Noncommutative formal geometry of a contractive quantum plane

Slide presentation

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Introduction

Recall that by a geometric object we mean a ringed space $(X, \mathcal{O}_X)$ of a topological space X and the structure sheaf $\mathcal{O}_X$ of local rings on X. To find out the geometry $(X, \mathcal{O}_X)$ of a noncommutative (associative) algebra A is a challenging task of noncommutative geometry. In this case, the global sections $\Gamma(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ should represent (or stay closer to) the original algebra A, that is

$$\begin{array}{ccc} A & \Longrightarrow & (X, \mathcal{O}_X) \\ \| & & \| \\ \Gamma(X, \mathcal{O}_X) & \Longleftarrow & (X, \mathcal{O}_X) \end{array}$$

Introduction

In algebraic geometry that relation defines an anti-equivalence between

$$\{\text{commutative rings}\} \Leftrightarrow \{\text{affine schemes}\}.$$

In the complex analytic geometry we have an anti-equivalence between

$$\{\text{locally compact topological spaces}\} \Leftrightarrow \{\text{commutative } C^*\text{-algebras}\}.$$

Noncommutative complex analytic geometry deals with the Banach space representations of a noncommutative complex algebra. A geometric space $(X, \mathcal{O}_X)$ of a finitely generated noncommutative complex algebra A consists of the spectrum X (analytic space) of A to be the set of all irreducible Banach space representations, and a noncommutative Fréchet $\widehat{\otimes}$-algebra (pre)sheaf $\mathcal{O}_X$ so that $\Gamma(X, \mathcal{O}_X)$ represents (or stay closer) the noncommutative algebra of all entire functions in the generators of A.

Introduction

The noncommutative algebra of all entire functions in the generators of A is well known in the literature as the Arens-Michael envelope of A (introduced by A. Ya. Helemskii). It turns out that the Arens-Michael envelope of a complex algebra A is the completion of A with respect to the family of all multiplicative seminorms defined on A.

If A is the algebra of all polynomial functions on a complex affine algebraic variety X, then its Arens-Michael envelope is the algebra of holomorphic functions on X (A. Yu. Pirkovskii, Trans. Moscow Math. Soc. 2008). In particular, the Arens-Michael envelope of the algebra $A = \mathbb{C}[x_1, \ldots, x_n]$ of all complex polynomials in n-variables is the Fréchet algebra $\mathcal{O}(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$.

Introduction

In the case of a noncommutative polynomial algebra A its Arens-Michael envelope represents the algebra of all entire functions in noncommuting variables generating A. Our main focus will be on the the quantum plane. The quantum plane (or just q-plane) is the free associative algebra

$$\mathfrak{A}_q = \mathbb{C}\langle x, y\rangle / \left(xy - q^{-1}yx\right), \quad q \in \mathbb{C}\setminus\{0,1\}$$

generated by x and y modulo $xy = q^{-1}yx$. The Arens-Michael envelope of $\mathfrak{A}_q$ is denoted by $\mathcal{O}_q\left(\mathbb{C}^2\right)$. If x and y are invertible additionally, then the algebra represents the quantum 2-torus. If $|q| \neq 1$, then we deal with the contractive quantum plane.

Introduction

The Arens-Michael envelope $\mathcal{O}_q\left(\mathbb{C}^2\right)$ representing the algebra of all noncommutative entire functions in x and y consists of the following absolutely convergent power series

$$\mathcal{O}_q\left(\mathbb{C}^2\right)=\left\{f=\sum_{i,k}a_{ik}x^iy^k:\|f\|_\rho=\sum_{i,k}|a_{ik}|\,\rho^{i+k}<\infty,\rho>0\right\}$$

if $|q|\leq 1$

(A. Yu. Pirkovskii 2008). The case of $|q|>1$ can be reduced to the case of $|q|<1$ by flipping the variables x and y, thereby whatever construction over the q-plane done for $|q|<1$ can be conveyed to the case of $|q|>1$ too.

Introduction

It turns out that if $|q| < 1$ then $\mathcal{O}_q\left(\mathbb{C}^2\right)$ is commutative modulo its Jacobson radical $\operatorname{Rad}\mathcal{O}_q\left(\mathbb{C}^2\right)$, that is, all irreducible Banach space representations (the spectrum X of $\mathfrak{A}_q$) are just continuous characters (trivial modules) and

$$\operatorname{Spec}\left(\mathcal{O}_q\left(\mathbb{C}^2\right)\right) = \mathbb{C}_{xy} = \mathbb{C}_x \cup \mathbb{C}_y,$$

where $\mathbb{C}_x = \mathbb{C} \times \{0\} \subseteq \mathbb{C}^2$, $\mathbb{C}_y = \{0\} \times \mathbb{C} \subseteq \mathbb{C}^2$, and we use the notation $\mathcal{O}_q\left(\mathbb{C}_{xy}\right)$ instead of $\mathcal{O}_q\left(\mathbb{C}^2\right)$. Moreover,

$$\mathcal{O}_q\left(\mathbb{C}_{xy}\right) / \operatorname{Rad}\mathcal{O}_q\left(\mathbb{C}_{xy}\right) = \mathcal{O}\left(\mathbb{C}_{xy}\right)$$

is the algebra of holomorphic functions on $\mathbb{C}_{xy}$.

The space $X = \operatorname{Spec} \mathcal{O}_q(\mathbb{C}_{xy})$ stands for the noncommutative "analytic" space of $\mathfrak{A}_q$, whose structure sheaf would consists of noncommutative Fréchet $\widehat{\otimes}$-algebras extending the algebra $\mathcal{O}_q(\mathbb{C}_{xy})$.

But there is a problem: the spectrum X is not uniquely defined by the algebra $\mathcal{O}_q(\mathbb{C}^2)$:

$$\begin{array}{ccc} \mathcal{F}_q(\mathbb{C}_{xy}) & \Longrightarrow & \operatorname{Spec} \mathcal{F}_q(\mathbb{C}_{xy}) \\ \uparrow & & \| \\ \mathcal{O}_q(\mathbb{C}_{xy}) & \Longrightarrow & X \end{array}$$

Introduction

So, to restore the geometry that stands for $\mathfrak{A}_q$, $|q| < 1$ one needs to pass to a certain formal completion of $\mathcal{O}_q(\mathbb{C}_{xy})$. It is important to define a Fréchet $\widehat{\otimes}$-algebra structure sheaf $\mathcal{F}_q$ on $\mathbb{C}_{xy}$ so that

$$X = \operatorname{Spec}\Gamma(X, \mathcal{F}_q)$$

and $\Gamma(X, \mathcal{F}_q)$ should be a Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-algebra (or bimodule). It turns out $\mathcal{O}_q(\mathbb{C}_{xy})$ is not suitable to have a self-satisfactory geometric construction but it is essential for the functional calculus problem.

The same phenomenon was detected in the case of the universal enveloping algebra

$$A = \mathcal{U}(\mathfrak{g})$$

of a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ whose Arens-Michael envelope $\mathcal{O}_{\mathfrak{g}}$ stands for the algebra of all noncommutative entire function in elements of $\mathfrak{g}$.

