Principal symbol mapping on Heisenberg groups and contact manifolds

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Pseudo-differential operators

For a smooth bounded (together with all its derivatives) function p on $\mathbb{R}^d \times \mathbb{R}^d$, define an operator $\mathrm{Op}(p) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ by setting

$$(\mathrm{Op}(p)f)(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle t,s\rangle} p(t,s)\hat{f}(s)ds.$$

If $m \leq 0$ and if

$$|\partial_t^\alpha \partial_s^\beta p(t,s)| = O(\langle s\rangle^{m-|\beta|}),$$

then $\mathrm{Op}(p)$ is called pseudo-differential (of order m). It extends to a bounded operator on $L_2(\mathbb{R}^d)$.

Classical pseudo-differential operators

Suppose there exists a sequence $\{p_n\}_{n\leq 0}$ of smooth such that

1. p_n is approximately homogeneous of degree n.
2. for every $n \leq 0$, the operator $\mathrm{Op}(p - p_0 - \cdots - p_n)$ is pseudo-differential of order $n-1$.

In this case, $\mathrm{Op}(p)$ is called classical and p_0 is called its principal symbol.

Nice properties of the principal symbol

Let P and Q be classical pseudo-differential operators. Let p_0 and q_0 be their principal symbols.

1. principal symbol of $P+Q$ is p_0+q_0 (obvious)
2. principal symbol of PQ is p_0q_0 (an outcome of some computation)
3. principal symbol of P^* is $\overline{p_0}$ (another computation)

Is principal symbol a $*$-homomorphism?

In the previous slide, we saw that principal symbol mapping preserves $*$-algebraic operations. So, it should be a $*$-homomorphism. This raises natural questions.

Question

What are the domain and the co-domain of this $*$-homomorphism? Is this $*$-homomorphism continuous in some reasonable topology?

The answer to this question appears to be positive. As we will see, the principal symbol mapping is a $*$-homomorphism of C^*-algebras. As such, it is a *topological* notion, not a smooth one!

Abstract principal symbol mapping

Theorem

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras and let π_1 and π_2 be their $*$-representations on the same Hilbert space H. Suppose that

1. $\mathcal{A}_1$ or $\mathcal{A}_2$ is commutative.
2. $[\pi_1(x), \pi_2(y)]$ is compact for every $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$.
3. if $\sum_{k=1}^n \pi_1(x_k)\pi_2(y_k)$ is compact, then $\sum_{k=1}^n x_k \otimes y_k = 0$.

Let Π be the C^*-algebra generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$. There exists a $*$-homomorphism $\mathrm{sym} : \Pi \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ such that

$$\mathrm{sym}(\pi_1(x)) = x \otimes 1, \quad \mathrm{sym}(\pi_2(y)) = 1 \otimes y.$$

Key lemma

Lemma

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras and let ρ_1 and ρ_2 be their $*$-representations on the same Hilbert space H. Suppose that

1. $\mathcal{A}_1$ or $\mathcal{A}_2$ is commutative.
2. $\rho_1(x)$ commutes with $\rho_2(y)$ for every $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$.
3. if $\sum_{k=1}^n \rho_1(x_k)\rho_2(y_k) = 0$, then $\sum_{k=1}^n x_k \otimes y_k = 0$.

Let Π_0 be the C^*-algebra generated by $\rho_1(\mathcal{A}_1)$ and $\rho_2(\mathcal{A}_2)$. There exists a $*$-isomorphism $\rho : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \Pi_0$ such that

$$\rho(x \otimes 1) = \rho_1(x), \quad \rho(1 \otimes y) = \rho_2(y).$$

Having this lemma at hands, we consider $\rho_k = q \circ \pi_k$, where q is the Calkin quotient mapping and set $\mathrm{sym} = \rho^{-1} \circ q$.

Example: Euclidean space

Set $\mathcal{A}_1 = C_0(\mathbb{R}^d)$, $\mathcal{A}_2 = C(\mathbb{S}^{d-1})$. Set

$$\pi_1(f) = M_f, \quad \pi_2(g) = g(\frac{\nabla}{\sqrt{\Delta}}).$$

These C^*-algebras satisfy the assumptions of the Abstract Principal Symbol Theorem. Hence, there exists a $*$-homomorphism

$$\mathrm{sym} : \Pi \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 = C_0(\mathbb{R}^d \times \mathbb{S}^{d-1})$$

such that

$$(\mathrm{sym}(M_f))(t, s) = f(t), \quad (\mathrm{sym}(g(\frac{\nabla}{\sqrt{\Delta}})))(t, s) = g(s).$$

Relation to classical PSDOs

Theorem

Let P be a classical PSDO with principal symbol p_0. We have $P \in \Pi$ and $\mathrm{sym}(P) = p_0$.

