Around Cowling's conjecture

Ignacio Vergara
EIMI

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Outline

1. **Weak amenability and the Haagerup property**
2. **Cowling's conjecture**
3. **Proper affine actions on $E \subset L^1$**

Main question

Let Γ be a countable group. We look at approximations of the identity:

$$\varphi_n : \Gamma \to \mathbb{C} \text{ such that } \varphi_n \to 1 \text{ pointwise.}$$

Γ amenable

φ_n finitely supported

φ_n positive definite

↙ ↘

Γ Haagerup

$\varphi_n \in c_0(\Gamma)$

φ_n positive definite

Γ weakly amenable

φ_n finitely supported

$\|\varphi_n\|_{B_2(\Gamma)} \leq C$

Question: How are these two properties related?

Positive definite functions

A function $\varphi : \Gamma \to \mathbb{C}$ is *positive definite* if

$$\sum_{i,j=1}^{n} \varphi(s_j^{-1} s_i) z_i \overline{z_j} \geq 0,$$

for all $s_1, \dots s_n \in \Gamma$, $z_1, \dots z_n \in \mathbb{C}$, $n \geq 1$.

Equivalently, φ is positive definite iff there exists a unitary representation $\pi : \Gamma \to \mathcal{B}(H)$ and $\xi \in H$ s.t.

$$\varphi(s) = \langle \pi(s)\xi, \xi \rangle, \quad \forall s \in \Gamma.$$

Remark: $\|\varphi\|_\infty = \varphi(e) = \|\xi\|^2$.

Herz–Schur multipliers

A function $\varphi : \Gamma \to \mathbb{C}$ is a *Herz–Schur multiplier* if there exist bounded maps $\xi, \eta : \Gamma \to H$ such that

$$\varphi(s^{-1}t) = \langle \xi(t), \eta(s) \rangle, \quad \forall s, t \in \Gamma. \qquad (*)$$

We endow the space of Herz–Schur multipliers $B_2(\Gamma)$ with the norm

$$\|\varphi\|_{B_2(\Gamma)} = \inf \left\{ \sup_{t\in\Gamma} \|\xi(t)\| \; \sup_{s\in\Gamma} \|\eta(s)\| \right\},$$

where the infimum is taken over all possible decompositions as in $(*)$.

Remarks:
1) The space $B_2(\Gamma)$ coincides with the algebra of completely bounded multipliers of the Fourier algebra $M_0A(\Gamma)$ (Bożejko–Fendler).
2) φ is positive definite iff $\|\varphi\|_{B_2(\Gamma)} = \varphi(e)$.

Let Γ be a countable group. We say that

Γ **is amenable** if there exist $\varphi_n : \Gamma \to \mathbb{C}$ s.t.

- $\varphi_n \to 1$ pointwise
- φ_n finitely supported
- φ_n positive definite

Γ **has the Haagerup property** if there exist $\varphi_n : \Gamma \to \mathbb{C}$ s.t.

- $\varphi_n \to 1$ pointwise
- φ_n vanishes at infinity
- φ_n positive definite

Γ **is weakly amenable:** if there exist $\varphi_n : \Gamma \to \mathbb{C}$ s.t.

- $\varphi_n \to 1$ pointwise
- φ_n finitely supported
- $\exists C \geq 1$, $\sup_n \|\varphi_n\|_{B_2(\Gamma)} \leq C$

Cowling–Haagerup constant: $\Lambda(\Gamma)$ is the infimum of all C such that the condition above holds.

Cowling's conjecture (v1)

A group Γ is weakly amenable with $\mathbf{\Lambda}(\Gamma) = 1$ iff it has the Haagerup property.

