An Improvement on Gauss's Circle Problem and Dirichlet's Divisor Problem

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Gauss's Circle Problem

- Error term function:

$$R(X) = \sum_{m^2+n^2 \le X} 1 - \pi X \tag{1}$$

- The sum counts the number of integer lattice points inside the circle of radius $\sqrt{X}$ centered at origin (including points on the boundary), and the second term is the area of the disk enclosed by the circle.
- Each lattice point is the left lower corner of some unit square. Through this correspondence, we see that the above sum is approximately the area of the disk. We subtract the main term from the sum and aim to bound the error term.

Dirichlet's Divisor Problem

- Error term function:

$$\Delta(X) = \sum_{1 \le n \le X} d(n) - X \log X - (2\gamma - 1)X, \tag{2}$$

  where

$$d(n) = \sum_{s|n, s \in \mathbb{N}} 1$$

  is the divisor function, and $\gamma = 0.5772\ldots$ is Euler's constant.
- The sum counts the number of integer lattice points under the hyperbola $xy = X$. The terms we subtract are the main terms of the sum derived using standard hyperbola method in number theory.

Our Goal

- We would like to show that

$$R(X) \lesssim_\epsilon X^{\theta+\epsilon}, \ \Delta(X) \lesssim_\epsilon X^{\theta+\epsilon},$$

with θ as small as possible. Here $\lesssim$ is a substitute for Vinogradov's notation.

- Using simple geometric observation, C. F. Gauss showed that $\theta = \frac{1}{2}$ in the Circle Problem.
- By the above mentioned hyperbola method, we can obtain $\theta = \frac{1}{2}$ in the Divisor Problem.

Conjectures

On the other direction, G. H. Hardy showed that both $R(X)$ and $\Delta(X)$ are

$$\Omega((X \log X)^{\frac{1}{4}}), \tag{3}$$

where $f = \Omega(g)$ means that for any constant $C > 0$, $|f| \geq C|g|$ infinitely often in the limit process. His theorems naturally lead to the following conjecture.

Conjecture

In Gauss's Circle Problem and Dirichlet's Divisor Problem,

$$\theta = \frac{1}{4}.$$

History

- $\theta = \frac{1}{2}$, Gauss($\approx$ 1800), Dirichlet(1849)
- $\theta = \frac{1}{3}$, Voronoi(1904), Sierpinski(1906), Landau(1913)
- $\theta = \frac{15}{46} = 0.3260\dots$, Titchmarsh(1934), Chih(1950), Richert(1953)
- $\theta = \frac{13}{40} = 0.3250\dots$, Hua(1942)
- $\theta = \frac{12}{37} = 0.3243\dots$, Chen(1963), Kolesnik(1969)
- $\theta = \frac{7}{22} = 0.3181\dots$, Iwaniec and Mozzochi(1988)
- $\theta = \frac{131}{416} = 0.3149\dots$, Huxley(2003)

Our result

Theorem (Li and Y.)

$$R(X) = O_\epsilon(X^{\theta^*+\epsilon}) \quad \text{and} \quad \Delta(X) = O_\epsilon(X^{\theta^*+\epsilon})$$

for all $\epsilon > 0$, *where*

$$\theta^* = 0.314483\cdots$$

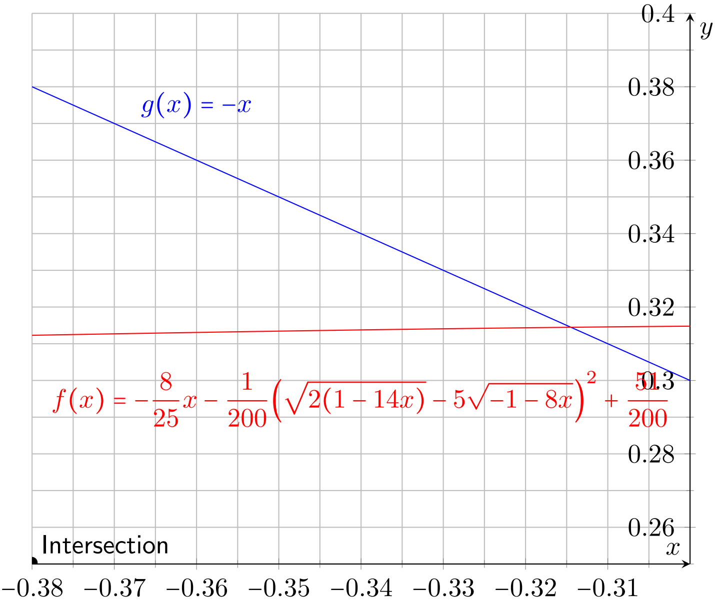
is defined below.