(1) Dosi, *Cohomology of Sheaves of Fréchet Algebras and Spectral Theory*, Funct. Anal. its Appl. (2005);

(2) Dosi, *Cartan-Slodkowski spectra, splitting elements and noncommutative spectral mapping theorems*, J. Funct. Anal., (2006);

(3) Dosi, *Taylor functional calculus for supernilpotent Lie algebra of operators*, J. Oper. Th. (2010).

The formal algebra stalk at zero

The space $\mathbb{C}[[x,y]]$ of all formal power series in variables x and y is a Fréchet space equipped with the direct product topology of $\prod_{i,k} \mathbb{C}x^i y^k$. If $f = \sum_{i,k} a_{ik} x^i y^k$ and $g = \sum_{i,k} b_{ik} x^i y^k$, then we put

$$f \cdot g = \sum_{m,n} \left(\sum_{s+t=m, i+j=n} a_{si} q^{it} b_{tj} \right) x^m y^n.$$

It defines an Arens-Michael-Fréchet $\widehat{\otimes}$-algebra structure on $\mathbb{C}[[x,y]]$, and

$$\mathfrak{A}_q \to \mathbb{C}[[x,y]]$$

is a (unital) algebra homomorphism. In particular, $\mathbb{C}[[x,y]]$ is a Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-algebra.

The formal stalk in x-direction

It can be treated as a formal stalk at zero - the intersection point of two complex lines $\mathbb{C}_x$ and $\mathbb{C}_y$. In the direction of the x-line $\mathbb{C}_x$ the algebra $\mathbb{C}[[x,y]]$ can be reconsidered as $\mathbb{C}[[x]][[y]]$. Every $h=\sum_{i,k} c_{ik}x^i y^k$ can be rewritten in the form $h=\sum_n h_n(x)\,y^n$ with $h_n(x)=\sum_i c_{in}x^i$. Then

$$f\cdot g=\sum_n\left(\sum_{i+j=n} f_i(x)\,g_j\left(q^i x\right)\right)y^n,$$

where $f_i(x)\,g_j\left(q^i x\right)$ is the multiplication in the algebra $\mathbb{C}[[x]]$, which is commutative.

The formal stalk in y-direction

In a similar way, $\mathbb{C}[[x,y]] = [[x]]\mathbb{C}[[y]]$ and every $h = \sum_{i,k} c_{ik} x^i y^k$ can be rewritten in the form $h = \sum_n x^m h_m(y)$ with $h_m(y) = \sum_i c_{mi} y^i$. Then

$$f \cdot g = \sum_m x^m \left(\sum_{s+t=m} f_s(q^t y) g_t(y) \right),$$

where $f_s(q^t y) g_t(y)$ is the multiplication in the algebra $\mathbb{C}[[y]]$, which is commutative.

Thus the formal q-multiplication in $\mathbb{C}[[x,y]]$ can be defined in two different ways by extending the multiplication of the q-plane $\mathfrak{A}_q$.

The formal stalk is local

The algebra $\mathbb{C}[[x,y]]$ has the continuous trivial character

$$(0,0):\mathbb{C}[[x,y]]\rightarrow\mathbb{C}$$

annihilating both variables x and y. The notation $(0,0)$ will be justified below as a point of the q-plane $\mathbb{C}_{xy}$. The algebra $\mathbb{C}[[x,y]]$ is local with its

$$\operatorname{Rad}\mathbb{C}[[x,y]]=\ker(0,0)$$

to be the closed two sided ideal generated by x and y. In particular,

$$\operatorname{Spec}(\mathbb{C}[[x,y]])=\{(0,0)\}.$$

The topology of the quantum plane

Fix $q \in \mathbb{C} \setminus \{0\}$, $|q| < 1$. A subset $S \subseteq \mathbb{C}$ is called a *q-spiraling set* if it contains the origin and $\{q^n x : n \in \mathbb{Z}_+\} \subseteq S$ for every $x \in S$. Thus S is a q-spiraling set iff $S_q = S$, where

$$S_q = \{0\} \cup \left(\cup_{n=1}^{\infty} q^n S\right)$$

is the *q-hull of* S. If $S = \{x\}$ is a singleton, then $\{x\}_q$ is a spiraling sequence which tends to zero including its limit point, that is,

$$\{x\}_q = \{q^n x : n \in \mathbb{Z}_+\} \cup \{0\}$$

is a compact set.

The topology of the quantum plane

A subset $U \subseteq \mathbb{C}$ is said to be *a q-open set* if it is an open subset of $\mathbb{C}$ in the standard topology, which is also a q-spiraling set. The whole plane $\mathbb{C}$ is q-open, and the empty set is assumed to be q-open set.

The family of all q-open subsets defines a new topology $\mathfrak{q}$ in $\mathbb{C}$, which is weaker than the original standard topology of the complex plane. Every open disk $B(0, r)$ centered at the origin is a q-open set. Thus the neighborhood filter base of the origin is the same in both $\mathfrak{q}$-topology and the standard topology.

The topology of the quantum plane

Notice that $\{0\}$ is a generic point of the topological space $(\mathbb{C}, \mathfrak{q})$ being dense in it. If $x \in \mathbb{C} \setminus \{0\}$ then its closure in $(\mathbb{C}, \mathfrak{q})$ is given by

$$\{x\}^{-\mathfrak{q}} = \left\{q^{-k}x : k \in \mathbb{Z}_+\right\}.$$

Thus $(\mathbb{C}, \mathfrak{q})$ satisfies the axiom T_0, and it turns out to be an irreducible topological space, which is not quasicompact.

If $K \subseteq \mathbb{C}$ is a compact subset then it is quasicompact in $(\mathbb{C}, \mathfrak{q})$, but not necessarily $\mathfrak{q}$-closed subset. All disks (open or closed) centered at the origin are quasicompact (nonclosed) subsets of $(\mathbb{C}, \mathfrak{q})$. They are all dense in $(\mathbb{C}, \mathfrak{q})$. Every closure $\{x\}^-$ of a point $x \in \mathbb{C}$ is not quasicompact. A nonempty subset $K \subseteq (\mathbb{C}, \mathfrak{q})$ is quasicompact iff so is its q-hull K_q. In this case, K is bounded automatically.

The standard sheaf

Let $\mathcal{O}$ be the standard Fréchet sheaf of stalks of the holomorphic functions on $\mathbb{C}$ and let $\mathrm{id} : \mathbb{C} \to (\mathbb{C}, \mathfrak{q})$ be the identity (continuous) mapping. Put

$$\mathcal{O}^{\mathfrak{q}} = \mathrm{id}_* \mathcal{O}$$

to be the direct image of $\mathcal{O}$ along the identity mapping. It is a Fréchet algebra sheaf on $(\mathbb{C}, \mathfrak{q})$. For every q-open set U and its quasicompact subset $K \subseteq U$ we define the related seminorm

$$\|f\|_K = \sup |f(K)|, \quad f \in \mathcal{O}(U)$$

on the algebra $\mathcal{O}^{\mathfrak{q}}(U)$. The family $\{\|\cdot\|_K\}$ of seminorms over all q-compact subsets $K \subseteq U$ (that is, $K = K_q$) defines the same original Fréchet topology of $\mathcal{O}(U)$, that is,

$$\mathcal{O}^{\mathfrak{q}}(U) = \mathcal{O}(U)$$

as the Fréchet algebras.

