathbb{T})$ for any $1 < p < \infty$.

Non-Abelian groups

G: countable discrete group.

Left regular representation

$\lambda : \mathrm{G} \to \mathcal{U}(\ell_2(\mathrm{G}))$ with $\lambda_g \varphi(h) = \varphi(g^{-1}h)$. $\mathcal{L}(\mathrm{G}) = \{\lambda_g\}''_{g\in\mathrm{G}}$.

Non-abelian Fourier transform

For $\widehat{f} \in \ell_1(\mathrm{G}), f := \sum_{\mathrm{G}} \widehat{f}(g)\lambda_g$ is a bounded linear map $\ell_2(\mathrm{G}) \to \ell_2(\mathrm{G})$.

Non-commutative L_p-spaces

$L_p(\widehat{\mathrm{G}}) := L_p(\mathcal{L}(\mathrm{G}), \tau) =$ "$\{f : \tau(|f|^p)^{\frac{1}{p}} < \infty\}$" with $\tau(f) = \widehat{f}(e)$.

Fourier multipliers on $\mathcal{L}(\mathrm{G})$:

$m \in \ell_\infty(\mathrm{G}) \rightsquigarrow T_m f := \sum_{\mathrm{G}} m(g)\widehat{f}(g)\lambda_g$.

Fourier multipliers on groups

Question

Problem: What kind of conditions to put on m to make $\| T_m : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| < \infty$?

Idea 1 : Using a cocycle.
An affine representation of G is an orthogonal representation $\alpha : G \to O(H)$ over a real Hilbert space H together with a mapping $b : G \to H$ satisfying the cocycle law

$$b(gh) = \alpha_g(b(h)) + b(g)$$

- G unimodular locally compact group. b injective finite-dimensional. [Junge/Mei/Parcet '14 '18 + González Pérez/Junge/Parcet '17] $\mathrm{SL}_2(\mathbb{R})$ no fin-dim orthogonal actions / $\mathrm{SL}_n(\mathbb{R})(n \geq 3)$ has (T): Bad cohomology.

Idea 2: If G is a Lie group, one can use the natural differential structure on G to be formulate the condition on m.

Theorem (Parcet-Ricard-de la Salle '18)

Assume that $m \in \mathcal{C}^{[\frac{n^2}{2}]+1}(\mathrm{SL}_n(\mathbb{R})\backslash\{e\})(n \geq 3)$ *satisfies*

$$L(g)^{|\gamma|}|d_g^\gamma m(g)| \leq C \qquad \text{for all } |\gamma| \leq [\frac{n^2}{2}] + 1.$$

Then T_m *is bounded on* $L_p(\widehat{\mathrm{SL}_n(\mathbb{R})})$ *for any* $1 < p < \infty$.

Remark

d_g *is the Lie derivative. The length function* $L(g)$ *is locally Euclidean and asymptotically exponential...*

Idea 3: If G is a free product, study multipliers on G coming from free product of that on building blocks.

Consider $\mathrm{G} = \mathbb{F}_\infty$, a free group with infinite generators $g_1, g_2, \cdots$. For $g \in \mathbb{F}_\infty$, g can be written as a reduced word

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_n}^{k_n} \quad \text{with } i_1 \neq i_2 \neq \cdots \neq i_n, k_j \in \mathbb{Z}/\{0\}.$$

Given a complex function m on $\mathbb{Z}$, we define $T_{\tilde{m}} : \mathbb{C}[\mathbb{F}_\infty] \to \mathbb{C}[\mathbb{F}_\infty]$ by

$$T_{\tilde{m}}(\lambda(g)) = m(k_1)\lambda(g).$$

If T_m is bounded on $L_p(\mathbb{T})$, is $T_{\tilde{m}}$ bounded on $L_p(\mathbb{F}_\infty)$?

- Mei/Ricard '17+Mei/Ricard/Xu '21: Hilbert transforms and Fourier multipliers on free products
- Gonzalez/Parcet/X. : '22 Hilbert transforms on graph of groups.