Definition

$\theta^* = 0.3144831759741\cdots$ is defined in such a way that $-\theta^*$ is unique solution to the equation

$$-\frac{8}{25}x - \frac{1}{200}\left(\sqrt{2(1-14x)} - 5\sqrt{-1-8x}\right)^2 + \frac{51}{200} = -x \tag{4}$$

on the interval $[-0.35, -0.3]$.



$g(x) = -x$
$f(x) = -\frac{8}{25}x - \frac{1}{200}\left(\sqrt{2(1-14x)} - 5\sqrt{-1-8x}\right)^2 + \frac{51}{200}$
Intersection
y
x
0.4
0.38
0.36
0.34
0.32
0.3
0.28
0.26
−0.38 −0.37 −0.36 −0.35 −0.34 −0.33 −0.32 −0.31

Hardy-Littlewood Circle Method

Currently, the most efficient method for estimating exponential sums on $\mathbb{Z}$ is the Bombieri-Iwaniec method, otherwise known as the discrete Hardy-Littlewood method. It was initiated by the two authors in their famous paper "On the order of $\left|\zeta(\frac{1}{2}+it)\right|$". In this paper, they obtained

$$\zeta\left(\frac{1}{2}+it\right)=O_\epsilon\left(t^{\frac{9}{56}+\epsilon}\right).$$

Let us first introduce Hardy-Littlewood Circle method. It is most often used in additive number theory and additive combinatorics. Two famous examples are "3-term APs in Primes" and "Ternary Goldbach Theorem".

3-term Arithmetic Progressions in Primes

Theorem

Fix $A > 0$. For $x \geq 2$, we have

$$\Big|\{(p_1, p_2, p_3) : p_i \leq x \text{ and prime}, p_1 + p_3 = 2p_2\}\Big| = \frac{C_1 x^2}{(\log x)^3} + O_A\Big(\frac{x^2}{(\log x)^{A+3}}\Big),$$

where

$$C_1 = \prod_{p \geq 3}\Big(1 - \frac{1}{(p-1)^2}\Big).$$

Analytic expression:
Define $f(\alpha) = \sum_{n \leq x} 1_{\mathbb{P}}(n) e(n\alpha)$, where $\mathbb{P}$ is the set of primes, then

$$\Big|\{(p_1, p_2, p_3) : p_i \leq x \text{ and prime}, p_1 + p_3 = 2p_2\}\Big| = \int_0^1 f(\alpha)^2 f(-2\alpha) d\alpha,$$

by the simple fact that $\int_0^1 e(n\alpha) d\alpha = 1$ if $n = 0$ and $= 1$ otherwise.

Ternary Goldbach Theorem

Theorem

Fix $A > 0$. For $N \in \mathbb{N}$ sufficiently large, we have

$$\Big|\{(p_1,p_2,p_3) : p_i \text{ prime}, p_1+p_2+p_3 = N\}\Big| = \frac{G_N x^2}{(\log x)^3} + O_A\Big(\frac{x^2}{(\log x)^{A+3}}\Big),$$

where

$$G_N = \prod_{p|N}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{p\nmid N}\Big(1+\frac{1}{(p-1)^3}\Big).$$

Analytic expression:
Define $g(\alpha) = \sum_{n\le N} 1_{\mathbb{P}}(n)e(n\alpha)$, then

$$\Big|\{(p_1,p_2,p_3) : p_i \text{ prime}, p_1+p_2+p_3 = N\}\Big| = \int_0^1 g(\alpha)^3 e(-N\alpha)d\alpha.$$