Evidence:

- Examples: Free groups, Coxeter groups, Baumslag–Solitar groups,...
- Non-examples: $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, lattices in higher rank Lie groups ($\mathbf{\Lambda}(\Gamma) = \infty$), lattices in $\mathrm{Sp}(n,1)$ ($\mathbf{\Lambda}(\Gamma) > 1$),...
- If $\mathbf{\Lambda}(\Gamma) = 1$, we can find a finitely supported approximation of 1 (φ_n) such that

$$\|\varphi_n\|_{B_2(\Gamma)} \to 1.$$

In particular, it is "asymptotically positive definite":

$$\|\varphi_n\|_{B_2(\Gamma)} - \varphi_n(e) \to 0.$$

Cowling's conjecture (v1) does not hold

Wreath products: Let Γ, Λ be two groups. Γ acts on $\bigoplus_\Gamma \Lambda$ by shifts.

$$\Lambda \wr \Gamma = \Gamma \ltimes \bigoplus_{\Gamma} \Lambda.$$

Theorem (Ozawa–Popa 2007, Ozawa 2011)

If Λ is not trivial and Γ is not amenable, then $\Lambda \wr \Gamma$ is not weakly amenable.

Theorem (Cornulier–Stalder–Valette 2009)

If both Λ and Γ have the Haagerup property, then so does $\Lambda \wr \Gamma$.

Conclusion: Many examples of non-weakly amenable groups satisfying the Haagerup property, e.g. $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$.
Remark: Even more is true. There are non-exact groups with the Haagerup property (Osajda 2014).

Cowling's conjecture (v2)

Every weakly amenable group Γ with $\boldsymbol{\Lambda}(\Gamma) = 1$ has the Haagerup property.

Status: OPEN.

New approach: actions on $E \subset L^1$.

Theorem (Bekka–Cherix–Valette 1993)

A countable group Γ has the Haagerup property iff it has a proper isometric action $\Gamma \curvearrowright^{\sigma} H$ on a Hilbert space.

(Here proper means that, for all $v \in H$, $\|\sigma(s)v\| \to \infty$ as $s \to \infty$.)

Sketch of proof.

$\boxed{\Leftarrow}$ If $\Gamma \curvearrowright^{\sigma} H$, we define

$$\varphi_n(s) = e^{-\frac{1}{n}\|\sigma(s)0\|^2}, \quad \forall s \in \Gamma.$$

$\boxed{\Rightarrow}$ Choosing (φ_n) and $\alpha_n > 0$ carefully,

$$\psi(s) = \sum_n \alpha_n(1 - \varphi_n(s))$$

is a proper conditionally negative definite function.

GNS $\rightsquigarrow$ There is an isometric action $\Gamma \curvearrowright^{\sigma} H$ such that

$$\psi(s) = \|\sigma(s)0\|^2, \quad \forall s \in \Gamma.$$

□

Actions on L^p-spaces

Theorem (Chatterji–Druţu–Haglund 2009)

Let Γ be a countable group.

- If Γ has the Haagerup property, then it has a proper isometric action on an L^p-space for every $p > 0$.
- If Γ has a proper isometric action on a subset of an L^p-space (for $0 < p \leq 2$), then it has the Haagerup property.

Corollary

Γ has the Haagerup property if and only if it has a proper isometric action on a subspace of an L^1-space.

Goal (imprecise): Given Γ with $\Lambda(\Gamma) = 1$, construct a proper action on $E \subset L^1$, which is "close" to being isometric.

Theorem (Mazur–Ulam 1932)

Let $f : E \to F$ be a surjective isometry between normed spaces. Then f is affine:

$$f(v) = Uv + w, \quad \forall v \in E,$$

where $U : E \to F$ is a linear isometry and $w \in F$.

As a consequence, every isometric action $\Gamma \curvearrowright^{\sigma} E$ is given by

$$\sigma(s)v = \pi(s)v + b(s), \quad \forall s \in \Gamma, \, \forall v \in E,$$

where π is an isometric representation, and $b : \Gamma \to E$ is a cocycle:

$$b(st) = \pi(s)b(t) + b(s), \quad \forall s, t \in \Gamma.$$

Goal (updated): Given Γ with $\Lambda(\Gamma) = 1$, construct a proper affine action on $E \subset L^1$, given by $\sigma(s)v = \pi(s)v + b(s)$, where π is uniformly bounded:

$$\sup_{s \in \Gamma} \|\pi(s)\| < \infty.$$

Main tool: Knudby's work on semigroups of multipliers

Theorem (Knudby 2013)