This follows from the next intermediate theorem.

Theorem

Let $h \in C_0(\mathbb{R}^d \times \mathbb{S}^{d-1})$ be a smooth mapping. Define T_h by setting

$$(T_h f)(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle t,s\rangle} h(t, \frac{s}{|s|}) \hat{f}(s) ds.$$

We have $T_h \in \Pi$ and $\mathrm{sym}(T_h) = h$.

Sketch of the proof

Let $\{Y_{n,j}\}_{1\leq j\leq N_n}$ be the eigenbasis of the spherical Laplacian (i.e., spherical harmonics). We write

$$h(t,s)=\sum_{n\geq 0}\sum_{j=1}^{N_n}a_{h,j}(t)Y_{n,j}(s).$$

Since h is smooth, it follows that the series converges absolutely in the uniform norm. We write

$$(T_hf)(t)=(2\pi)^{-\frac{d}{2}}\sum_{n\geq 0}\sum_{j=1}^{N_n}\int_{\mathbb{R}^d}e^{i\langle t,s\rangle}a_{h,j}(t)Y_{n,j}(\frac{s}{|s|})\hat{f}(s)ds.$$

Thus,

$$T_h=\sum_{n\geq 0}\sum_{j=1}^{N_n}\pi_1(a_{n,j})\pi_2(Y_{n,j}).$$

Heisenberg group

The Heisenberg group $\mathbb{H}^d$ is $\mathbb{C}^d \times \mathbb{R}$ equipped with the product

$$(z,t) \times (z',t') = \big(z + z', t + t' + \Im(\sum_{j=1}^{d} z_j \bar{z}'_j)\big).$$

Clearly, $\mathbb{H}^d$ is a stratified Lie group of degree 2. Its first stratum is $\mathbb{C}^d \times \{0\}$ and the second one is $\{0\} \times \mathbb{R}$.
It is helpful to denote $\Re(z_l) = x_l$ and $\Im(z_l) = y_l$, $1 \le l \le d$.

Differential calculus on Heisenberg group

The $2d+1$ vector fields

$$X_l = \frac{\partial}{\partial x_l} - y_l \frac{\partial}{\partial t}, \quad Y_l := \frac{\partial}{\partial y_l} + x_l \frac{\partial}{\partial t}, \quad 1 \le l \le d, \quad T = \frac{\partial}{\partial t},$$

form a natural basic for the Lie algebra of left-invariant vector fields on $\mathbb{H}^d$. For convenience, we set $X_{d+l} = Y_l$, $1 \le l \le d$, and $X_{2d+1} = T$. The standard sub-Laplacian Δ on $\mathbb{H}^d$ is defined by

$$\Delta = -\sum_{l=1}^{2d} X_l^2.$$

Principal symbol on Heisenberg group

Set $\mathcal{A}_1 = C_0(\mathbb{H}^d)$ and $\mathcal{A}_2 = C^*(\{R_k\}_{k=1}^{2d})$, where Heisenberg Riesz transforms R_k are defined as $R_k = X_k \Delta^{-\frac{1}{2}}$. Set $\pi_1(f) = M_f$ and $\pi_2(g) = g$. These C^*-algebras satisfy the assumptions of the Abstract Principal Symbol Theorem. Hence, there exists a $*$-homomorphism

$$\mathrm{sym} : \Pi \to \mathcal{A}_1 \otimes_{\mathrm{min}} \mathcal{A}_2 = C_0(\mathbb{H}^d, \mathcal{A}_2)$$

such that

$$(\mathrm{sym}(\pi_1(f)))(p) = f(p), \quad (\mathrm{sym}(\pi_2(g)))(p) = g.$$

Let H be a Hilbert space and $B(H)$ be the $*$-algebra of all bounded operators on H. For every compact $A \in B(H)$, let $\mu(A)$ be the sequence of its singular values (taken with multiplicities).
We identify l_∞ with diagonal subalgebra in $B(H)$. Let $\mathcal{L}_{1,\infty}$ be the principal ideal generated by the sequence $\{\frac{1}{k+1}\}_{k\geq 0}$. Equivalently,

$$\mathcal{L}_{1,\infty} = \{A : \ \mu(k,A) = O(\frac{1}{k+1})\}.$$

We equip $\mathcal{L}_{1,\infty}$ with the natural quasi-norm

$$\|A\|_{1,\infty} = \sup_{k\geq 0}(k+1)\mu(k,A).$$

Clearly, $\mathcal{L}_{1,\infty}$ is a quasi-Banach space.