Major and Minor Arcs

- By Dirichlet's Approximation, each α is close to some reduced fraction $\frac{a}{q}$. When we talk about closeness, we usually take $\frac{1}{qx}, \frac{1}{qN}$-neighborhood of such fractions. Thus we have cut $\mathbb{T}$ into many arcs described by $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{qx}, \frac{1}{qN}$.
- When q is small, due to the "uniform" distribution of primes mod q, $f(\alpha), g(\alpha)$ is large in absolute value when $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{qx}, \frac{1}{qN}$, so we define them to be major arcs.
- The rest are called minor arcs. $f(\alpha), g(\alpha)$ are small in absolute value.
- In the above expressions, we have $f^3 = f^2 \cdot f$. When we compute L^2 norms, we have square root cancellation. Combining L^2 estimates with L^∞ bounds on minor arcs, we can show that the contributions on minor arcs are relatively small.
- At last, we explicitly compute the contributions from major arcs. The trick is to first work with fractions $\alpha = \frac{a}{q}$, and then to show that the functions f, g do not change much in the small neighborhood of these fractions.

von Mongoldt function

- In practice, we replace the indictor function $1_{\mathbb{P}}$ by von Mongoldt function Λ, where

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^b; \\ 0, & \text{otherwise.} \end{cases}$$

- Due to the scarcity of prime powers p^b, $b \geq 2$, we have

$$1_{\mathbb{P}}(n) \approx \frac{\Lambda(n)}{\log n}.$$

- We use Vaughan's Identity to treat the exponential sum $\sum_{n \leq x} \Lambda(n) e(n\alpha)$.

Discrete Circle Method

- In the Circle and Divisor Problems, we encounter the following exponential sum:

$$S = \sum_{h \sim H} \sum_{m \sim M} e\Big(\frac{hT}{m}\Big),$$

where $H = M^{\beta}$ for $\frac{1}{3} \le \beta \le \frac{1}{2}$, and $T = M^2$.

- Here h appears as a linear term, so we focus on the sum over m first. Our best knowledge of estimating an exponential sum

$$\sum_{m \sim M} e(F(m))$$

on $\mathbb{Z}$ is when F is a polynomial. So we use Taylor expansion to replace the function $\frac{T}{m}$ by quadratic polynomials.

- We dissect the interval $[M, 2M]$ evenly into intervals I_j of length N. On each interval I_j, we fix some $m_j \in \mathbb{N}$ to be chosen later as the "center" of Taylor expansion. Then

$$\frac{hT}{m_j + n} = h\Big(\frac{T}{m_j} - \frac{T}{m_j^2}n + \frac{T}{m_j^3}n^2\Big) + O\Big(\frac{HTN^3}{M^4}\Big)$$
$$= \frac{hT}{m_j} - \frac{hT}{m_j^2}n + \frac{hT}{m_j^3}n^2 + O(1).$$

- The original sum becomes

$$S = \sum_{I_j} \sum_{h \sim H} e\Big(\frac{hT}{m_j}\Big) \sum_{n, n+m_j \in I_j} e\Big(-\frac{hT}{m_j^2}n + \frac{hT}{m_j^3}n^2\Big).$$

- We notice that the linear term $-\frac{T}{m_j^2} \sim 1$, so we approximate it by rationals $\frac{a}{q}$. Observe that now each interval I_j corresponds to a unique $\frac{a}{q}$, that means the sum over I_j can be rewritten as sum over $\frac{a}{q}$. In the end, we will look for the cancellation between different arcs I_j, and we exploit the information of rational approximation of linear coefficient.

Rational Approximation

- The sum over n becomes

$$\sum_{|n|\le N} e\Big(\frac{ah}{q}n + \frac{hT}{m_j^3}n^2\Big)$$

and we apply Poisson summation to it.

Lemma

Let f be a continuous function compactly supported on $\mathbb{R}$ and let c, d be integers, $c \ge 1$. We then have

$$\sum_{n\equiv d(\mathrm{mod}\ c)} f(n) = \frac{1}{c}\sum_k e\Big(-\frac{dk}{c}\Big)\hat{f}\Big(\frac{k}{c}\Big).$$

- After that, we use stationary phase approximation to replace each integral by an exponential function. Thus we have a new exponential sum and h appears in the denominator of the new phase function. Now we exchange the sum, and apply Poisson summation to h. The power of two Poisson summations is that we replace long sums by a short sums.