HNN extensions

Base group: Γ **Subgroup of Γ:** A
Isomorphism of A: $\theta : A \to B \subset \Gamma$

Construction of HNN extension of Γ:
Let Γ be a group, and let $\theta : A \to B \subset \Gamma$ be an isomorphism between two subgroups of Γ. Let t be a symbol not in Γ. The HNN extension of Γ (relative to A, B and θ) is

$$\Gamma *_\theta = \{\Gamma, t : tat^{-1} = \theta(a), \forall a \in A\}.$$

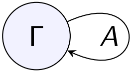
The new generator t is called the **stable letter**.

Remark

- *If $A = \Gamma$, and θ is an automorphism of Γ, then $\Gamma *_\theta = \langle t \rangle \ltimes_\theta \Gamma$.*
- *$\Gamma *_\theta = \langle t \rangle \ltimes H$, where H is the normal subgroup generated by conjugates $\Gamma_n = t^n \Gamma t^{-n}$, $n \in \mathbb{Z}$.*

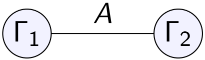
Understanding HNN extensions and amalgamated free products from Bass-Serre theory

- Fundamental groups for graph of groups which look like



are **HNN extensions of Γ**.

- If there are multiple loops connecting to Γ, we call the corresponding fundamental group a **multiple HNN extension**.
- Fundamental groups for graph of groups which look like



are **amalgamated free products of Γ_1 and Γ_2**.

- Fundamental groups of graphs of groups can always be written as iterated amalgamated free products or multiple HNN-extensions of amalgamated free products.

Normal forms of HNN extensions

Normal forms of group elements in HNN extensions:

$$g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n, \quad g_i \in \Gamma, \epsilon_i = \pm 1.$$

The above normal form is **unique** if

- $g = e$ iff $n = 0$, $g_0 = e$.
- if $\epsilon_i = -1$, g_i is a representation of a coset of A in G.
- if $\epsilon_i = 1$, g_i is a representation of a coset of B in G.
- no subsequence $t^{\epsilon} e t^{-\epsilon}$.

Theorem (Connections to groups acting on trees)

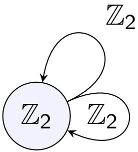
Let G be a group acting on a tree X without inversion. G can be identified with the fundamental group of a certain graph of groups (G, Y), where $Y = \mathrm{G} \backslash X$, i.e.

$$\mathrm{G} = \pi_1(G, Y, x_0),$$

where x_0 is a vertex of Y.

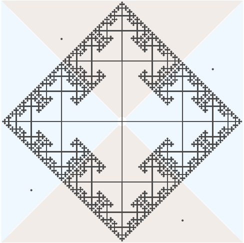
Examples–$\mathbb{F}_2 \ltimes \mathbb{Z}_2$

- $\mathbb{F}_2 \ltimes \mathbb{Z}_2$ is the fundamental group of



  $\mathbb{F}_2 \ltimes \mathbb{Z}_2$ is a multiple HNN-extension based on $\Gamma = \mathbb{Z}_2$ associated with $A = A' = \mathbb{Z}_2$.

- $\mathbb{F}_2 \ltimes \mathbb{Z}_2$ acts on the Cayley tree of $\mathbb{F}_2$ by means of the quotient map.



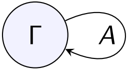
Examples–Baumslag-Solitar groups

The group presentation of $\mathrm{B}(m,n)$ is given by

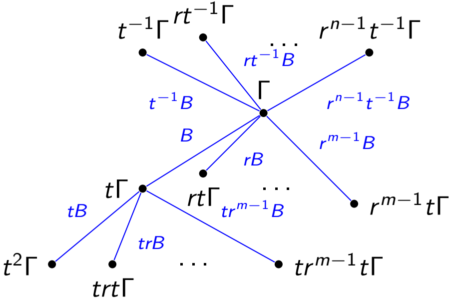
$$\langle r, t : tr^m t^{-1} = r^n \rangle.$$

$\mathrm{B}(m,n)$ is an HNN-extension based on $\Gamma = \mathbb{Z} = \langle r \rangle$ associated with $A = m\mathbb{Z}$ and $B = n\mathbb{Z}$.

