A dual and a conjugate system for q-Gaussians

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joint work with Akihiro Miyagawa
arXiv:2203.00547

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Consider operators a_i and their adjoints a_i^* for $i \in I$; for $I = [d]$ or $I = \mathbb{N}$

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Cuntz relations ($q = 0$)

$$a_i a_j^* = \delta_{ij} 1$$

q-relations for $-1 \le q \le 1$

$$a_i a_j^* - q a_j^* a_i = \delta_{ij} 1$$

Bozejko, Speicher 1991

- there exists a realization of the q-relations on a Hilbert space for all $-1 \leq q \leq 1$, such that a_i is adjoint to a_i^*
- this is a Fock representation, i.e., there is vacuum vector Ω such that

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Bozejko, Kümmerer, Speicher 1997

- operatoralgebraic and probabilistic properties of the q-Gaussian operators and algebras
- q-Gaussian functor $\mathcal{H} \mapsto \Gamma_q(\mathcal{H})$

The q C^*-algebras and von Neumann algebras

- $C^*(a_i, a_i^* \mid i \in I)$

- $C^*(a_i + a_i^* \mid i \in I)$

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  - for $|q| < \sqrt{2} - 1$ isomorphic to $q = 0$ (Jorgensen,Schmitt,Werner 1995)
  - for $|q| < 0.44$ isomorphic to $q = 0$ (Dykema, Nica 1993)
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  - for all $|q| < 1$ isomorphic to $q = 0$ (Kuzmin, March 2022)
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  - for $q = 0$: the free Gaussian functor of Voiculescu
  - for q sufficiently small (depending on d, $d < \infty$): isomorphic to $q = 0$ (Guionnet, Shlyakhtenko 2014)
  - for all $-1 < q < 1$, $q \neq 0$, $d = \infty$: not isomorphic to $q = 0$ (Borst, Caspers, Klisse, Wasilewski, Feb 2022)
- vN$(a_i, a_i^* \mid i \in I)$ isomorphic to $B(\mathcal{H})$
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The q-Fock space

Fix $q \in [-1,1]$ and consider Hilbert space $\mathcal{H}$. The q-Fock space

$$\mathcal{F}_q(\mathcal{H}) = \overline{\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}}^{\langle\cdot,\cdot\rangle_q} \qquad\qquad (\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega)$$

is completion of algebraic Fock space with respect to inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q = \delta_{nm} \sum_{\sigma \in S_n} \prod_{r=1}^{n} \langle f_r, g_{\sigma(r)} \rangle q^{i(\sigma)}$$

- $i(\sigma) = \#\{(k,l) \mid 1 \leq k < l \leq n; \sigma(k) > \sigma(l)\}$ is number of inversions

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- $i(\sigma) = \#\{(k,l) \mid 1 \leq k < l \leq n; \sigma(k) > \sigma(l)\}$ is number of inversions
- inner product is positive definite, and has a kernel only for $q = 1$ and $q = -1$ (Bozejko, Speicher 1991)
- for $q = 1$ and $q = -1$ first divide out the kernel, thus leading to the symmetric and anti-symmetric Fock space, respectively

Creation and annihilation operators

- $a^*(f)\Omega = f$ and $a^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$
- its adjoint is given by $a(f)\Omega = 0$ and

$$a(f) f_1 \otimes \cdots \otimes f_n = \sum_{r=1}^{n} q^{r-1} \langle f, f_r \rangle f_1 \otimes \cdots \otimes f_{r-1} \otimes f_{r+1} \otimes \cdots \otimes f_n$$

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- those operators satisfy the q-commutation relations

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- prominent special cases:
  - $q = 1$: CCR relations
  - $q = 0$: Cuntz relations
  - $q = -1$: CAR relations
- with the exception of the case $q = 1$, the operators $a^*(f)$ are bounded

q-Gaussian Distribution

- consider q-Gaussian operators

$$X(f) = a(f) + a^*(f) \qquad f \in \mathcal{H}_{\text{real}}$$

- consider vacuum expectation state

$$\tau(T) = \langle \Omega, T\Omega \rangle_q, \quad \text{for} \quad T \in \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$$

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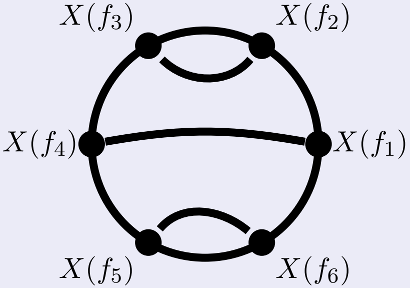
- is given by q-deformed version of the Wick/Isserlis formula

$$\tau\left(X(f_1)\cdots X(f_n)\right) = \sum_{\pi \in \mathcal{P}_2(k)} q^{cr(\pi)} \prod_{(l,r)\in\pi} \langle f_l, f_r \rangle,$$

where $cr(\pi)$ denotes number of crossings of pairing π

Contribution of Pairing to Moment $\tau[X(f_1)X(f_2)X(f_3)X(f_4)X(f_5)X(f_6)]$

non-crossing



$$\langle f_1, f_4\rangle \cdot \langle f_2, f_3\rangle \cdot \langle f_5, f_6\rangle$$