Double Large Sieve Inequality

- After two steps Poisson summation, we have the following form:

$$S = V \sum_{\frac{a}{q}} \sum_{l \sim L} \sum_{k \sim K} e\Big(\vec{x}_{k,l} \cdot \vec{y}_{\frac{a}{q}}\Big),$$

where $L \ll H$, $K \ll N$, and

$$\vec{x}_{k,l} = \Big(l, kl, l\sqrt{k}, \frac{l}{\sqrt{k}}\Big).$$

Theorem (Double large sieve inequality)

Suppose that we have two sets of vectors $\mathcal{X}, \mathcal{Y}$, where every $\vec{x} = (x_1, \dots, x_4) \in \mathcal{X}$ satisfies

$$|x_i| \le X_i, \ \ \forall i = 1, \dots, 4, \tag{5}$$

and every $\vec{y} = (y_1, \dots, y_4) \in \mathcal{Y}$ satisfies

$$|y_j| \le Y_j, \ \ \forall y = 1, \dots, 4. \tag{6}$$

Moreover, we require that $X_i Y_i \ge 1$ for every i.

Double Large Sieve Inequality continued

Theorem (Double large sieve inequality continued)

Then

$$\Big|\sum_{\vec{x}\in\mathcal{X}}\sum_{\vec{y}\in\mathcal{Y}} e\Big(\vec{x}\cdot\vec{y}\Big)\Big|$$

$$\lesssim\Big(\prod_{i=1}^{n} X_iY_i\Big)^{\frac{1}{p}}|\mathcal{Y}|^{1-\frac{2}{p}}\Big\|\sum_{\vec{x}\in\mathcal{X}} e\Big(\vec{x}\cdot\vec{t}\Big)\Big\|_{L^p_{\#}(D)}\Big(\sum_{\vec{y}_1,\vec{y}_2\in\mathcal{Y}} 1_{\vec{y}_1-\vec{y}_2\in 2B}\Big)^{\frac{1}{p}},$$

where $p\geq 2$, *and*

$$B=\prod_{i=1}^{n}\Big[-\frac{1}{10X_i},\frac{1}{10X_i}\Big],$$

$$D=\prod_{j=1}^{n}[-Y_j,Y_j].$$

Second Spacing Problem

The second spacing problem seeks for the number of rational pairs $\left(\frac{a}{q}, \frac{a_1}{q_1}\right)$ s.t. $\vec{y}_{\frac{a}{q}} - \vec{y}_{\frac{a_1}{q_1}} \in 2B$, which is equivalent to

$$\left\| \frac{\overline{a}}{q} - \frac{\overline{a_1}}{q_1} \right\| \leq \Delta_1, \quad \text{where } \Delta_1 \ll 1, \tag{7}$$

$$\left\| \frac{\overline{a}c}{q} - \frac{\overline{a_1}c_1}{q_1} \right\| \leq \Delta_2, \tag{8}$$

$$\left| \frac{\mu_1 q_1^3}{\mu q^3} - 1 \right| \leq \Delta_3, \tag{9}$$

$$|\kappa - \kappa_1| \leq \Delta_4, \tag{10}$$

We resort to Huxley's result for this part. His novel observation is that this problem is related to finding the distribution of integers points near a C^3-curve. Improvement in this direction can be directly applied in estimating $\left|\zeta(\frac{1}{2} + it)\right|$.

First Spacing Problem I

- Relations between variables:

$$KL \geq \frac{1}{\eta} \geq K \gg L.$$

$$G_p^* = \Big\| \sum_{k \sim K} \sum_{l \sim L} a_{kl} e(lx_1 + klx_2 + l\sqrt{k}x_3 + \frac{l}{\sqrt{k}}x_4) \Big\|_{L^p_{\#}\left(B(K,1,\frac{1}{\eta L\sqrt{K}},\frac{1}{\eta L\sqrt{K}})\right)},$$

where $|a_{kl}| \leq 1$, and $\#$ means we normalize the measure space. Let ψ be a smooth bump function supported on $[-1,1]$,

$$\frac{1}{T}\int \psi\Big(\frac{t}{T}\Big) e(xt)dt = \widehat{\psi}(Tx) \approx \begin{cases} 1, & \text{if } |x| \lesssim \frac{1}{T}; \\ 0, & \text{if } |x| \gg \frac{1}{T}. \end{cases}$$

If we let $p = 2$, we realize that the last condition

$$\Big| \frac{l}{\sqrt{k}} - \frac{l_1}{\sqrt{k_1}} \Big| \lesssim \eta L \sqrt{K}$$

is a weak condition, so we ignore it.