- $\mathrm{B}(m,n)$ is the fundamental group of



- $\mathrm{B}(m,n)$ acts on its Bass-Serre tree:



Lyndon's length function

A real-valued function on G is called a **Lyndon's length function** if it satisfies the following three axioms:

- $L(e) = 0$.
- $L(g) = L(g^{-1})$ for any $g \in \mathrm{G}$.
- If $\rho(g, h) = \frac{1}{2}(L(g) + L(h) - L(gh^{-1}))$, then for all $g, h, \omega \in \mathrm{G}$, we have

$$\rho(g, h) \geq \min(\rho(g, \omega), \rho(\omega, h)).$$

Theorem (Bozejko '89)

Every Lyndon's length function on a discrete group is negative-definite and for each $\lambda \geq 0$ the function $\gamma_\lambda(x) = e^{-\lambda L(x)}$ is positive-definite.

Length functions on groups acting on $\mathbb{R}$-trees

G: a group acting on an $\mathbb{R}$-tree (X, d) by isometries.

A **based length function** L_{x_0} associated to a base point $x_0 \in X$ is defined by

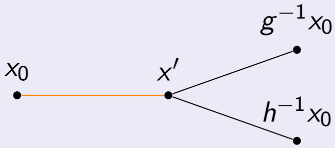
$$L_{x_0}(g) = d(g \cdot x_0, x_0).$$

Theorem (Chiswell '76)

L_{x_0} defined above gives a Lyndon's length function on G. Conversely, given a Lyndon's length function L on G, there exists an $\mathbb{R}$-tree on which G acts and such that $L = L_{x_0}$ for some point $x_0 \in X$.

Remark

The function $\rho(g, h)$ corresponds to the largest common path of the path from x_0 to $g^{-1}x_0$ and that from x_0 to $h^{-1}x_0$.



Examples

- $\mathrm{G} = \mathbb{F}_n$, the word length and block length are both Lyndon's Length functions.

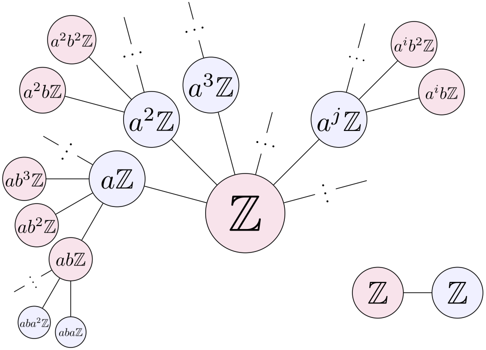


Figure: Bass-Serre tree of $\mathbb{F}_2$, pink $\mathbb{Z} = \langle a \rangle$ and blue $\mathbb{Z} = \langle b \rangle$

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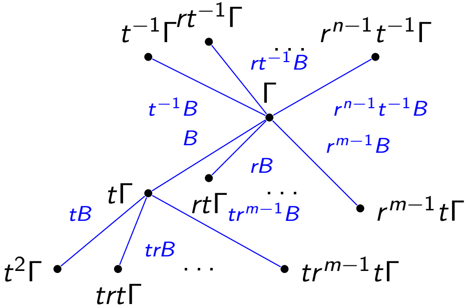
Examples

- $\mathrm{G} = \Gamma *_\theta$ and consider its action on its Bass-Serre tree X.



For any $x_0 \in X$, the based length function L_{x_0} is given by

$$L_{x_0}(g) = n,$$

for any $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n, \ \ g_i \in \Gamma, \epsilon_i = \pm 1.$

In other words, L counts the number of loops of g goes through, as an element in fundamental group of the loop of group Γ.

Khintchine type inequalities for free groups/amalgamated free products

- **Haagerup's inequality ('79)**: Let W_d be the subset of $\mathbb{F}_n$ with word length d.

$$\|\sum_{w\in W_d} x_w \lambda(w)\|_{C_r^*(\mathbb{F}_n)} \approx (\sum_{w\in W_d} |x_w|^2)^{\frac{1}{2}}, x_w \in \mathbb{C}.$$

- **Voiculescu ('98), Junge ('05)**: Let w_1 be the subset of $\Gamma = \Gamma_1 *_A \Gamma_2 *_A \cdots *_A \Gamma_n$ with block length 1. Let $x = \sum_{w\in w_1} x_w \lambda(w)$ and $L_i x$ be the part of x which is in $C_r^*(\Gamma_i)$.

$$\|x\|_{C_r^*(\Gamma)} \approx \max_i \|L_i(x)\|_{C_r^*(\Gamma_i)} + \|E_A(x^*x)\|^{\frac{1}{2}} + \|E_A(xx^*)\|^{\frac{1}{2}}.$$

- **Ricard-Xu ('06)**: The corresponding equivalence for w_d ($d \in \mathbb{N}$) in Γ.