If $\Lambda(\Gamma) = 1$, then there exists $\psi : \Gamma \to [0, \infty)$ proper such that

$$\left\| e^{-r\psi} \right\|_{B_2(\Gamma)} \leq 1, \quad \forall r > 0.$$

Sketch of proof. Carefully choosing an approximation of 1 with $\|\varphi_n\|_{B_2(\Gamma)} = 1$, and $\alpha_n \nearrow \infty$,

$$\psi = \sum_n \alpha_n (1 - \varphi_n)$$

is well defined and proper.

$$\begin{aligned} \left\| e^{-r\psi} \right\|_{B_2(\Gamma)} &\leq \prod_n \left\| e^{-r\alpha_n(1-\varphi_n)} \right\|_{B_2(\Gamma)} \\ &\leq \prod_n e^{-r\alpha_n} e^{r\alpha_n \|\varphi_n\|_{B_2(\Gamma)}} \leq 1 \end{aligned}$$

□

Theorem (Knudby 2013)

Let $\psi : \Gamma \to \mathbb{R}$ be a symmetric function. TFAE:

a) For all $r > 0$, $\left\| e^{-r\psi} \right\|_{B_2(\Gamma)} \leq 1$.

b) There is a Hilbert space H and maps $R, S : \Gamma \to H$ such that

$$\psi(y^{-1}x) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2, \quad \forall x, y \in \Gamma.$$

Idea of the proof of (a) $\Longrightarrow$ (b).

- Construct a sequence $A_n = \left(\begin{array}{c|c} b_n & e^{-\psi/n} \\ \hline e^{-\psi/n} & c_n \end{array} \right) \geq 0.$
- This defines a sequence of pos. def. kernels κ_n on $\Gamma \sqcup \overline{\Gamma}$.
- $\forall x \in \Gamma, \forall y \in \overline{\Gamma}, \quad n(1 - \kappa_n(x, \overline{y})) \to \psi(y^{-1}x)$.
- Ultraproduct $\rightsquigarrow \psi(y^{-1}x) = \| T(x) - T(\overline{y})\|^2, \quad \forall x \in \Gamma, \forall y \in \overline{\Gamma}$.
- Define

$$R(x) = \tfrac{1}{2}(T(x) + T(\overline{x}))$$
$$S(x) = \tfrac{1}{2}(T(x) - T(\overline{x})), \quad \forall x, y \in \Gamma.$$

□

Corollary

If $\Lambda(\Gamma) = 1$, then there exist a proper function $\psi : \Gamma \to [0, \infty)$, and maps $R, S : \Gamma \to H$ such that

$$\psi(y^{-1}x) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2, \quad \forall x, y \in \Gamma.$$

Remark: $\forall x \in \Gamma,\ 4\|S(x)\|^2 = \psi(e)$.

Theorem (V. 2021)

Let Γ be a countable group with $\Lambda(\Gamma) = 1$. Then there exist a measure space (Ω, μ), a closed subspace $E \subset L^1(\Omega, \mu)$, and a representation π of Γ on E such that

$$\sup_{s \in \Gamma} \|\pi(s)\| \leq 1 + \sqrt{\psi(e)},$$

and π admits a proper cocycle.

Construction of $E \subset L^1(\Omega, \mu)$

Assuming $\mathbf{\Lambda}(\Gamma) = 1$, let $\psi : \Gamma \to [0, \infty)$ and $R, S : \Gamma \to H$ given by Knudby's theorem:

$$\psi(y^{-1}x) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2, \quad \forall x, y \in \Gamma.$$

Let V be the space of finitely supported functions $v : \Gamma \to \mathbb{R}$ such that $\sum v(x) = 0$.
For $v, w \in V$, we define

$$\langle v, w \rangle_R = -\sum_{x,y \in \Gamma} v(x)w(y)\|R(x) - R(y)\|^2 + \sum_{x \in \Gamma} v(x)w(x).$$

Let $\mathcal{H}_R$ be the completion of V for $\langle \cdot, \cdot \rangle_R$. We define $E = \mathcal{H}_R \cap \ell^1(\Gamma)$ and

$$\|v\|_E = \|v\|_R + \|v\|_1.$$

Construction of $E \subset L^1(\Omega, \mu)$

Lemma

There exists a measure space (Ω, μ) such that E can be linearly and isometrically embedded into $L^1(\Omega, \mu)$.