Traces on $\mathcal{L}_{1,\infty}$

Let $\mathcal{I}$ be an ideal in $B(H)$. Linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is called a trace if it is unitarily invariant. In other words,

$$\varphi(AB) = \varphi(BA), \quad A \in \mathcal{I}, \quad B \in B(H).$$

For example, let $\mathcal{I} = \mathcal{L}_{1,\infty}$. For a given ultrafilter ω, let

$$\mathrm{Tr}_\omega(A) = \lim_{n\to\omega} \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A), \quad 0 \leq A \in \mathcal{L}_{1,\infty}.$$

The functional Tr_ω happens to be additive on $\mathcal{L}_{1,\infty}^{+}$. Hence, it extends to a linear functional on $\mathcal{L}_{1,\infty}$. This linear functional is unitarily invariant and is, therefore, a (Dixmier) trace on $\mathcal{L}_{1,\infty}$.

1. Every Dixmier trace is positive
2. Every positive trace is continuous
3. There exist discontinuous traces
4. There exist positive traces which are not Dixmier traces
5. Every trace on $\mathcal{L}_{1,\infty}$ vanishes on $\mathcal{L}_1$ (and, hence, on finite rank operators)
6. There are $2^{2^{\mathbb{N}}}$ (Dixmier) traces on $\mathcal{L}_{1,\infty}$

Connes Trace Theorem on $\mathbb{R}^d$

Let Π be as on p.8. Operator $A \in \Pi$ is called compactly supported if there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $A = M_\phi A = AM_\phi$.

Theorem

For every (compactly supported) $A \in \Pi$, we have

$$\varphi(A(1+\Delta)^{-\frac{d}{2}}) = c_d \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{sym}(A).$$

Connes Trace Theorem on $\mathbb{H}^d$

Let Π be as on p.13. Operator $A \in \Pi$ is called compactly supported if there exists $\phi \in C_c^\infty(\mathbb{H}^d)$ such that $A = M_\phi A = AM_\phi$.
While the algebra $L_\infty(\mathbb{S}^{d-1})$ is finite, the algebra $\mathrm{VN}_{\mathrm{hom}}(\mathbb{H}^d)$ is not. In fact, it is $B(H) \otimes \mathbb{C}^2$. The algebra $\mathrm{VN}(\mathbb{H}^d)$ is $B(H)\bar{\otimes}L_\infty(\mathbb{R})$. It is equipped with the natural trace $\tau = \mathrm{Tr} \otimes r^d dr$.

Theorem

For every (compactly supported) $A \in \Pi$, we have

$$\varphi(A(1+\Delta)^{-\frac{d}{2}}) = c_d \tau(\mathrm{sym}(A) e^{-\Delta}).$$

Is principal symbol equivariant under diffeomorphisms?

Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Define a unitary mapping $U_\Phi : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ by setting

$$U_\Phi \xi = \mathrm{det}(J_\Phi)^{\frac{1}{2}} \cdot (\xi \circ \Phi).$$

Higson asked whether (a) $U_\Phi^{-1} A U_\Phi$ for every $A \in \Pi$ and (b) what is the principal symbol of $U_\Phi^{-1} A U_\Phi$?
The answer to both questions appears to be positive and plays the crucial role in defining the principal symbol on manifolds.

Principal symbol is equivariant under diffeomorphisms.