The standard sheaf

But $\mathcal{O}^{\mathfrak{q}}$ and $\mathcal{O}$ are different sheaves having quite different stalks. The stalks of the sheaves $\mathcal{O}^{\mathfrak{q}}$ and $\mathcal{O}$ at zero coincide, whereas

$$\mathcal{O}^{\mathfrak{q}}_{\lambda} = \mathcal{O}\left(\{\lambda\}_q\right) = \mathcal{O}_0 + \sum_{n\in\mathbb{Z}_+} \mathcal{O}_{q^n\lambda}$$

at every $\lambda \in \mathbb{C}\setminus\{0\}$. The algebra $\mathcal{O}^{\mathfrak{q}}_{\lambda}$ is not local for $\lambda \in \mathbb{C}\setminus\{0\}$. It has an ideal of those stalks $\langle U, f\rangle \in \mathcal{O}^{\mathfrak{q}}_{\lambda}$ with $f\left(\{\lambda\}_q\right) = \{0\}$.

The sheaf filtration

The sheaf $\mathcal{O}^{\mathfrak{q}}$ has the following filtration $\{\mathfrak{m}_d\}$ of closed ideal subsheaves. If $U \subseteq (\mathbb{C}, \mathfrak{q})$ is a q-open subset, then it contains the origin and we put

$$\mathfrak{m}_d(U) = \left\{ f(z) \in \mathcal{O}^{\mathfrak{q}}(U) : z^{-d} f(z) \in \mathcal{O}^{\mathfrak{q}}(U) \right\}$$

to be a closed ideal of $\mathcal{O}^{\mathfrak{q}}(U)$, where $d \in \mathbb{Z}_+$. Notice that $\mathfrak{m}_0 = \mathcal{O}^{\mathfrak{q}}$, and $\mathfrak{m}_d(U)$ consists of those $f(z) \in \mathcal{O}^{\mathfrak{q}}(U)$ such that

$$f(0) = f'(0) = \cdots = f^{(d-1)}(0) = 0.$$

The sheaf filtration

The ideal $\mathfrak{m}_d(U)$ is the principal ideal of $\mathcal{O}^{\mathfrak{q}}(U)$ generated by z^d, that is,

$$\mathfrak{m}_d(U) = z^d \mathcal{O}^{\mathfrak{q}}(U).$$

The linear mapping

$$\mathfrak{m}_d(U) \longrightarrow \mathcal{O}^{\mathfrak{q}}(U), \quad f(z) \mapsto z^{-d} f(z)$$

implements a topological isomorphism of the Fréchet spaces preserving the multiplication operator by z. Moreover,

$$\mathcal{O}^{\mathfrak{q}}(U) = \mathfrak{m}_d(U) \oplus \mathbb{C}1 \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^{d-1}$$

is a topological direct sum of the subalgebras $\mathfrak{m}_d(U)$ and (polynomial) $\mathbb{C}1 \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^{d-1}$.

The sheaf filtration

Thus $\mathfrak{m}_d(U)$ defines a new Fréchet $\mathcal{O}$-module sheaf on $(\mathbb{C}, \mathfrak{q})$, which is an isomorphic copy of $\mathcal{O}^{\mathfrak{q}}$. We use the notation $\mathcal{O}^{\mathfrak{q}}(d)$ for this Fréchet sheaf called the d-shift of $\mathcal{O}^{\mathfrak{q}}$. Thus

$$\mathcal{O}^{\mathfrak{q}} = \mathcal{O}^{\mathfrak{q}}(d) \oplus \mathbb{C}1 \oplus \mathbb{C}z \oplus \cdots \oplus \mathbb{C}z^{d-1}$$

is a direct sum of the Fréchet sheaves for every $d \in \mathbb{Z}_+$. In particular,

$$\mathcal{O}^{\mathfrak{q}}(0) = \mathcal{O}^{\mathfrak{q}}.$$

The topology of the q-plane

The spectrum $\mathbb{C}_{xy}$ being the union $\mathbb{C}_x \cup \mathbb{C}_y$ can be equipped with the final topology so that both embeddings

$$(\mathbb{C}_x, \mathfrak{q}) \hookrightarrow \mathbb{C}_{xy} \hookleftarrow (\mathbb{C}_y, \mathfrak{q})$$

are continuous, which is called the $\mathfrak{q}$-*topology of* $\mathbb{C}_{xy}$. The topology base in $\mathbb{C}_{xy}$ consists of all open subsets $U = U_x \cup U_y$ with q-open sets $U_x \subseteq \mathbb{C}_x$ and $U_y \subseteq \mathbb{C}_y$. In this case,

$$\mathbb{C}_{xy} = \mathbb{C}_x \cup \mathbb{C}_y$$

is the union of two irreducible components, whose intersection is a unique generic point.

The sheaf on the x-direction

Consider the Fréchet sheaf $\mathcal{O}^{\mathfrak{q}}$ and the constant Fréchet sheaf $\mathbb{C}[[y]]$ over the topological space $(\mathbb{C}_x, \mathfrak{q})$. Put

$$\mathcal{O}^{\mathfrak{q}}[[y]] = \mathcal{O}^{\mathfrak{q}} \widehat{\otimes} \mathbb{C}[[y]]$$

to be their projective tensor product. The space $\mathcal{O}^{\mathfrak{q}}[[y]](U_x)$ of all its sections over a q-open subset U_x is the Fréchet space $\mathcal{O}(U_x)[[y]]$ equipped with the defining family $\left\{\|\cdot\|_{K,m} : K \subseteq U_x, m \in \mathbb{Z}_+\right\}$ of seminorms, where

$$\|f\|_{K,m} = \sum_{n=0}^{m} \|f_n\|_K, \quad f \in \mathcal{O}(U_x)[[y]],$$

and $K \subseteq U_x$ is a compact subset.