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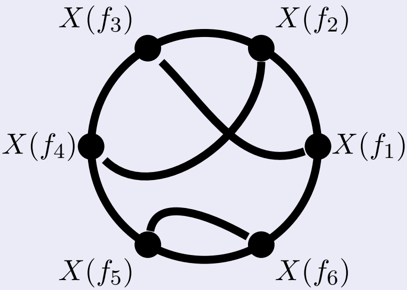
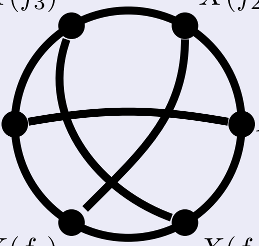
one crossing



$$q \cdot \langle f_1, f_3 \rangle \cdot \langle f_2, f_4 \rangle \cdot \langle f_5, f_6 \rangle$$

Contribution of Pairing to Moment $\tau[X(f_1)X(f_2)X(f_3)X(f_4)X(f_5)X(f_6)]$

three crossings



$$q^3 \cdot \langle f_1, f_4\rangle \cdot \langle f_2, f_5\rangle \cdot \langle f_3, f_6\rangle$$

The q-Gaussian von Neumann algebras $\Gamma_q(\mathcal{H})$

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- it is a non-injective, prime, strongly solid II_1-factor for all $-1 < q < 1$ (Ricard 2005, Nou 2004, Avsec 2011)

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- combinatorial description is nice and concrete
- analytic description is more abstract and mostly perturbative around the case $q = 0$

Regularity properties of generators

Consider $X_1, \dots, X_d$ selfadjoint operators in a tracial vN-algebra (M, τ)

Conjugate system: $\xi_1, \dots, \xi_d \in L^2(X_1, \dots, X_d)$

$$\tau(\xi_i Q(X_1, \dots, X_d)) = \tau \otimes \tau[\partial_i Q(X_1, \dots, X_d)], \quad \text{i.e.} \quad \xi_i = \partial_i^* \Omega \otimes \Omega$$

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Normalized dual system: $D_1, \dots, D_d$

- unbounded operators on $L^2(X, \tau)$ with $\mathbb{C}\langle X \rangle \subset$ domain
- $D_i \Omega = 0$, $1 \in \mathrm{dom}(D_i^*)$
- $[D_i, X_j] = \delta_{ij} P_\Omega$

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Theorem

If $(D_1, \ldots, D_d)$ is a normalized dual system, then a conjugate system is given by

$$\partial_i^* \Omega \otimes \Omega = \xi_i = D_i^* \Omega$$

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- note that the non-commutative derivative ∂_f :

$$\mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle \to \mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle \otimes \mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle$$

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has an easy (independent of q) description on the algebra generated by all $X(f)$

- in order to calculate its adjoint ∂_f^*, however, we have to understand the behaviour of ∂_f as an unbounded operator on the Hilbert space

$$\partial_f : L^2(M, \tau) \to L^2(M, \tau) \otimes L^2(M, \tau)$$

Embedding of vN-algebra into Fock space

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$$T \mapsto T\Omega$$

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Wick product

stochastic integral

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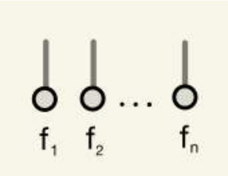
$$T \mapsto T\Omega$$

$$X(f_1)\cdots X(f_n) \mapsto X(f_1)\cdots X(f_n)\Omega$$



$$W(f_1 \otimes \cdots \otimes f_n) \mapsto f_1 \otimes \cdots \otimes f_n$$

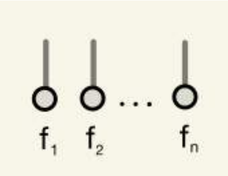
Wick product

stochastic integral

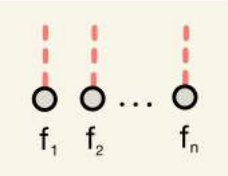
Embedding of vN-algebra into Fock space

$$\Gamma_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}) = L^2(\Gamma_q(\mathcal{H}), \tau)$$
$$T \mapsto T\Omega$$

$$X(f_1)\cdots X(f_n) \mapsto X(f_1)\cdots X(f_n)\Omega$$



$$W(f_1 \otimes \cdots \otimes f_n) \mapsto f_1 \otimes \cdots \otimes f_n$$



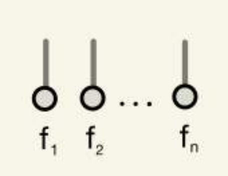
Wick product

stochastic integral

From $X(\cdots)$ to $W(\cdots)$ and back

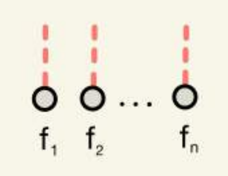
There are combinatorial relations between

$$X(f_1)X(f_2)\cdots X(f_n) =$$



and

$$W(f_1 \otimes f_2 \otimes \cdots \otimes f_n) =$$

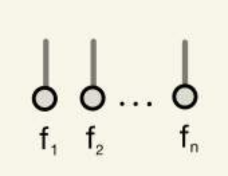


... in both directions.