Let $\Xi_\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be defined by setting

$$\Xi_\Phi(t, s) = (\Phi^{-1}(t), J_\Phi^*(\Phi^{-1}(t))s), \quad t, s \in \mathbb{R}^d.$$

Let us view the functions on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ as homogeneous functions on $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem

Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Suppose Φ is affine outside some ball. For every compactly supported $A \in \Pi$, we have $U_\Phi^{-1} A U_\Phi \in \Pi$ and

$$\mathrm{sym}(U_\Phi^{-1} A U_\Phi) = \mathrm{sym}(A) \circ \Xi_\Phi.$$

Globalization theorem I

Definition

Let X be a manifold with an atlas $\{(\mathcal{U}_i, h_i)\}_{i\in\mathbb{I}}$. Let $\mathfrak{B}$ be the Borel σ-algebra on X and let ν be a countably additive measure on $\mathfrak{B}$. We say that $\{\mathcal{A}_i\}_{i\in\mathbb{I}}$ are local algebras in $B(L_2(X,\nu))$ if

1. elements of $\mathcal{A}_i$ are compactly supported in $\mathcal{U}_i$;
2. if $T \in \mathcal{A}_i$ is compactly supported in $\mathcal{U}_i \cap \mathcal{U}_j$, then $T \in \mathcal{A}_j$;

(plus some more assumptions).

Definition

Let $\mathcal{B}$ be a $*$-algebra. We say that $\{\mathrm{hom}_i : \mathcal{A}_i \to \mathcal{B}\}_{i\in\mathbb{I}}$ are local $*$-homomorphisms if $\mathrm{hom}_i = \mathrm{hom}_j$ *on* $\mathcal{A}_i \cap \mathcal{A}_j$ (plus some more assumptions).

Globalization theorem II

Definition

We say that $T \in \mathcal{A}$ if

1. for every $i \in \mathbb{I}$ and for every $\phi \in C_c(\mathcal{U}_i)$, we have $M_\phi T M_\phi \in \mathcal{A}_i$;
2. for every $\psi \in C(X)$, the commutator $[T, M_\psi]$ is compact;

Theorem

Let X be a compact manifold. $\mathcal{A}$ is a C^*-subalgebra in $B(L_2(X,\nu))$. There exists a unique $*$-homomorphism $\mathrm{hom} : \mathcal{A} \to \mathcal{B}$ such that $\mathrm{hom}|_{\mathcal{A}_i} = \mathrm{hom}_i$.

Principal symbol on compact manifolds

Borel measure ν on X is said to be a continuous density if $\nu \circ h_i^{-1}$ is absolutely continuous with respect to the Lebesgue measure for every $i \in \mathbb{I}$, its Radon-Nikodym derivative is continuous and does not vanish at any point.

Set $W_i f = f \circ h_i^{-1}$. Let Π_i consist of all A compactly supported in $\mathcal{U}_i$ such that $W_i T W_i^{-1} \in \Pi$.

Let H_i be the coordinate mappings of the cotangent bundle. We have $H_i \circ H_j^{-1} = \Xi_{h_i \circ h_j^{-1}}$. Set $\mathrm{sym}_i(A) = \mathrm{sym}(W_i A W_i^{-1}) \circ H_i$.

Definition

Algebras $\{\Pi_i\}_{i \in \mathbb{I}}$ and mappings $\{\mathrm{sym}_i\}_{i \in \mathbb{I}}$ satisfy the assumptions of Globalization Theorem. Denote the corresponding $\mathcal{A}$ by Π_X and the corresponding hom by sym_X. This is the principal symbol mapping on X.

Heisenberg diffeomorphisms

Let $N \subset \mathbb{H}^d$ be the hyper-plane orthogonal to $(0, \cdots, 0, 1)$. Consider the surface pN for every $p \in \mathbb{H}^d$. This surface happens to be a plane passing through p (otherwise we would consider its tangent plane at p). Consider the Euclidean shift N_p of the latter plane by p.

Definition

Diffeomorphism is called a Heisenberg one if $J_\Phi(p) : N_p \to N_{\Phi(p)}$ for every $p \in \mathbb{H}^d$.

Is principal symbol equivariant under Heisenberg diffeomorphism?

Let Π be the C^*-algebra on p.13. Let $\Phi : \mathbb{H}^d \to \mathbb{H}^d$ be the Heisenberg diffeomorphism. Define a unitary mapping $U_\Phi : L_2(\mathbb{H}^d) \to L_2(\mathbb{H}^d)$ by setting

$$U_\Phi \xi = \det(J_\Phi)^{\frac{1}{2}} \cdot (\xi \circ \Phi).$$

Higson asked whether (a) $U_\Phi^{-1} A U_\Phi$ for every $A \in \Pi$ and (b) what is the principal symbol of $U_\Phi^{-1} A U_\Phi$?
The answer to both questions appears to be positive and plays the crucial role in defining the principal symbol on contact manifolds.