The sheaf on the x-direction

It turns out that $\mathcal{O}^{\mathfrak{q}}[[y]]$ is a Fréchet $\widehat{\otimes}$-algebra sheaf equipped with the formal q-multiplication. Namely, if $f = \sum_n f_n(x) y^n$ and $g = \sum_n g_n(x) y^n$ are sections from $\mathcal{O}^{\mathfrak{q}}[[y]](U_x)$, then we put

$$f \cdot g = \sum_n \left(\sum_{i+j=n} f_i(x) g_j\left(q^i x\right) \right) y^n.$$

Notice that $\left\{q^i x : i \in \mathbb{Z}_+\right\} \cup \{0\} = \{x\}_q \subseteq U_x$ whenever $x \in U_x$, and $f_i(x) g_j\left(q^i x\right)$ is the multiplication from the commutative algebra $\mathcal{O}(U_x)$. Moreover,

$$\left\{ \|\cdot\|_{K,m} : K \subseteq U_x, \quad m \in \mathbb{Z}_+ \right\}$$

is a defining family of multiplicative seminorms of $\mathcal{O}^{\mathfrak{q}}(U_x)[[y]]$ whenever $K \subseteq U_x$ is running over all q-compact subsets and $m \in \mathbb{Z}_+$ (the Arens-Michael-Fréchet algebra)

The sheaf on the x-direction

The canonical mapping

$$\begin{aligned} I(U_x) &: \mathcal{O}_q(\mathbb{C}_{xy}) \to \mathcal{O}^{\mathfrak{q}}(U_x)[[y]], \\ f &= \sum_{i,k} a_{ik} x^i y^k \mapsto I(U_x)(f) = \sum_n \left(\sum_i a_{in} x^i \right) y^n \end{aligned}$$

is a continuous algebra homomorphism. In particular, $\mathcal{O}(U_x)[[y]]$ is a Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-algebra (or Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-bimodule) and

$$\mathrm{Spec}(\mathcal{O}^{\mathfrak{q}}(U_x)[[y]]) = U_x.$$

The sheaf on the y-direction

One can also consider the Fréchet $\widehat{\otimes}$-algebra sheaf $[[x]]\,\mathcal{O}^{\mathfrak{q}}$ over the topological space $(\mathbb{C}_y, \mathfrak{q})$ with the formal q-multiplication

$$f \cdot g = \sum_n x^n \left(\sum_{i+j=n} f_i\left(q^j y\right) g_j\left(y\right) \right) \in [[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right),$$

where $f = \sum_n x^n f_n\left(y\right)$, $g = \sum_n x^n g_n\left(y\right)$, The canonical mapping

$$\begin{aligned} r\left(U_x\right) &: \mathcal{O}_q\left(\mathbb{C}_{xy}\right) \rightarrow [[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right), \\ f &= \sum_{i,k} a_{ik} x^i y^k \mapsto r\left(U_y\right)\left(f\right) = \sum_n x^n \left(\sum_k a_{nk} y^k \right) \end{aligned}$$

is a continuous algebra homomorphism. So $[[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right)$ is a Fréchet $\mathcal{O}_q\left(\mathbb{C}_{xy}\right)$-algebra and

$$\operatorname{Spec}\left([[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right)\right) = U_y.$$

The sheaves on q-plane

Both sheaves $\mathcal{O}^{\mathfrak{q}}[[y]]$ and $[[x]]\mathcal{O}^{\mathfrak{q}}$ are identified with the related Fréchet $\widehat{\otimes}$-algebra sheaves on $\mathbb{C}_{xy}$ as the direct images along the canonical inclusions $\mathbb{C}_x \hookrightarrow \mathbb{C}_{xy}$ and $\mathbb{C}_y \hookrightarrow \mathbb{C}_{xy}$, respectively. Moreover, both Fréchet $\widehat{\otimes}$-algebras $\mathcal{O}_q(\mathbb{C}_{xy})$ and $\mathbb{C}[[x, y]]$ equipped with the formal q-multiplication are identified with the constant sheaves on $\mathbb{C}_{xy}$.
Let $U \subseteq \mathbb{C}_{xy}$ be a q-open subset. The following topological algebra decompositions

$$\begin{aligned} \mathcal{O}^{\mathfrak{q}}(U_x)[[y]] &= \mathcal{O}^{\mathfrak{q}}(U_x) \oplus \operatorname{Rad}\mathcal{O}^{\mathfrak{q}}(U_x)[[y]], \\ [[x]]\mathcal{O}^{\mathfrak{q}}(U_y) &= \mathcal{O}^{\mathfrak{q}}(U_y) \oplus \operatorname{Rad}[[y]]\mathcal{O}^{\mathfrak{q}}(U_y) \end{aligned}$$

hold. In this case, $\operatorname{Rad}\mathcal{O}^{\mathfrak{q}}(U_x)[[y]] = \prod_{n\in\mathbb{N}} \mathcal{O}^{\mathfrak{q}}(U_x)\, y^n$ and $\operatorname{Rad}[[x]]\mathcal{O}^{\mathfrak{q}}(U_y) = \prod_{n\in\mathbb{N}} x^n\mathcal{O}^{\mathfrak{q}}(U_y)$.

The sheaves on q-plane

Let $U = U_x \cup U_y \subseteq \mathbb{C}_{xy}$ be a q-open subset. The canonical maps

$$\begin{array}{ccc} \mathcal{O}^{\mathfrak{q}}\left(U_x\right)[[y]] & & [[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right) \\ & {}_{s(U_x)}\searrow \qquad \swarrow_{t(U_y)} & \\ & \mathbb{C}\left[[x,y]\right] & \end{array}$$

$$s\left(U_x\right) : \sum_n f_n\left(x\right) y^n \mapsto \sum_{s,i} \frac{f_i^{(s)}\left(0\right)}{s!} x^s y^i,$$

$$t\left(U_y\right) : \sum_n x^n g_n\left(y\right) \mapsto \sum_{t,j} \frac{g_t^{(j)}\left(0\right)}{j!} x^t y^j,$$

are continuous algebra homomorphisms. They are formal evaluations of the stalks at zero from both directions $\mathcal{O}^{\mathfrak{q}}[[y]]$ and $[[x]]\,\mathcal{O}^{\mathfrak{q}}$.

The sheaves on q-plane

Thus there are Fréchet $\widehat{\otimes}$-algebra sheaf morphisms

$$
\begin{array}{ccccc}
 & & \mathcal{O}_q\left(\mathbb{C}_{xy}\right) & & \\
 & {}^{l}\swarrow & & \searrow^{r} & \\
\mathcal{O}^{\mathfrak{q}}\left[\left[y\right]\right] & & & & \left[\left[x\right]\right]\mathcal{O}^{\mathfrak{q}} \\
 & {}_{s}\searrow & & \swarrow_{t} & \\
 & & \mathbb{C}\left[\left[x,y\right]\right] & &
\end{array}
$$

that makes the diagram commutative.