From $X(\cdots)$ to $W(\cdots)$ and back

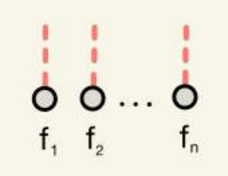
There are combinatorial relations between

$$X(f_1 \otimes f_2 \otimes \cdots \otimes f_n) =$$



and

$$W(f_1 \otimes f_2 \otimes \cdots \otimes f_n) =$$



... in both directions.

From $X(\cdots)$ to $W(\cdots)$

$$X(f_1 \otimes f_2) \quad = \quad W(f_1 \otimes f_2) \quad + \quad \langle f_1, f_2 \rangle W(\Omega)$$

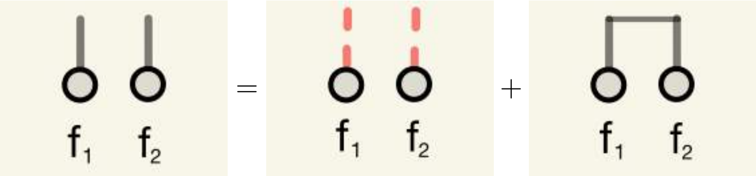
From $X(\cdots)$ to $W(\cdots)$ and back

$$X(f_1 \otimes f_2) = W(f_1 \otimes f_2) + \langle f_1, f_2 \rangle W(\Omega)$$

$$W(f_1 \otimes f_2) = X(f_1 \otimes f_2) - \langle f_1, f_2 \rangle X(\Omega)$$

From $X(\cdots)$ to $W(\cdots)$ and back

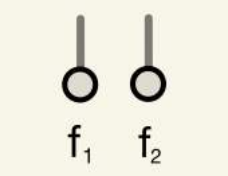
$$X(f_1 \otimes f_2) \quad = \quad W(f_1 \otimes f_2) \quad + \quad \langle f_1, f_2 \rangle W(\Omega)$$



$$W(f_1 \otimes f_2) \quad = \quad X(f_1 \otimes f_2) \quad - \quad \langle f_1, f_2 \rangle X(\Omega)$$

From $X(\cdots)$ to $W(\cdots)$ and back

$$X(f_1 \otimes f_2) = W(f_1 \otimes f_2) + \langle f_1, f_2 \rangle W(\Omega)$$



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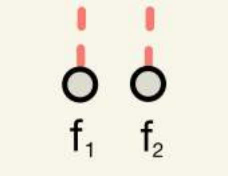

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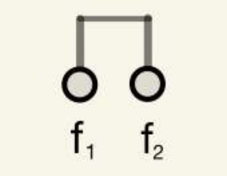
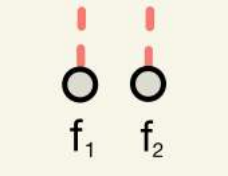


$$W(f_1 \otimes f_2) = X(f_1 \otimes f_2) - \langle f_1, f_2 \rangle X(\Omega)$$



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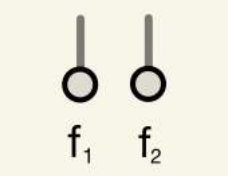

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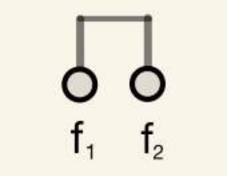
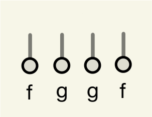
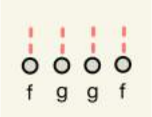


From $X(\cdots)$ to $W(\cdots)$



$= \quad 1$



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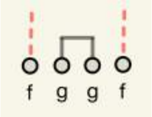
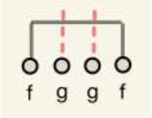


$+q^2$



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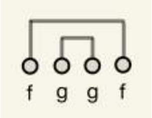
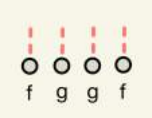


From $X(\cdots)$ to $W(\cdots)$ and back



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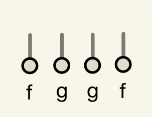
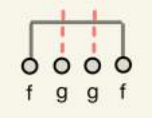


$-q^2$



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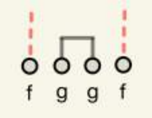