First Spacing Problem II

- We aim to estimate

$$G_p = \Big\| \sum_{k\sim K} \sum_{l\sim L} a_{kl} e(lx_1 + klx_2 + l\sqrt{k}x_3) \Big\|_{L^p_{\#}\left(B(K,1,\frac{1}{\eta L\sqrt{K}})\right)}.$$

- If we let $p = 2$, then it is easy to derive that $G_2 = (KL)^{\frac{1}{2}}$. In general, if $p = 2n$ is an even integer, and we let $a_{kl} = 1$, then G_p^p is equal to the number of integer solutions of the following system:

$$l_1 + \cdots + l_n = l_{n+1} + \cdots + l_{2n}, \tag{11}$$

$$k_1 l_1 + \cdots + k_n l_n = k_{n+1} l_{n+1} + \cdots + k_{2n} l_{2n}, \tag{12}$$

$$l_1\sqrt{k_1} + \cdots + l_n\sqrt{k_n} = l_{n+1}\sqrt{k_{n+1}} + \cdots + l_{2n}\sqrt{k_{2n}} + O(\eta L\sqrt{K}), \tag{13}$$

$$k_i \sim K, l_i \sim L, \quad \forall i = 1, \ldots, 2n. \tag{14}$$

First Spacing Problem III

- There are many trivial solutions to the above system. If we set $k_{i+n} = k_i$ and $l_{i+n} = l_i$ for all $i = 1, ..., n$, the system always holds, no matter what values $k_1, \ldots, k_n, l_1, \ldots, l_n$ we take. So the number of solutions is $\gtrsim K^n L^n$, which implies

$$G_p \gtrsim K^{\frac{1}{2}} L^{\frac{1}{2}}, \tag{15}$$

when $p = 2n \geq 2$. This is true for any real $p \geq 2$. By computation, we realize that square root cancellation is only possible when $p \leq p_0$ where p_0 is a number determined by the relations between η, K, L and it is a little bit > 4. Our first job is obtain square root cancellation at $p = 4$.

$$l_1 + l_2 = l_3 + l_4, \tag{16}$$

$$k_1 l_1 + k_2 l_2 = k_3 l_3 + k_4 l_4, \tag{17}$$

$$l_1\sqrt{k_1} + l_2\sqrt{k_2} = l_3\sqrt{k_3} + l_4\sqrt{k_4} + O(\eta L\sqrt{K}), \tag{18}$$

$$k_i \sim K, l_i \sim L, \quad \forall i = 1, ..., 4. \tag{19}$$

First Spacing Problem IV

Definition (The truncated cone and its neighborhood)

Let

$$\mathcal{C} = \{(\xi_1, \xi_2, \xi_3) : \xi_1^2 + \xi_2^2 = \xi_3^2, \quad \xi_3 \sim 1\} \tag{20}$$

be the truncated cone, and let $\mathcal{N}_\eta(\mathcal{C})$ be a η-neighborhood of $\mathcal{C}$ in $\mathbb{R}^3$.

- $$\begin{cases} \xi_1 = \frac{l}{L}\frac{\sqrt{k}}{\sqrt{K}} \sim 1, \\ \xi_2 = \frac{l}{L}\frac{k/K - 1}{2}, \\ \xi_3 = \frac{l}{L}\frac{k/K + 1}{2} \sim 1. \end{cases} \tag{21}$$

  we find that

$$G_p = \left\| \sum_{\vec{\xi} \in \Gamma} a_{\vec{\xi}}\, e\left(\xi_1 y_1 + \xi_2 y_2 + \sqrt{\xi_1^2 + \xi_2^2}\, y_3\right) \right\|_{L^p_{\#}\left(B(\frac{1}{\eta}, KL, KL)\right)}. \tag{22}$$

Cone Decoupling I

At this step, we apply the small cap decoupling theorem of Guth and Maldague to (22). Before stating their result, we introduce some concepts first. A generic plate σ of dimensions $\eta^{\beta_2} \times \eta^{\beta_1} \times \eta$ is a rectangular box in a η-neighborhood of $\mathcal{C}$ such that η^{β_2} is the length in the null direction, η^{β_1} is the length in the circular direction, and η is the thickness of the plate. $\mathcal{N}_\eta(\mathcal{C})$ can be covered by essentially pairwise disjoint generic plates σ of dimensions $\eta^{\beta_2} \times \eta^{\beta_1} \times \eta$, where $\beta_2 \in [0, 1]$, $\beta_1 \in [\frac{1}{2}, 1]$. By the essential disjointness, we mean that those plates may have finite overlaps but can be divided into finitely many sets, each of which contains disjoint plates. Thus we can view the collection of those essentially disjoint generic plates as a partition of $\mathcal{N}_\eta(\mathcal{C})$. Given a Schwartz function f, we define f_σ by setting its Fourier transform

$$\widehat{f_\sigma} = \widehat{f} \cdot \chi_\sigma,$$

where χ_σ is the characteristic function of σ. We now are ready to state Guth and Maldague's theorem.