The Fréchet algebra sheaf on the q-plane

Now we focus on sections from both directions that are compatible at zero. We define a new sheaf $\mathcal{F}_q$ of Fréchet $\widehat{\otimes}$-algebras on $\mathbb{C}_{xy}$ to be the fibered product

$$\mathcal{F}_q = \mathcal{O}^{\mathfrak{q}}\left[\left[y\right]\right] \underset{\mathbb{C}[[x,y]]}{\times} \left[\left[x\right]\right]\mathcal{O}^{\mathfrak{q}}$$

of the Fréchet $\widehat{\otimes}$-algebra sheaves $\mathcal{O}^{\mathfrak{q}}\left[\left[y\right]\right]$ and $\left[\left[x\right]\right]\mathcal{O}^{\mathfrak{q}}$ over the constant sheaf $\mathbb{C}\left[\left[x,y\right]\right]$. It is uniquely given by the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{F}_q & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ \mathcal{O}^{\mathfrak{q}}\left[\left[y\right]\right] & & & & \left[\left[x\right]\right]\mathcal{O}^{\mathfrak{q}} \\ & {}_{s}\searrow & & \swarrow_{t} & \\ & & \mathbb{C}\left[\left[x,y\right]\right] & & \end{array}$$

The Fréchet algebra sheaf on the q-plane

Let $U \subseteq \mathbb{C}_{xy}$ be a q-open subset. Then $\mathcal{F}_q(U)$ consists of those couples

$$(f, g) \in \mathcal{O}^{\mathfrak{q}}(U_x)[[y]] \oplus [[x]]\mathcal{O}^{\mathfrak{q}}(U_y)$$

such that

$$\frac{f_k^{(i)}(0)}{i!} = \frac{g_i^{(k)}(0)}{k!} \text{ for all } i, k \in \mathbb{Z}_+.$$

There is a unique natural Fréchet $\widehat{\otimes}$-algebra sheaf morphism

$$\mathcal{O}_q(\mathbb{C}_{xy}) \to \mathcal{F}_q$$

given by the morphisms l and r.

The fibered product of Fréchet algebras

Let X, Y, Z be objects with morphisms $s : X \to Z$, $t : Y \to Z$ from $\mathfrak{Fa}$. The *fibered product* $X \underset{Z}{\times} Y$ *of* X *and* Y *over* Z or *the morphism couple* (s, t) is the defined to be the pullback of the morphisms s and t in the category $\mathfrak{Fa}$. Thus $X \underset{Z}{\times} Y$ is a Fréchet $\widehat{\otimes}$-algebra equipped with the projections p and q that make the diagram

$$
\begin{array}{ccccc}
 & & X \underset{Z}{\times} Y & & \\
 & {}^{p}\swarrow & & \searrow{}^{q} & \\
X & & & & Y \\
 & {}_{s}\searrow & & \swarrow{}_{t} & \\
 & & Z & &
\end{array}
$$

commutative.

The fibered product of Fréchet algebras

It possesses the following universal-injective property: if

$$\begin{array}{ccccc} & & W & & \\ & {\scriptstyle p'}\swarrow & & \searrow{\scriptstyle q'} & \\ X & & & & Y \\ & {\scriptstyle s}\searrow & & \swarrow{\scriptstyle t} & \\ & & Z & & \end{array}$$

is another similar commutative diagram in $\mathfrak{Fa}$ then there is a unique morphism $u : W \to X \underset{Z}{\times} Y$ such that $pu = p'$ and $qu = q'$.

The fibered product of Fréchet algebras

The fibered product $X \underset{Z}{\times} Y$ of the morphisms $s : X \to Z$, $t : Y \to Z$ from $\mathfrak{Fa}$ does exist and

$$X \underset{Z}{\times} Y = \{(x, y) \in X \oplus Y : s(x) = t(y)\}$$

is a closed subalgebra of the direct sum $X \oplus Y$ of the Fréchet $\widehat{\otimes}$-algebras.

The Fréchet algebra sheaf on the q-plane

The standard sheaf $\mathcal{O}^{\mathfrak{q}}$ of stalks of holomorphic functions on $\mathbb{C}_{xy}$ can also be treated as the fibered product

$$\mathcal{O}^{\mathfrak{q}} = \mathcal{O}_x^{\mathfrak{q}} \underset{\mathbb{C}}{\times} \mathcal{O}_y^{\mathfrak{q}}.$$

For every q-open subset $U \subseteq \mathbb{C}_{xy}$ we have

$$\begin{aligned} \mathcal{O}^{\mathfrak{q}}(U) &= \mathcal{O}^{\mathfrak{q}}(U_x) \underset{\mathbb{C}}{\times} \mathcal{O}^{\mathfrak{q}}(U_y) \\ &= \{(f_0, g_0) \in \mathcal{O}^{\mathfrak{q}}(U_x) \oplus \mathcal{O}^{\mathfrak{q}}(U_y) : f_0(0) = g_0(0)\} \end{aligned}$$

to be a closed subalgebra of the Fréchet sum $\mathcal{O}^{\mathfrak{q}}(U_x) \oplus \mathcal{O}^{\mathfrak{q}}(U_y)$. Notice

$$\operatorname{Spec}(\mathcal{O}^{\mathfrak{q}}(U)) = U.$$

The Fréchet algebra sheaf on the q-plane

The following diagrams

$$\begin{array}{ccccc} & & \mathcal{F}_q & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ \mathcal{O}^{\mathfrak{q}}\left[\left[y\right]\right] & & & & \left[\left[x\right]\right]\mathcal{O}^{\mathfrak{q}} \\ & {}_{s}\searrow & & \swarrow_{t} & \\ & & \mathbb{C}\left[\left[x,y\right]\right] & & \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{O}^{\mathfrak{q}} & & \\ & {}^{p_0}\swarrow & & \searrow^{q_0} & \\ \mathcal{O}^{\mathfrak{q}}_x & & & & \mathcal{O}^{\mathfrak{q}}_y \\ & {}_{s_0}\searrow & & \swarrow_{t_0} & \\ & & \mathbb{C} & & \end{array}$$

are linked in the following way.

The Fréchet algebra sheaf on the q-plane

There are canonical projections and the trivial character

$$\begin{array}{rcl} \mathcal{O}^{\mathfrak{q}}\left(U_x\right)[[y]] & \rightarrow & \mathcal{O}^{\mathfrak{q}}\left(U_x\right), \quad \sum f_n\left(x\right) y^n \mapsto f_0\left(z\right) \\ [[x]]\,\mathcal{O}^{\mathfrak{q}}\left(U_y\right) & \rightarrow & \mathcal{O}^{\mathfrak{q}}\left(U_y\right), \quad \sum x^n g_n\left(y\right) \mapsto g_0\left(w\right), \\ (0,0) & : & \mathbb{C}[[x,y]] \rightarrow \mathbb{C}. \end{array}$$

Based on the universal-injective property of the fibered products given by these morphisms, we obtain a unique continuous algebra homomorphism

$$\Lambda\left(U\right) : \mathcal{F}_q\left(U\right) \rightarrow \mathcal{O}^{\mathfrak{q}}\left(U\right), \quad \Lambda\left(U\right)\left(f, g\right) = \left(f_0, g_0\right).$$

The first challenging problem: is it true that $\ker \Lambda\left(U\right) = \mathrm{Rad}\, \mathcal{F}_q\left(U\right)$? If yes, how can we describe it ?