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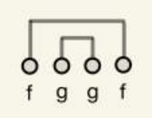
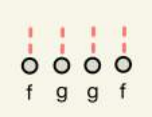


From $X(\cdots)$ to $W(\cdots)$ and back



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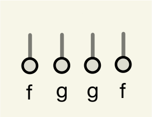

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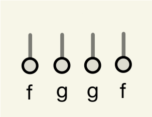

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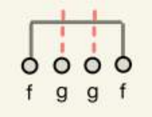

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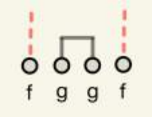

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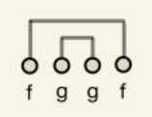

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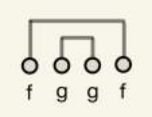
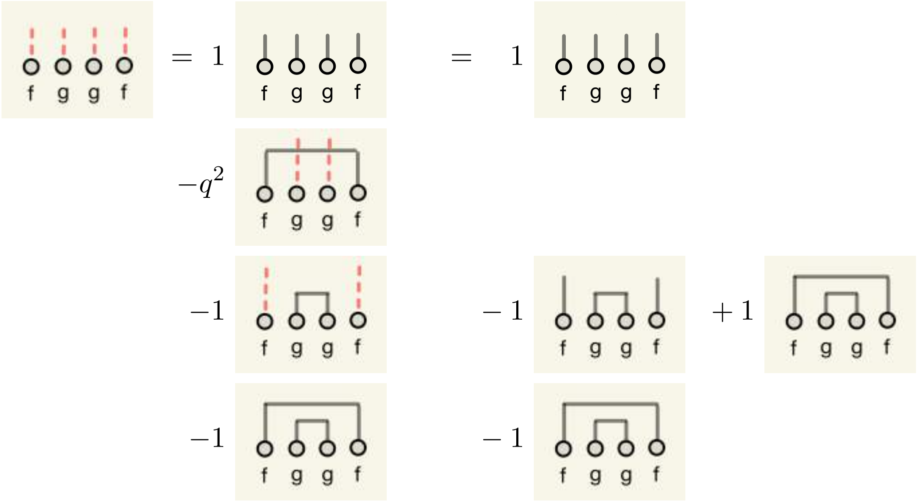
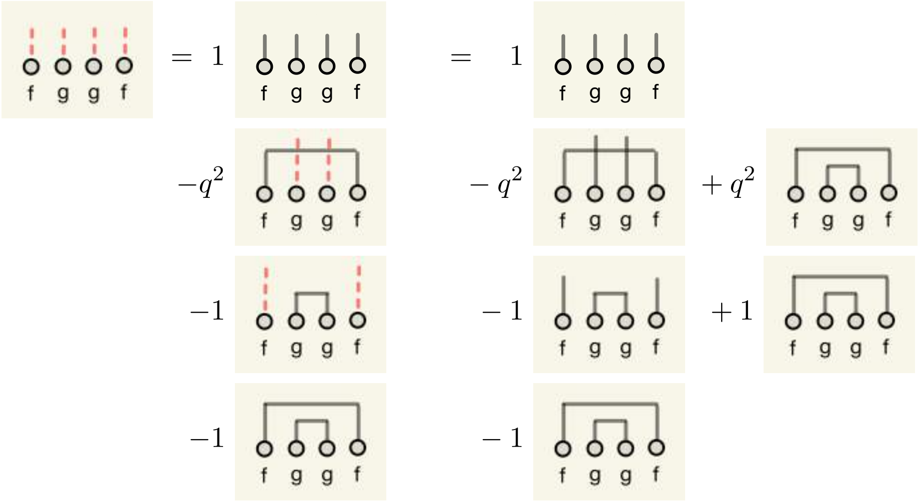


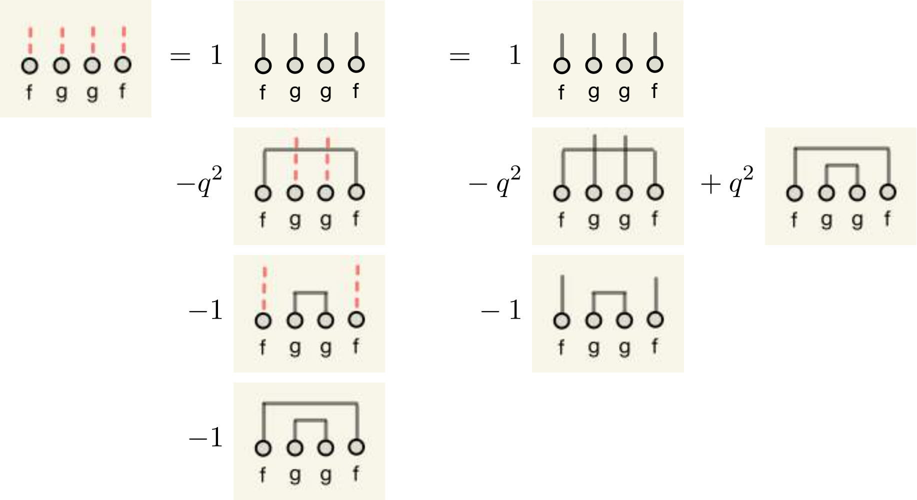
From $X(\cdots)$ to $W(\cdots)$ and back



