On the isomorphism class of q-Gaussian C*-algebras for infinite variables

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Outline

Operator algebras

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In general, if you have a state on a C^*-algebra then, using the GNS construction, you can build a von Neumann algebra – many constructions exist on both levels.

Free Gaussian algebras

Free group factors

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For appropriate operators $(a_i)_{i\in[n]}$ satisfying $a_i a_j^* = \delta_{ij}\mathbb{1}$ the von Neumann algebra generated by $(a_i + a_i^*)_{i\in[n]}$ will be isomorphic to $\mathrm{L}(\mathbb{F}_n)$.

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The corresponding C^*-algebras are not the same. How to construct such operators?

Fock spaces

Full Fock space

Let H be a Hilbert space. The Hilbert space $\mathcal{F}(\mathsf{H}) := \bigoplus_{n=0}^{\infty} \mathsf{H}^{\otimes n}$ is called the (full) Fock space, where $\mathsf{H}^{\otimes 0} := \mathbb{C}\Omega$.

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Creation/annihilation operators

For $v \in \mathrm{H}$ we can define the creation operator $a^*(v)$ and the annihilation operator $a(v)$ via $a^*(v)(w_1 \otimes \cdots \otimes w_n) := v \otimes w_1 \otimes \cdots \otimes w_n$ and $a(v)(w_1 \otimes \cdots \otimes w_n) := \langle v, w_1 \rangle w_2 \otimes \cdots \otimes w_n$ (with $a(v)\Omega := 0$).

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Field operators

If H is the complexification of a real Hilbert space $\mathrm{H}_{\mathbb{R}}$ then we define $s(v) := a^*(v) + a(v)$ for $v \in \mathrm{H}_{\mathbb{R}}$.

Commutation relations

The creation and annihilation operators satisfy the relation $a(v)a^*(w) = \langle v, w \rangle \mathbb{1}$.

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Canonical (anti)commutation relations

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Could it be that for each $q \in [-1, 1]$ we can find operators satisfying $a(v)a^*(w) - qa^*(w)a(v) = \langle v, w\rangle \mathbb{1}$?

q-Gaussian algebras

Bożejko-Speicher

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Let $\mathrm{H}_{\mathbb{R}}$ be a real Hilbert space and let H be its complexification. Then the von Neumann algebra generated by $(a(v) + a^*(v))_{v \in \mathrm{H}_{\mathbb{R}}}$ is denoted by $\Gamma_q(\mathrm{H}_{\mathbb{R}})$ and called the q-**Gaussian algebra**.

Some more details

Formulas

The creation operator stays the same: $a^*(v)(w_1 \otimes \cdots \otimes w_n) = v \otimes w_1 \otimes \cdots \otimes w_n)$ and the commutation relations force $a(v)(w_1 \otimes \cdots \otimes w_n) = \sum_{i=1}^{n} q^{i-1} \langle v, w_i \rangle w_1 \otimes \dots \widehat{w_i} \cdots \otimes w_n$.

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The trace

The formula $\tau(x) := \langle \Omega, x\Omega \rangle$ defines a trace on $\Gamma_q(\mathrm{H}_{\mathbb{R}})$.

Are they all the same?

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Free transport

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Nothing is yet known in the case $\dim(\mathsf{H}_{\mathbb{R}}) = \infty$ – that's the plan for today.

Approximation properties

Approximately finite dimensional

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For vNas AFD = injective! (Connes)

Aproximation properties of free groups

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Haagerup showed that $C^*_\lambda(\mathbb{F}_n)$ has the metric approximation property, which is a strong form of having finite dimensional approximations.

Akemann and Ostrand discovered the surprising fact that the left-right action $C^*_\lambda(\mathbb{F}_n) \otimes C^*_\rho(\mathbb{F}_n) \mapsto \mathrm{B}(\ell^2(\mathbb{F}_n))$ becomes min-continuous if we take the quotient onto the Calkin algebra.

The Akemann-Ostrand property

A vNA M has the **Akemann-Ostrand property** if there exist weakly dense C^*-subalgebras $A \subset \mathrm{M}$ and $B \subset \mathrm{M}^{\mathrm{op}}$ such that the left-right action $A \otimes B \mapsto \mathrm{B}(L^2(\mathrm{M}))/\,\mathrm{K}(L^2(\mathrm{M}))$ is min-continuous.

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Proof strategy

Embed A into a nuclear C^*-algebra $\widetilde{A}$ such that $[\widetilde{A}, B] \subset K$.

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Ozawa proved that if M has the AO property (with *A* locally reflexive) then M is **solid**, i.e. the commutant of any diffuse subalgebra is injective.

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Prime factors

A II_1-factor M is called **prime** if it cannot be decomposed as a non-trivial tensor product. The unique injective II_1-factor is not prime. Non-injective solid factors are prime (e.g. the free group factors).

Free vs q-Gaussian algebras

AO for free Gaussian algebras

The C^*-algebra generated by field operators embeds into the one generated by creation operators, which is nuclear (Toeplitz-Cuntz). The commutator of the left creation operator with the right annihilation operator is of finite rank, so all commutators are compact.

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In this case the commutator is equal to the operator q^N (multiplication by q^n on the n-th tensor power in the Fock space), which is not compact if $\dim(\mathrm{H}_{\mathbb{R}}) = \infty$.

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$\Gamma_q(\ell^2)$ might not have the Akemann-Ostrand property but it is hard to check it for all possible C^*-subalgebras A and B.

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Theorem (Borst, Caspers, Klisse, W.)

For $q \neq 0$ the q-Gaussian C^*-algebra $A_q(\ell^2)$ is not isomorphic to the free Gaussian C^*-algebra.

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Theorem (Borst, Caspers, Klisse, W.)

For $q \neq 0$ the q-Gaussian C^*-algebra $A_q(\ell^2)$ is not isomorphic to the free Gaussian C^*-algebra.

Proof.

One can show that these algebras have a unique trace, so they remember the von Neumann algebras, therefore an isomorphism would transfer the AO property. □

Crucial lemma

Lemma

$B \subset A \subset \mathrm{M}$, infinitely many mutually orthogonal subspaces $H_i \subset L^2(\mathrm{M})$, left and right invariant. Suppose that, for some $\delta > 0$ and $b_j, c_j \in B$ we have

$$\inf_i \| \sum_j b_j c_j^{\mathrm{op}} \|_{\mathrm{B}(H_i)} \geqslant (1+\delta) \| \sum_j b_j \otimes c_j^{\mathrm{op}} \|_{B \otimes_{\min} B^{\mathrm{op}}}.$$

Then M does not have AO with respect to A.

The case of q-Gaussians

Write $\ell^2 = \mathbb{R}^d \oplus \mathsf{H}_\mathbb{R}$ and take $\mathsf{M} := \Gamma_q(\ell^2)$, $A := A_q(\ell^2)$, $B := A_q(\mathbb{R}^d)$, and $H_i := \overline{Bf_iB}$, where $(f_i)_{i\in\mathbb{N}}$ is an ONB of $\mathsf{H}_\mathbb{R}$.

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If k is large enough then for an orthonormal basis (ξ_j) of $(\mathbb{R}^d)^{\otimes k}$, we can choose $b_j := W(\xi_j)^*$ and $c_j := W(\xi_j)$ and use Nou's Khintchine inequality.

Thank you for your attention!