The Fréchet algebra sheaf on the q-plane

The filtration given by the subsheaves of the z^d-principal ideals of $\mathcal{O}^{\mathfrak{q}}$ on $\mathbb{C}_x$ defines the family $\left\{\mathcal{O}_x^{\mathfrak{q}}(d) : d \in \mathbb{Z}_+\right\}$ of Fréchet space sheaves. In a similar way, $\left\{\mathcal{O}_y^{\mathfrak{q}}(d) : d \in \mathbb{Z}_+\right\}$ is given by the filtration of w^d-principal ideals of $\mathcal{O}^{\mathfrak{q}}$ on $\mathbb{C}_y$. There are continuous (evaluation) linear maps

$$\begin{array}{ccccc} \mathcal{O}_x^{\mathfrak{q}}(d) & & & & \mathcal{O}_y^{\mathfrak{q}}(d) \\ & s_d \searrow & & \swarrow t_d & \\ & & \mathbb{C} & & \end{array}$$

$$s_d : f(z) \mapsto (d!)^{-1} f^{(d)}(0), \quad t_d : g(w) \mapsto (d!)^{-1} g^{(d)}(0)$$

that define

$$\mathcal{O}^{\mathfrak{q}}(d) = \mathcal{O}_x^{\mathfrak{q}}(d) \underset{\mathbb{C}}{\times} \mathcal{O}_y^{\mathfrak{q}}(d)$$

to be a Fréchet space sheaf on $\mathbb{C}_{xy}$, and $\mathcal{O}^{\mathfrak{q}}(0) = \mathcal{O}_x^{\mathfrak{q}}(0) \underset{\mathbb{C}}{\times} \mathcal{O}_y^{\mathfrak{q}}(0) = \mathcal{O}^{\mathfrak{q}}$.

The Fréchet algebra sheaf on the q-plane

For $d \in \mathbb{N}$, $f \in \mathcal{O}_x^{\mathfrak{q}}(d)(U_x)$, $g \in \mathcal{O}_y^{\mathfrak{q}}(d)(U_y)$ we have

$$\begin{aligned} s_0(U_x)\left(z^{-d} f(z)\right) &= \left(z^{-d} f(z)\right)|_{z=0} = (d!)^{-1} f^{(d)}(0) \\ &= s_d(U_x)(f(z)), \\ t_0(U_y)\left(w^{-d} g(w)\right) &= t_d(U_y)(g(w)). \end{aligned}$$

Thus the isomorphisms

$$\begin{aligned} \mathcal{O}_x^{\mathfrak{q}}(d)(U_x) &\rightarrow \mathcal{O}^{\mathfrak{q}}(U_x), \quad f(z) \mapsto z^{-d} f(z), \\ \mathcal{O}_y^{\mathfrak{q}}(d)(U_y) &\rightarrow \mathcal{O}^{\mathfrak{q}}(U_y), \quad g(w) \mapsto w^{-d} g(w) \end{aligned}$$

are compatible with the evaluations maps.

Using the universal-injective property, we deduce that

$$\mathcal{O}^{\mathfrak{q}}(d)(U) \to \mathcal{O}^{\mathfrak{q}}(U), \quad (f(z), g(w)) \mapsto \left(z^{-d} f(z), w^{-d} g(w)\right)$$

is a topological isomorphism of the Fréchet spaces. Thus

$$\{\mathcal{O}^{\mathfrak{q}}(d) : d \in \mathbb{Z}_{+}\}$$

are isomorphic copies of the Fréchet sheaf $\mathcal{O}^{\mathfrak{q}}$ on $\mathbb{C}_{xy}$.

The decomposition theorem

If $U \subseteq \mathbb{C}_{xy}$ is q-open, then $\Lambda(U) : \mathcal{F}_q(U) \to \mathcal{O}^{\mathfrak{q}}(U)$ is a retraction in $\mathfrak{Fs}$, which allows us to identify $\mathcal{O}^{\mathfrak{q}}(U)$ with a complemented subspace of $\mathcal{F}_q(U)$.

Theorem 1. The following decomposition holds

$$\mathcal{F}_q(U) = \mathcal{O}^{\mathfrak{q}}(U) \oplus \operatorname{Rad} \mathcal{F}_q(U)$$

into a topological direct sum of the closed subspaces. Moreover,

$$\operatorname{Rad} \mathcal{F}_q(U) = \prod_{d \in \mathbb{N}} \mathcal{O}^{\mathfrak{q}}(d)(U)$$

up to a topological isomorphism of the Fréchet spaces, and

$$\operatorname{Spec}(\mathcal{F}_q(U)) = U.$$

The deformation quantization

Thus one needs to take the structure sheaf $\mathcal{O}^{\mathfrak{q}}$ of the commutative space $(\mathbb{C}_{xy}, \mathcal{O}^{\mathfrak{q}})$ and use its deformation quantization

$$\mathcal{F}_q = \prod_{d \in \mathbb{Z}_+} \mathcal{O}^{\mathfrak{q}}(d)$$

which results in the noncommutative analytic q-space $(\mathbb{C}_{xy}, \mathcal{F}_q)$ of $\mathcal{O}_q(\mathbb{C}_{xy})$-algebras such that

$$\mathbb{C}_{xy} = \operatorname{Spec} \Gamma(\mathbb{C}_{xy}, \mathcal{F}_q).$$

In this case,

$$\mathcal{F}_q = \mathcal{O}^{\mathfrak{q}} \oplus \operatorname{Rad} \mathcal{F}_q.$$

But $\Gamma(\mathbb{C}_{xy}, \mathcal{F}_q)$ is larger than $\mathcal{O}_q(\mathbb{C}_{xy})$ (a bit).

The global sections

It turns out that

$$\begin{aligned}\mathcal{O}_q\left(\mathbb{C}_{xy}\right) &= \left\{f=\sum_{i,k} a_{ik}x^i y^k : \left\|f\right\|_\rho = \sum_{i,k}\left|a_{ik}\right|\rho^{i+k}<\infty, \rho>0\right\} \\ &\subseteq \left\{\begin{array}{c} f=\sum_{i,k} a_{ik}x^i y^k : \sum_i \left|a_{ik}\right|\rho^i<\infty, \sum_k \left|a_{ik}\right|\rho^k<\infty, \\ \rho>0, i,k\in\mathbb{Z}_+ \end{array}\right\} \\ &= \Gamma\left(\mathbb{C}_{xy},\mathcal{F}_q\right).\end{aligned}$$

The global sections

For example, the formal series

$$f = \sum_{i,k} \frac{i^k k^i}{i!k!} x^i y^k \in \Gamma\left(\mathcal{F}_q, \mathbb{C}_{xy}\right) \setminus \mathcal{O}_q\left(\mathbb{C}_{xy}\right).$$

Indeed, for $\rho = 1$ we have

$$\|f\|_1 = \sum_{i,k} \frac{i^k k^i}{i!k!} \geq \sum_n \left(\frac{n^n}{n!}\right)^2 = \infty,$$

whereas

$$\sum_i \frac{i^k k^i}{i!k!} \rho^i < \infty \quad \text{and} \quad \sum_k \frac{i^k k^i}{i!k!} \rho^k < \infty$$

for all $\rho > 0$.