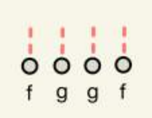
From $X(\cdots)$ to $W(\cdots)$ and back

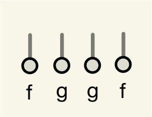


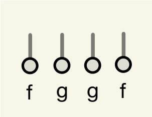
From $X(\cdots)$ to $W(\cdots)$ and back

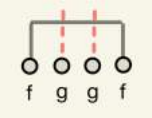


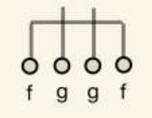
From $X(\cdots)$ to $W(\cdots)$ and back

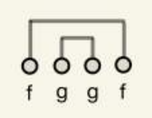


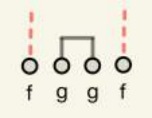
$= 1$ 

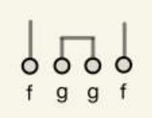
$= 1$ 

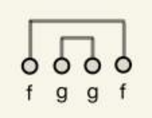
$-q^2$ 

$-\, q^2$ 

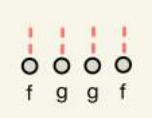
$+\, q^2$ 

-1 

$-\, 1$ 

-1 

From $X(\cdots)$ to $W(\cdots)$ and back



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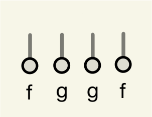

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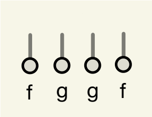

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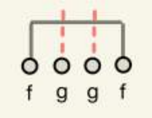
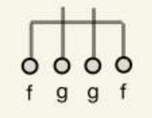
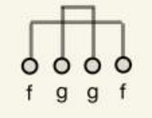


$-\ q^2$



$+\ {\color{red}q^2}$



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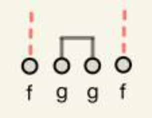

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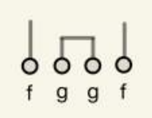

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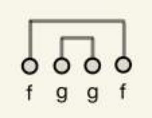


Regularity properties of generators

Consider $X_1, \dots, X_d$ selfadjoint operators in a tracial vN-algebra (M, τ)

Conjugate system: $\xi_1, \dots, \xi_d \in L^2(X_1, \dots, X_d)$

$\tau(\xi_i Q(X_1, \dots, X_d)) = \tau \otimes \tau[\partial_i Q(X_1, \dots, X_d)]$, i.e. $\xi_i = \partial_i^* \Omega \otimes \Omega$

- note that the non-commutative derivative ∂_f :

$$\mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle \to \mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle \otimes \mathbb{C}\langle X(f) \mid f \in \mathcal{H}\rangle$$

$$X(f_1)\cdots X(f_n) \mapsto \sum_{k=1}^{n} \langle f, f_k\rangle X(f_1)\cdots X(f_{k-1}) \otimes X(f_{k+1})\cdots X(f_n)$$

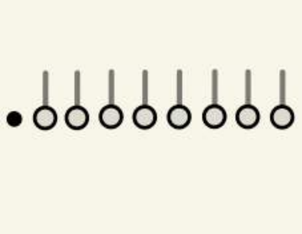
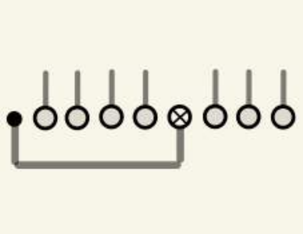
  has an easy (independent of q) description on the algebra generated by all $X(f)$

- in order to calculate its adjoint ∂_f^* we have to understand the behaviour of ∂_f as an unbounded operator on the Hilbert space

$$\partial_f : L^2(M, \tau) \to L^2(M, \tau) \otimes L^2(M, \tau)$$

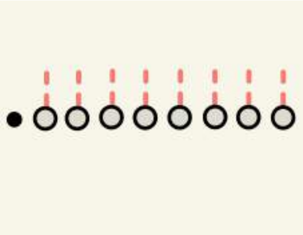
Non-commutative derivative on the algebra

$$\partial_f X(f_1 \otimes \cdots \otimes f_n) = \sum_k \langle f, f_k \rangle X(f_1 \otimes \cdots \otimes f_{k-1}) \otimes X(f_{k+1} \otimes \cdots \otimes f_n)$$