Khintchine type inequalities on (multiple) HNN extensions

Theorem (X.)

Let $\mathrm{G} = \Gamma *_\theta$ *be an (multiple) HNN extension of a discrete group* Γ*. Let* $W_1 = \{g : L(g) = 1\}$, $x = \sum_{w \in W_1} x_w \lambda(w)$, $L_i^+ x = \sum_{w \in \Gamma t_i \Gamma} x_w \lambda(w)$ *and* $L_i^- x = \sum_{w \in \Gamma t_i^{-1} \Gamma} x_w \lambda(w)$*. Then we have, for any* $x \in W_1$,

$$\|x\|_{\mathcal{L}(\Gamma)} \approx \max_i \left(\|L_i^+(x)\|_{\mathcal{L}(\Gamma)}, \|L_i^-(x)\|_{\mathcal{L}(\Gamma)} \right) + \|E_\Gamma(x^* x)\|^{\frac{1}{2}} + \|E_\Gamma(x x^*)\|^{\frac{1}{2}}.$$

A sketch of proof:

- It is easy to check "$\gtrsim$".
- To prove "$\lesssim$", the main idea is the decomposition

$$x = \sum_i L_i^+ x + \sum_i L_i^- x = \sum_i (P_i^+ + (1 - P_i^+)) L_i^+ x + \sum_i (P_i^- + (1 - P_i^-)) L_i^- x,$$

where P_i^+ and P_i^- are the projections from $L_2(\Gamma)$ to the subspaces $\overline{\text{span}}\{\lambda(g) : g \in \Gamma t_i \Gamma\}$ and $\overline{\text{span}}\{\lambda(g) : g \in \Gamma t_i^{-1} \Gamma\}$ respectively.

Remark

The constants in the equivalence are independent of the number of stable letters.

$$
\begin{aligned}
&\|\sum_i (P_i + (1-P_i))L_i x\| \\
&\le \|\sum_i P_i L_i x\| + \|\sum_i (1-P_i)L_i x\| \\
&= \|\sum_i L_i x^* P_i L_i x\|^{\frac{1}{2}} + \|\sum_{i,j} (1-P_i)L_i x L_j x^* (1-P_j)\|^{\frac{1}{2}} \\
&\le \|\sum_i L_i x^* L_i x\|^{\frac{1}{2}} + \|\sum_i E_\Gamma (L_i x L_i x^*)(1-P_j)\|^{\frac{1}{2}} \\
&\le \|\sum_i L_i x^* L_i x\|^{\frac{1}{2}} + \|\sum_i E_\Gamma (L_i x L_i x^*)\|^{\frac{1}{2}} \\
&= \|\sum_i L_i x^* L_i x(\sum_i P_i^{\pm} + Q)\|^{\frac{1}{2}} + \|\sum_i E_\Gamma (L_i x L_i x^*)\|^{\frac{1}{2}} \\
&\quad \ldots
\end{aligned}
$$

Proposition (González Pérez-Parcet-X. '22)

Let $T_m : L_2(\widehat{\mathrm{G}}) \to L_2(\widehat{\mathrm{G}})$ be bounded. If

$$(m(gh) - m(g))(\overline{m(g^{-1})} - \overline{m(h)}) = 0, \forall g \in G - \Gamma, \forall h \in \mathrm{G}$$

and m is left Γ-invariant relative to a multiplicative character $\mathcal{X}$ on Γ ($m(kg) = \mathcal{X}(k)m(g), \forall k \in \Gamma, g \in \mathrm{G}$), then $\| T_m : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| \lesssim p^{\beta}$, $\beta = \log_2(1 + \sqrt{2})$ for any $2 \leq p < \infty$.