Cone Decoupling II

Theorem (Small cap decoupling, Guth and Maldague, 2022)

Let $\beta_1 \in [\frac{1}{2}, 1]$ *and* $\beta_2 \in [0, 1]$. *For* $p \geq 2$ *and any Schwartz function* $f : \mathbb{R}^3 \to \mathbb{C}$ *with Fourier transform supported in* $\mathcal{N}_\eta(\mathcal{C})$, *we have*

$$\int_{\mathbb{R}^3} |f|^p \lesssim_\epsilon \eta^{-\epsilon} D^p_{\beta_1, \beta_2, p} \sum_\sigma \| f_\sigma \|^p_{L^p(\mathbb{R}^3)}, \tag{23}$$

where the decoupling constant $D_{\beta_1, \beta_2, p}$ *is given by*

$$D_{\beta_1, \beta_2, p} = \eta^{-(\beta_1+\beta_2)(\frac{1}{2}-\frac{1}{p})} + \eta^{-(\beta_1+\beta_2)(1-\frac{2}{p})+\frac{1}{p}} + \eta^{-(\beta_1+\beta_2-\frac{1}{2})(1-\frac{2}{p})}.$$

This is essentially sharp since we have examples in each extremal case.

Application of small cap decoupling

Applying the theorem to the right side of (22), we get the following lemma immediately.

Lemma

Let $\beta_1 \in [\frac{1}{2}, 1]$ *and* $\beta_2 \in [0, 1]$. *For* $p \geq 4$,

$$G_p \lesssim_\epsilon \eta^{-\epsilon} D_{\beta_1,\beta_2,p} \Big(\sum_\sigma \Big\| \sum_{\vec{\xi} \in \sigma} F_{\vec{\xi}}(x_1, x_2, x_3) \Big\|^p_{L^p_\#\left(B(\frac{1}{\eta}, KL, KL)\right)} \Big)^{\frac{1}{p}}, \tag{24}$$

where

$$F_{\vec{\xi}}(x_1, x_2, x_3) = a_{\vec{\xi}}\, e\Big(\xi_1 x_1 + \xi_2 x_2 + \sqrt{\xi_1^2 + \xi_2^2}\, x_3\Big).$$

Define

$$E_p(\beta_1, \beta_2) := \Big(\sum_\sigma \Big\| \sum_{(k,l) \in \mathcal{R}_\sigma} a_{kl} e\Big(l x_1 + k l x_2 + l\sqrt{k} x_3\Big) \Big\|^p_{L^p_\#\left(B(1,1,\frac{1}{\eta L \sqrt{K}})\right)} \Big)^{\frac{1}{p}}. \tag{25}$$

First Spacing Problem V

- A plate σ corresponds to a small rectangle in the big box $\{(k,l) : k \sim K, l \sim L\}$.
- When $p = 4$, if we express the norm in terms of algebraic equations, then we have

$$l_1 + l_2 = l_3 + l_4, \tag{26}$$

$$k_1 l_1 + k_2 l_2 = k_3 l_3 + k_4 l_4, \tag{27}$$

$$l_1\sqrt{k_1} + l_2\sqrt{k_2} = l_3\sqrt{k_3} + l_4\sqrt{k_4} + O(\eta L\sqrt{K}), \tag{28}$$

$$k_i \sim K, l_i \sim L, \quad \forall i = 1, ..., 4, \tag{29}$$

$$\text{diam}(k_1, \cdots, k_4) \lesssim \eta^{\beta_1} K, \tag{30}$$

$$\text{diam}(l_1, \cdots, l_4) \lesssim \eta^{\beta_2} L. \tag{31}$$

Here $\text{diam}(k_1, \ldots, k_4) \lesssim \eta^{\beta_1} K$ means $|k_i - k_j| \lesssim \eta^{\beta_1} K$ for any $1 \le i, j \le 4$. We call the last two conditions "localization conditions". With them, we can drop condition (28) and still get essentially sharp estimate.