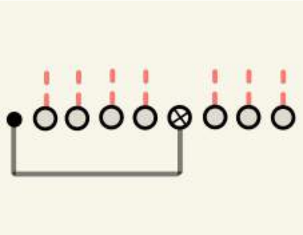
 $= \cdots \quad +$  $+ \quad \cdots$

Non-commutative derivative on the Fock space

$$\partial_f W(f_1 \otimes \cdots \otimes f_n) = \sum_{\pi} (-1)^{\cdots} q^{\cdots} \delta_\pi W(\text{singl. left}) \otimes W(\text{singl. right})$$



$=$ $\cdots$



$\cdots$

Non-commutative derivative on the Fock space

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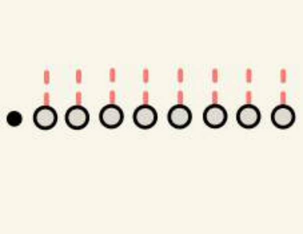
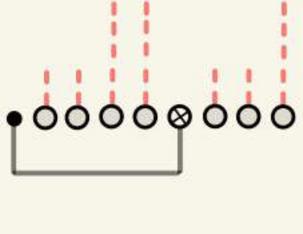


$= \quad \cdots$



$\cdots$

Non-commutative derivative on the Fock space

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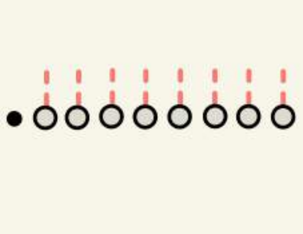
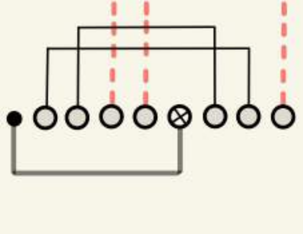


$= \quad \cdots \quad -$



$\cdots$

Non-commutative derivative on the Fock space

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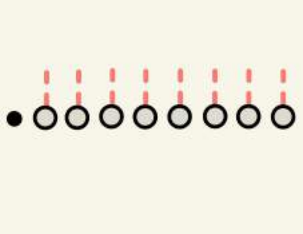

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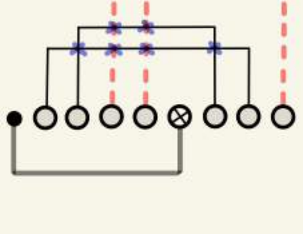

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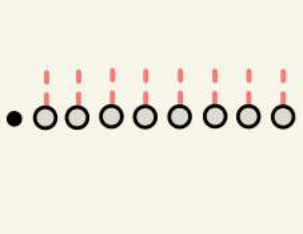

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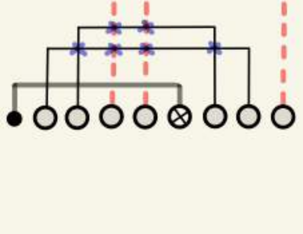

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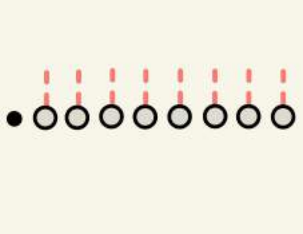

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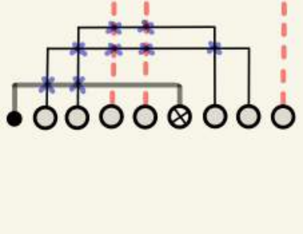

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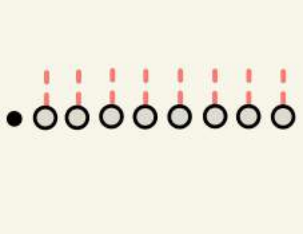

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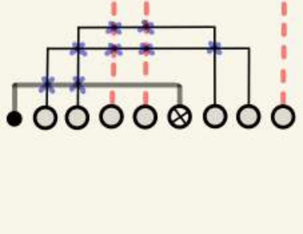

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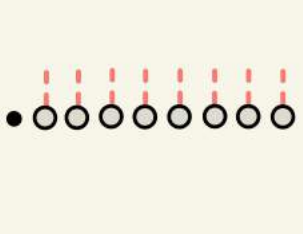

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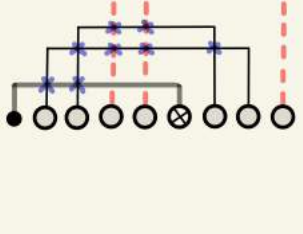
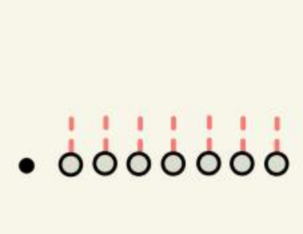


$\cdots$

Note: For conjugate variable we only need the vacuum part of ∂

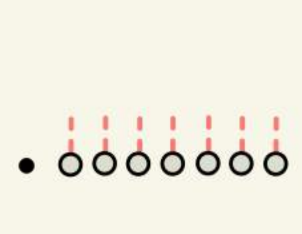
Non-commutative derivative on the Fock space - vacuum part