Proof.

• Since $\mathcal{H}_R$ is an L^2-space, it can be linearly and isometrically embedded into $L^1(\Omega', \mu')$ for some (Ω', μ').
• Let $J' : \mathcal{H}_R \to L^1(\Omega', \mu')$ be such an embedding and define $J : E \to L^1(\Omega', \mu') \oplus_1 \ell^1(\Gamma)$ by

$$Jv = J'v + v.$$

• Then

$$\|Jv\| = \|J'v\|_{L^1(\Omega')} + \|v\|_1 = \|v\|_R + \|v\|_1 = \|v\|_E.$$

• Define $\Omega = \Omega' \sqcup \Gamma$ and $\mu = \mu' +$counting meas. Then

$$L^1(\Omega', \mu') \oplus_1 \ell^1(\Gamma) \cong L^1(\Omega, \mu).$$

□

Construction of $\pi : \Gamma \to \mathcal{B}(E)$

Γ acts on V by shifts: $\quad \pi(s)v(x) = v(s^{-1}x), \quad \forall s, x \in \Gamma,\ \forall v \in V.$

Lemma

π extends to a representation on E with

$$\sup_{s\in\Gamma} \|\pi(s)\| \leq 1 + \sqrt{\psi(e)}.$$

Idea of the proof. For all $s \in \Gamma$, $v \in V$,

$$\begin{aligned}
\|\pi(s)v\|_R^2 - \|v\|_R^2 &= \sum_{x,y\in\Gamma} v(x)v(y)\left(\|R(sx) - R(sy)\|^2 - \|R(x) - R(y)\|^2\right) \\
&= \sum_{x,y\in\Gamma} v(x)v(y)\left(\|S(x) + S(y)\|^2 - \|S(sx) + S(sy)\|^2\right) \\
&\leq \psi(e) \sum_{x,y\in\Gamma} |v(x)||v(y)| = \psi(e)\|v\|_1^2.
\end{aligned}$$

$\square$

Construction of the cocycle $b : \Gamma \to E$

For all $s \in \Gamma$, define

$$b(s) = \delta_s - \delta_e \in V.$$

Lemma

$b : \Gamma \to E$ is a proper cocycle for π.

Proof. • b is a cocycle:

$$b(st) = \delta_{st} - \delta_s + \delta_s - \delta_e = \pi(s)b(t) + b(s).$$

• b is proper:

$$\begin{aligned} \|b(s)\|_E &= \left(2\|R(s) - R(e)\|^2 + 2\right)^{\frac{1}{2}} + 2 \\ &= \sqrt{2}\left(\psi(s) - \|S(s) + S(e)\|^2 + 1\right)^{\frac{1}{2}} + 2 \\ &\geq \sqrt{2}\,(\psi(s) - \psi(e) + 1)^{\frac{1}{2}} + 2. \end{aligned}$$

□

Recall that

$$\psi = \sum_n \alpha_n (1 - \varphi_n).$$

One can choose (α_n) and (φ_n) so that $\psi(e)$ is as small as we want. Hence we can rewrite the main theorem as follows.

Theorem

Let Γ be a countable group with $\mathbf{\Lambda}(\Gamma) = 1$ and let $\varepsilon > 0$. Then there exist a measure space (Ω, μ), a closed subspace $E \subset L^1(\Omega, \mu)$, and a representation π of Γ on E such that

$$\sup_{s \in \Gamma} \|\pi(s)\| \leq 1 + \varepsilon,$$

and π admits a proper cocycle.

Thank you.

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