First Spacing Problem VI

Lemma

$$E_4(\beta_1, \beta_2) \lesssim_\epsilon K^{\frac{1}{4}+\epsilon} L^{\frac{1}{4}} \left(\eta^{2(\beta_1+\beta_2)} K^2 L + \eta^{2\beta_1} K^2 + \eta^{2\beta_2} L^2 \right)^{\frac{1}{4}}. \tag{32}$$

Proof.

The trick we repeatedly use is that if we have an equation

$$ab = cd,$$

where $a, b, c, d \in \mathbb{Z}$ and $abcd \neq 0$, then once we fix a, b, then c, d are essentially determined. This is because $d(n) \lesssim_\epsilon n^\epsilon$. □

We plug (11) back into (24) and optimize our choice of β_1, β_2 to derive that

$$G_4 \lesssim_\epsilon K^{\frac{1}{2}+\epsilon} L^{\frac{1}{2}}.$$

This is essentially sharp. This result was proven by Huxley in his book, with a relatively complicated method. Then we turn to look at $p \geq 4$.

The Case When $p \geq 4$

- Now p is not an even integer any more, we cannot relate the norm with number of solutions of some Diophantine system. It is really hard to derive essentially sharp estimate for a non-even p, but we can still get pretty good one.
- The trick is to write $|f|^p = |f|^4 \cdot |f|^{p-4}$. Since $p-4$ is small, we do not lose much.
- Each σ determines a small rectangle in the big box $\{(k,l) : k \sim K, l \sim L\}$, and we simply count the number of pairs (k,l) s.t. $\vec{\xi} \in \sigma$, which is

$$\lesssim \eta^{\beta_1+\beta_2} KL.$$

Estimation of E_p I

Lemma

For $p \geq 4$,

$$E_p(\beta_1, \beta_2) \lesssim (\eta^{\beta_1+\beta_2} KL)^{1-\frac{4}{p}} E_4(\beta_1, \beta_2)^{\frac{4}{p}}. \tag{33}$$

Proof.

For each σ,

$$\Big\| \sum_{(k,l)\in\mathcal{R}_\sigma} a_{kl} e(lx_1 + klx_2 + l\sqrt{k}x_3) \Big\|_{L^\infty} \lesssim \eta^{\beta_1+\beta_2} KL.$$

Accordingly,

$$\begin{aligned} &\Big\| \sum_{(k,l)\in\mathcal{R}_\sigma} a_{kl} e\Big(lx_1 + klx_2 + l\sqrt{k}x_3\Big) \Big\|^p_{L^p_\#\left(B(1,1,\frac{1}{\eta L\sqrt{K}})\right)} \\ \leq &\Big\| \sum_{(k,l)\in\mathcal{R}_\sigma} a_{kl} e(lx_1 + klx_2 + l\sqrt{k}x_3) \Big\|^{p-4}_{L^\infty} \end{aligned} \tag{34}$$

□

Estimation of E_p II

Proof.

$$\times \Big\| \sum_{(k,l)\in\mathcal{R}_\sigma} a_{kl} e\Big(lx_1 + klx_2 + l\sqrt{k}x_3\Big)\Big\|^4_{L^4_\#\left(B(1,1,\frac{1}{\eta L\sqrt{K}})\right)}$$

$$\lesssim (\eta^{\beta_1+\beta_2} KL)^{p-4} \tag{35}$$

$$\times \Big\| \sum_{(k,l)\in\mathcal{R}_\sigma} a_{kl} e\Big(lx_1 + klx_2 + l\sqrt{k}x_3\Big)\Big\|^4_{L^4_\#\left(B(1,1,\frac{1}{\eta L\sqrt{K}})\right)}.$$

Thence, by comparing the definition of $E_p(\beta_1, \beta_2)$, (25) and (34), we find that

$$E_p(\beta_1, \beta_2)^p \lesssim (\eta^{\beta_1+\beta_2} KL)^{p-4} E_4(\beta_1, \beta_2)^4. \tag{36}$$

The inequality (33) follows by taking the $\frac{1}{p}$-th power on both sides of (36). □

G_p

- At last, we plug (33) into (24) and optimize with respect to β_1, β_2 to attain the following result:

$$G_{4.29} \lesssim_\epsilon K^\epsilon (KL)^{0.505}.$$

- Combining the results in the first spacing problem and second spacing problem, we finally reach an improvement for the Circle Problem and the Divisor Problem.

Thank you!

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