$=$

Non-commutative derivative on the Fock space - vacuum part

$$\langle \partial_f \eta_1 \otimes f \otimes \eta_2, \Omega \otimes \Omega \rangle = \qquad (m = 3)$$



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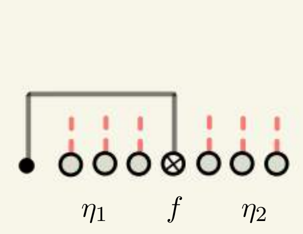

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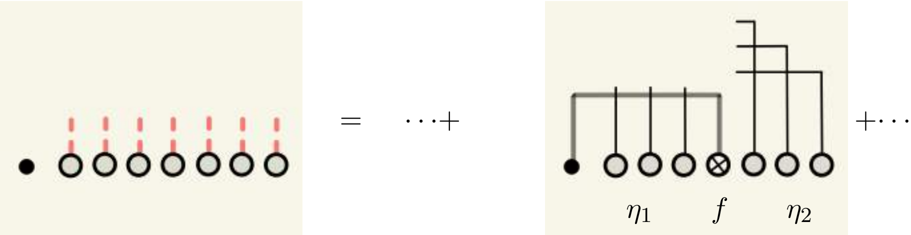

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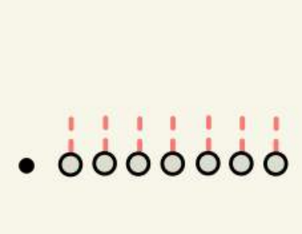

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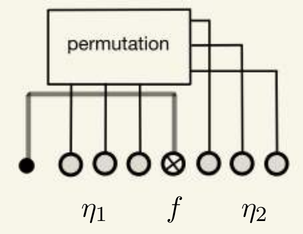

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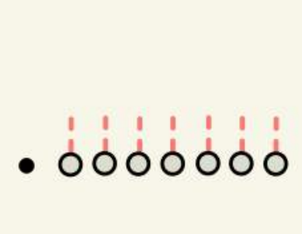
$$\langle \partial_f \eta_1 \otimes f \otimes \eta_2, \Omega \otimes \Omega \rangle = (-1)^m q^{\frac{(m+1)m}{2}} \qquad (m=3)$$



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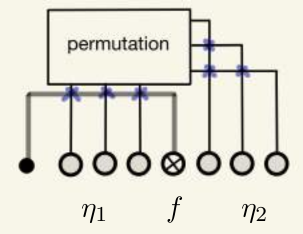

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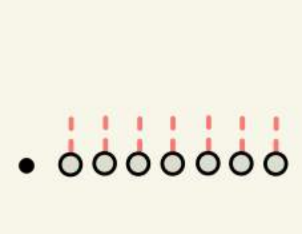

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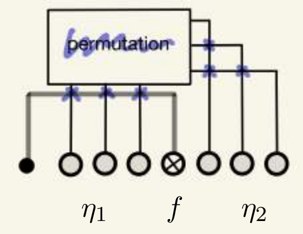

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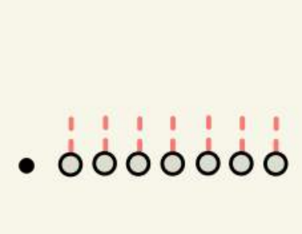

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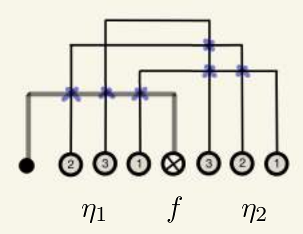

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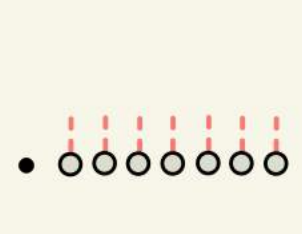

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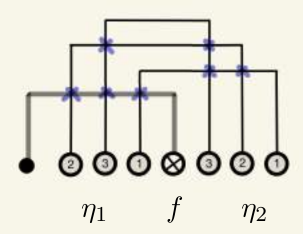

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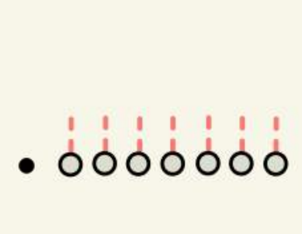

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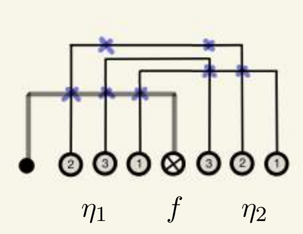

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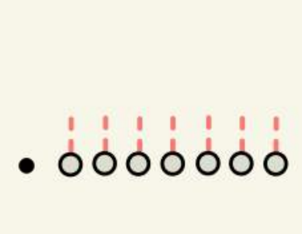