The topological homology

The canonical embedding $\mathfrak{A}_q \to \mathcal{O}_q(\mathbb{C}_{xy})$ is a localization in the sense of Taylor (by A. Yu. Prikovskii (2008)). Using the Takhtajan resolution, we obtain its resolution $\mathcal{R}\left(\mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2}\right)$:

$$0 \to \mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2} \xrightarrow{d^0} \mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2} \oplus \mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2} \xrightarrow{d^1} \mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2} \to 0,$$

whose differentials can be written as

$$d^0 = \begin{bmatrix} R_y \otimes 1 - q1 \otimes L_y \\ 1 \otimes L_x - qR_x \otimes 1 \end{bmatrix}, \quad d^1 = \begin{bmatrix} 1 \otimes L_x - R_x \otimes 1 & 1 \otimes L_y - R_y \otimes 1 \end{bmatrix},$$

where L and R indicate to the left and right regular (anti) representations of the algebra $\mathcal{O}_q(\mathbb{C}_{xy})$.

The topological homology

It follows that every Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-algebra $\mathcal{A}$ possesses a similar resolution. By applying the functor $\mathcal{A} \underset{\mathcal{O}_q(\mathbb{C}_{xy})}{\widehat{\otimes}} \circ \underset{\mathcal{O}_q(\mathbb{C}_{xy})}{\widehat{\otimes}} \mathcal{A}$ to the resolution $\mathcal{R}\left(\mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2}\right)$, we obtain that

$$\mathcal{A} \underset{\mathcal{O}_q(\mathbb{C}_{xy})}{\widehat{\otimes}} \mathcal{R}\left(\mathcal{O}_q(\mathbb{C}_{xy})^{\widehat{\otimes}2}\right) \underset{\mathcal{O}_q(\mathbb{C}_{xy})}{\widehat{\otimes}} \mathcal{A} = \mathcal{R}\left(\mathcal{A}^{\widehat{\otimes}2}\right).$$

Therefore $\mathcal{R}\left(\mathcal{A}^{\widehat{\otimes}2}\right)$ is a free $\mathcal{A}$-bimodule resolution of $\mathcal{A}$ with the same differentials, that is, the complex

$$\mathcal{R}\left(\mathcal{A}^{\widehat{\otimes}2}\right) \xrightarrow{\pi} \mathcal{A} \to 0$$

is admissible. The potential candidates for $\mathcal{A}$ are the following algebras $\mathcal{O}(U_x)[[y]]$, $[[x]]\mathcal{O}(U_y)$, $\mathcal{F}_q(U)$ and $\mathbb{C}[[x,y]]$.

Taylor spectrum

Let X be a left Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-module, which means that there is a pair $T, S \in \mathcal{L}(X)$ with $TS = q^{-1}ST$. A right Fréchet $\mathcal{O}_q(\mathbb{C}_{xy})$-module Y is in the transversality relation with respect to X if

$$\mathrm{Tor}_k^{\mathcal{O}_q(\mathbb{C}_{xy})}(Y, X) = \{0\}, \quad k \geq 0.$$

In this case we write $Y \perp X$ (see A. Ya. Helemskii, Homology Banach. Top. Alg.). Every $\gamma \in \mathbb{C}_{xy}$ being a continuous character of $\mathcal{O}_q(\mathbb{C}_{xy})$ defines the trivial $\mathcal{O}_q(\mathbb{C}_{xy})$-module $\mathbb{C}(\gamma)$. *The resolvent set* $\mathrm{res}(T, S)$ is defined to be a set of those $\gamma \in \mathbb{C}_{xy}$ such that $\mathbb{C}(\gamma) \perp_{\mathcal{O}_q(\mathbb{C}_{xy})} X$. The set

$$\sigma(T, S) = \mathrm{Spec}(\mathcal{O}_q(\mathbb{C}_{xy})) \setminus \mathrm{res}(T, S)$$

is called the joint (Taylor) spectrum of the operator pair (T, S).

The homology groups $\mathrm{Tor}_k^{\mathcal{O}_q(\mathbb{C}_{xy})}(\mathbb{C}(\gamma), X)$ can be calculated by means of the obtained resolution. We come up with the following parametrized over $\mathbb{C}_{xy}$ Fréchet space complex

$$0 \to X \xrightarrow{d_\gamma^0} X \oplus X \xrightarrow{d_\gamma^1} X \to 0$$

with the differentials

$$d_\gamma^0 = \begin{bmatrix} \gamma(y) - qS \\ T - q\gamma(x) \end{bmatrix}, \; d_\gamma^1 = \begin{bmatrix} T - \gamma(x) & S - \gamma(y) \end{bmatrix}.$$

Thus $\sigma(T,S) = \sigma_x(T,S) \cup \sigma_y(T,S)$, where $\sigma_x(T,S) = \sigma(T,S) \cap \mathbb{C}_x$ and $\sigma_y(T,S) = \sigma(T,S) \cap \mathbb{C}_y$.

Theorem 2. Let X be a left Banach $\mathfrak{A}_q$-module given by an operator pair $T, S \in \mathcal{B}(X)$ with $TS = q^{-1}ST$, and let $U \subseteq \mathbb{C}_{xy}$ be a q-open subset. Then

$$\begin{array}{rcccl} \mathcal{O}(U_x)[[y]] & \perp & X & \Leftrightarrow & U_x \cap \sigma_x(T,S) = \varnothing, \\ [[x]]\mathcal{O}(U_y) & \perp & X & \Leftrightarrow & U_y \cap \sigma_y(T,S) = \varnothing. \\ \mathbb{C}[[x,y]] & \perp & X & \Leftrightarrow & (0,0) \notin \sigma(T,S). \end{array}$$

Is it possible to get the same result for $\mathcal{F}_q(U)$ by passing to the fibered product of the exact complexes? The answer is NO (in the general case).

French général

If

$$\begin{array}{ccccc} X_1 & & & & Y_1 \\ & s_1 \searrow & & \swarrow t_1 & \\ & & Z_1 & & \end{array} \qquad \begin{array}{ccccc} X_2 & & & & Y_2 \\ & s_2 \searrow & & \swarrow t_2 & \\ & & Z_2 & & \end{array}$$

are morphisms in $\mathfrak{Fa}$, then by a morphism $\tau : (s_1, t_1) \to (s_2, t_2)$ of these couples we mean a triple $\tau = (f, g, l)$ of morphisms

$$f : X_1 \to X_2, \qquad g : Y_1 \to Y_2,$$

$$l : Z_1 \to Z_2$$

from $\mathfrak{Fa}$ such that $s_2 f = l s_1$ and $t_2 g = l t_1$. It is a new *category of the couples over* $\mathfrak{Fa}$. The morphism τ in turn defines the morphism

$$\begin{aligned} u &= f \times_l g : X_1 \underset{Z_1}{\times} Y_1 \longrightarrow X_2 \underset{Z_2}{\times} Y_2, \\ p_2 u &= f p_1, \quad q_2 u = g q_2 \end{aligned}$$

in $\mathfrak{Fa}$ of the fibered products by the universal-injective property of $X_2 \underset{Z_2}{\times} Y_2$.