Geometric Model

Let G be a group acting on an $\mathbb{R}$ tree X by homomorphisms.
Choose a vertex x_0 in X and write $X \backslash \{x_0\} = \sqcup_i X_i$.
For every connected component X_i, we choose a constant $C_i \in \mathbb{C}$ such that $\sup_i C_i < \infty$. Define a bounded function on X by

$$\widetilde{m}|_{X_i} \equiv C_i, \quad \text{and} \quad \widetilde{m}(x_0) = 0.$$

Then the function $\widetilde{m}$ induces a function on G by

$$m(g) = \widetilde{m}(g \cdot x_0) \quad \text{for any } g \in \mathrm{G}.$$

Let G be an HNN-extension of Γ. Consider its action on its Bass-Serre tree, if we choose x_0 to be the vertex labelled by Γ, the number of connected components of $X \backslash \{x_0\}$ is equal to $|\mathrm{G}/A| + |\mathrm{G}/B|$.

Corollary

Let Γ be a discrete abelian group and G be an HNN extension of Γ. For $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n, \ \ g_i \in \Gamma, \epsilon_i = \pm 1$, define

$$\chi(g) = \begin{cases} \chi_1(P_1(g_0)), & \text{if } e_1 = 1, \\ \chi_2(P_2(g_0)), & \text{if } e_1 = -1, \end{cases}$$

where P_1, P_2 are the canonical projections from G to G/A and to G/B, and χ_1 and χ_2 are two characters on G/A and G/B respectively. T_χ extends to a bounded Fourier multiplier on $L_p(\widehat{\mathrm{G}})$ for any $1 < p < \infty$

Given a A, B-invariant function m on Γ, we define a Fourier multiplier on $\mathbb{C}[\mathrm{G}]$ by

$$M_m(\lambda(g)) = m(g_0)\lambda(g),$$

for $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots g_{n-1} t^{\epsilon_n} g_n, \; g_i \in \Gamma, \epsilon_i = \pm 1$.

Corollary

Let Γ be a discrete abelian group and G be an HNN extension of Γ. If T_m is a bounded A, B- invariant Fourier multiplier on $L_p(\widehat{\Gamma})$, then M_m extends to a bounded Fourier multiplier on $L_p(\widehat{\mathrm{G}})$ for any $1 < p < \infty$.

Question: What if Γ is not abelian?

Γ abelian

$$T_\chi : \mathbb{C}(\Gamma *_\theta) \to \mathbb{C}(\Gamma *_\theta)$$
$$\lambda(g) \longmapsto \chi_1(P_1(g_0))\lambda(g)$$

Γ non-abelian

$$\alpha : \mathbb{C}(\Gamma *_\theta) \to \mathbb{C}(\Gamma \times \Gamma *_\theta)$$
$$\lambda(g) \longmapsto \lambda_\Gamma(P_1(g_0)) \otimes \lambda(g)$$

Lemma

For $1 < p < \infty$, we have $\|\alpha(x)\|_{L_p(\mathcal{L}(\Gamma)\overline{\otimes}\mathcal{L}(\Gamma *_\theta))} \approx \|x\|_{L_p(\mathcal{L}(\Gamma *_\theta))}$.

Idea of the proof:
Write $\Gamma \times \Gamma *_\theta = (\Gamma \times \Gamma) *_{\tilde{\theta}}$ + length reduction + Khintchine inequality for HNN extensions (length 1).

Theorem

Assume that $\Gamma *_\theta$ has QWEP. If the Fourier multiplier T_m is A and B-biinvariant and completely bounded on $L_p(\widehat{\Gamma})$, then M_m extends to a completely bounded Fourier multiplier on $L_p(\widehat{\Gamma *_\theta})$ for any $1 < p < \infty$.

Keys: 1. $\Gamma *_\theta$ has QWEP, $T_m \otimes \mathrm{Id}_{L_p(\widehat{\Gamma *_\theta})}$ is bounded on $L_p(\mathcal{L}(\Gamma)\overline{\otimes}\mathcal{L}(\Gamma *_\theta))$.
2. $(T_m \otimes \mathrm{Id})[\alpha(x)] = \alpha[M_m(x)]$.

Thank you!