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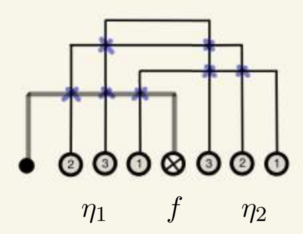

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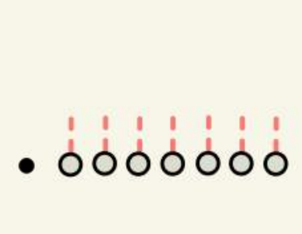

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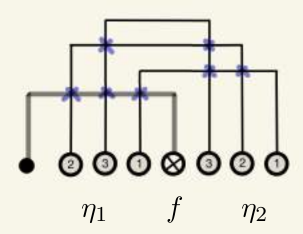

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key factor: $q^{\frac{(m+1)m}{2}}$

Results for all q with a similar flavor as ours

Diagramatics and Wick products

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Stochastic context

- Donati-Martin: Stochastic integration with respect to q-Brownian motion, 2003
- Deya, Schott: On multiplication in q-Wiener chaoses, 2018

Linear basis for concrete calculations

- Let $\{e_1, \dots, e_d\}$ be an ONB of $\mathcal{H}$. Then

$$\{e_{i(1)} \otimes \cdots \otimes e_{i(m)} \mid m \geq 0, 1 \leq i(1), \dots i(m) \leq d\}$$

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- Notation:

$$e_w := e_{i(1)} \otimes \cdots \otimes e_{i(m)} \qquad \text{for} \quad w = (i(1), \dots, i(m)) \in [d]^*$$

Now let's look on the conjugate variable ξ_i

We want $\xi_i = \partial_i^* \Omega \otimes \Omega$, so we need ξ_i with

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only if

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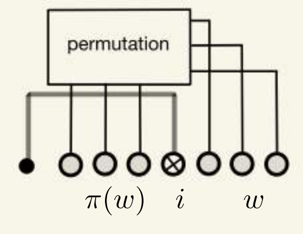

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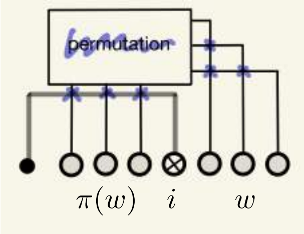

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$$\langle \partial_i e_v, \Omega \otimes \Omega \rangle_q = \langle \partial_i e_{\pi(w) i w}, \Omega \otimes \Omega \rangle_q = (-1)^m q^{\frac{(m+1)m}{2}} \langle e_{\pi(w)}, e_w \rangle_q$$

Now for the conjugate variable ξ_i

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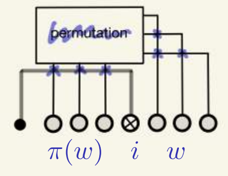

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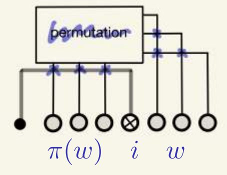

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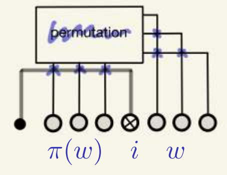

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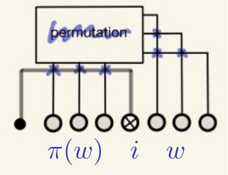

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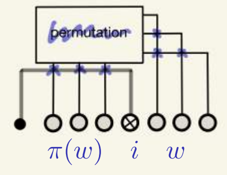

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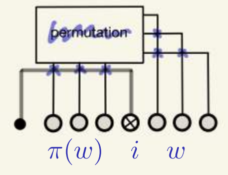

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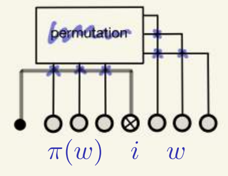

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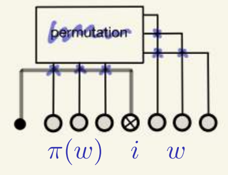

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&= \langle e_v, \underbrace{\sum_{m,w} (-1)^m q^{\frac{(m+1)m}{2}} r_{iw}^* e_w}_{\xi_i} \rangle_q
\end{aligned}$$

Theorem (Miyagawa, Speicher 2022): We have or all $-1 < q < 1$:

- a conjugate system $(\xi_1, \dots, \xi_d)$ for the q-Gaussians $(X_1, \dots, X_d)$ is given by

$$\xi_i = \sum_{w \in [d]^*} (-1)^{|w|} q^{\frac{(|w|+1)|w|}{2}} r_{iw}^* e_w$$

- the above sum converges in operator norm, thus $\xi_i \in \Gamma_d(\mathbb{R}^d)$
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- but note: in the end the factor $q^{\frac{(|w|+1)|w|}{2}}$ beats them all, even when taking the dimension d^m of $\mathcal{H}^{\otimes m}$ into account

Consequences for the q-Gaussians

For all $-1 < q < 1$ we have the following properties:

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