Proposition 3. Let

$$0 \longrightarrow (s_0, t_0) \xrightarrow{\tau_0} (s_1, t_1) \xrightarrow{\tau_1} (s_2, t_2) \longrightarrow 0$$

be an exact sequence of the morphism couples over $\mathfrak{Fa}$. Then the related sequence

$$0 \longrightarrow X_0 \underset{Z_0}{\times} Y_0 \xrightarrow{u_0} X_1 \underset{Z_1}{\times} Y_1 \xrightarrow{u_1} X_2 \underset{Z_2}{\times} Y_2$$

of the fibered products is exact, $\operatorname{im}(u_1)$ is closed, and

$$H^2 = X_2 \underset{Z_2}{\times} Y_2 / \operatorname{im}(u_1) = l_0^{-1} \left(\operatorname{im}\left[\begin{array}{cc} s_1 & t_1 \end{array}\right]\right) / \left(\operatorname{im}\left[\begin{array}{cc} s_0 & t_0 \end{array}\right]\right),$$

where $\operatorname{im}\left[\begin{array}{cc} s_i & t_i \end{array}\right] = \operatorname{im}(s_i) + \operatorname{im}(t_i)$, $i = 0, 1$. Thus the identification is a topological isomorphism iff $\operatorname{im}\left[\begin{array}{cc} s_0 & t_0 \end{array}\right]$ is closed.

The main results

Nonetheless using our decomposition theorem one can prove the following transversality assertion for the sheaf $\mathcal{F}_q$ too.

Theorem 4. Let X be a left Banach $\mathfrak{A}_q$-module given by an operator pair $T, S \in \mathcal{B}(X)$ with $TS = q^{-1}ST$, and let $U \subseteq \mathbb{C}_{xy}$ be a nonempty q-open subset. Then

$$\mathcal{F}_q(U) \perp X \quad \Leftrightarrow \quad U \cap \sigma(T, S) = \varnothing.$$

Theorem 5. Let $U \subseteq \mathbb{C}_{xy}$ be q-open. If the left $\mathcal{O}_q(\mathbb{C}_{xy})$-module action on a Banach space X is extended up to a left Banach $\mathcal{F}_q(U)$-module structure on X, then

$$\exists n \in \mathbb{N}, \quad (TS)^n = 0, \text{ and} \quad \sigma(T) \subseteq U_x, \quad \sigma(S) \subseteq U_y.$$

Moreover, the left Banach $\mathcal{O}_q(\mathbb{C}_{xy})$-module X is lifted to a left Banach $\mathcal{F}_q(\mathbb{C}_{xy})$-module structure on X if and only if TS is a nilpotent operator.

Noncommutative holomorphic functional calculus

Theorem 6. Let X be a left Banach $\mathfrak{A}_q$-module given by an operator pair $T, S \in \mathcal{B}(X)$ with $TS = q^{-1}ST$, and $U \subseteq \mathbb{C}_{xy}$ a q-open subset. Then

$$\sigma(T,S)^{-\mathfrak{q}} \subseteq U \Longrightarrow \exists \ \mathcal{F}_q(U) \to \mathcal{B}(X), \quad x \mapsto T, y \mapsto S,$$
$$\text{(noncommutative holomorphic functional calculus on } U\text{)}$$

a continuous algebra homomorphism. Thus the left $\mathfrak{A}_q$-module module structure of X can be lifted up to a left Banach $\mathcal{F}_q(U)$-module one on X whenever $\sigma(T,S)^{-\mathfrak{q}} \subseteq U$.

Examples

Let X be a Fréchet space with an operator tuple $T = (T_1, \ldots, T_n)$ and S on X with $T_i S = q^{-1} S T_{i+1}$, $1 \leq i \leq n-1$. Then $X^{\oplus n}$ is a left Fréchet $\mathcal{O}_q(\mathbb{C}^2)$-module given by

$$T = \begin{bmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_n \end{bmatrix}, \quad S_n = \begin{bmatrix} 0 & S & & 0 \\ & 0 & \ddots & \\ & & \ddots & S \\ 0 & & & 0 \end{bmatrix}.$$

One can easily verify that $TS_n = q^{-1} S_n T$. A particular case of this construction is the case of $T_i = T_j = T$ and its diagonal inflation $T^{\oplus n}$.

Examples

For example, $T_i, S \in \mathcal{B}(C(K,X))$ with

$$T_i(\mathbf{f}(z)) = q^{i-1}\mathbf{f}(z), \quad S(\mathbf{f}(z)) = \mathbf{f}(qz), \quad 1 \le i \le n$$

where, $K \subseteq \mathbb{C}$ is a q-compact set. Then $C(K,X)^{\oplus n} = C(K, X^{\oplus n})$ and

$$\begin{aligned} T &= 1 \oplus q \oplus \cdots \oplus q^{n-1} \in \mathcal{B}\left(C\left(K, X^{\oplus n}\right)\right), \\ S_n &= r_n S_0, \quad r_n \in \mathcal{B}\left(X^{\oplus n}\right), r_n(\zeta_1, \ldots, \zeta_n) = (\zeta_2, \ldots, \zeta_n, 0) \end{aligned}$$

with $S_0 \in \mathcal{B}(C(K, X^{\oplus n}))$, $S_0(\mathbf{f}(z)) = \mathbf{f}(qz)$. If $K = \{0\}$ and $X = \mathbb{C}$, then we come up with the matrices

$$T = \begin{bmatrix} 1 & & & 0 \\ & q & & \\ & & \ddots & \\ 0 & & & q^{n-1} \end{bmatrix}, \quad S_n = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

in the matrix algebra M_n (see O. Aristov 2024).

Examples

If (T, S_n) is the operator pair of $T = (T_1, \ldots, T_n)$ and S on X with $T_i S = q^{-1} S T_{i+1}$, $1 \leq i \leq n-1$, then

$$\left(\sigma\left(q^{-1}T_1\right) \cup \sigma\left(T_n\right)\right) \times \{0\} \subseteq \sigma\left(T, S_n\right) \subseteq \left(\cup_{i=1}^{n} \sigma\left(q^{-1}T_i\right) \cup \sigma\left(T_i\right)\right) \times \{0\}$$

In particular, if $T_i = T_j = T$ for all i, j, then

$$\sigma\left(T^{\oplus n}, S_n\right) = \left(\sigma\left(q^{-1}T\right) \cup \sigma\left(T\right)\right) \times \{0\}.$$

In the case of the matrices, we have

$$\left\{\left(q^{-1}, 0\right), \left(q^{n-1}, 0\right)\right\} \subseteq \sigma\left(T, S_n\right) \subseteq \left\{\left(q^{i}, 0\right) : -1 \leq i \leq n-1\right\}.$$

In particular, for the $\mathfrak{q}$-closures in $(\mathbb{C}_{xy}, \mathfrak{q})$ we obtain that

$$\sigma\left(T, S_n\right)^{-} = \left\{\left(q^{n-1}, 0\right)\right\}^{-} = \left\{\left(q^{i}, 0\right) : i \leq n-1\right\}$$

Thanks

Thanks !