Horizontal Jacobian of a Heisenberg diffeomorphism

Set

$$HJ_{\Phi} = (X_l \Phi_k)_{k,l=1}^{2d}.$$

Just like the Jacobian, the horizontal Jacobian satisfies the composition rule

$$HJ_{\Phi_1 \circ \Phi_2} = (HJ_{\Phi_1} \circ \Phi_2) \cdot HJ_{\Phi_2}.$$

We have

$$V_{\Phi}^{-1} X_j V_{\Phi} = \sum_{l=1}^{2d} M_{X_j \Phi_l \circ \Phi^{-1}} X_l, \quad 1 \leq j \leq 2d,$$

where $V_{\Phi}\xi = \xi \circ \Phi$.

Symplectic group appears

Matrix S is called symplectic if $S^*\Omega S = \Omega$. Here, $\Omega = \sum_{k=1}^{d} E_{k,k+d} - E_{k+d,k}$. Symplectic matrices form a group denoted by $\mathrm{Sp}(2d, \mathbb{R})$.

Theorem

Horizontal Jacobian at every point is a scalar multiple of symplectic matrix.

In what follows, we may assume (for simplicity of notations) that horizontal Jacobian is everywhere symplectic.

Principal symbol is equivariant under Heisenberg diffeomorphisms

If S is symplectic matrix, then we set $W_S\xi = \xi \circ S^*$. Define the automorphism π_S of $\mathrm{VN}(\mathbb{H}^d)$ by setting $\pi_S(A) = W_S^{-1}AW_S$.
If $S : \mathbb{H}^d \to \mathrm{Sp}(2d, \mathbb{R})$, then we define the automorphism π_S of $L_\infty(\mathbb{H}^d)\bar{\otimes}\mathrm{VN}(\mathbb{H}^d)$ by setting

$$(\pi_S A)(p) = \pi_{S(p)}(A(p)), \quad p \in \mathbb{H}^d.$$

Theorem

Let $\Phi : \mathbb{H}^d \to \mathbb{H}^d$ be a Heisenberg diffeomorphism affine outside some ball. Let $A \in \Pi$ be compactly supported. We have $U_\Phi^{-1}AU_\Phi \in \Pi$ and

$$\mathrm{sym}(U_\Phi^{-1}AU_\Phi) = \pi_{HJ_\Phi}(\mathrm{sym}(x)) \circ \Phi^{-1}.$$

Principal symbol on contact manifolds

Let X be a manifold with an atlas $\{(\mathcal{U}_i, h_i)\}_{i\in\mathbb{I}}$. We say that the atlas is a Heisenberg one if $h_j \circ h_i^{-1}$ is a Heisenberg diffeomorphism. Manifold is called contact if there is a Heisenberg atlas.
Let X be a compact contact manifold. Equivariance Theorem from p.28 and Globalization Theorem from p.22 deliver the principal symbol mapping for the contact manifolds.

Where does principal symbol belong?

The co-domain of the principal symbol is the C^*-algebra of the continuous sections of a certain bundle E_{hom} of C^*-algebras. Each level of the bundle is the same — $C^*(\{R_k\}_{k=1}^{2d})$.
This is a sub-algebra of the algebra of measurable sections of a certain bundle E of von Neumann algebras. Each level of the bundle is the same — $\mathrm{VN}(\mathbb{H}^d)$.
The latter is isomorphic, as a von Neumann algebra, to $L_\infty(\mathbb{H}^d)\bar{\otimes}\mathrm{VN}(\mathbb{H}^d)$. It carries a natural trace $\Lambda = \int_{\mathbb{H}^d} \otimes \tau$.
The latter algebra is infinite and we need some integrable weight to make the Connes Trace Formula true. This weight is delivered by the sub-Riemannian structure on X.

Sub-Riemannian structure on contact manifolds

Sub-Riemannian structure on X is a collection of smooth mapping $G_i : \mathcal{U}_i \to \mathrm{GL}^+(2d, \mathbb{R})$. For any $i, j \in \mathbb{I}$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$, we have

$$G_j(t) = HJ^*_{h_i \circ h_j^{-1}}(h_j(t)) \cdot G_i(t) \cdot HJ_{h_i \circ h_j^{-1}}(h_j(t)), \quad t \in \mathcal{U}_i \cap \mathcal{U}_j.$$

There exists an unbounded self-adjoint positive operator q_X affiliated to $L_\infty(E)$, such that

$$(q_X)_i \circ h_i^{-1} = -\sum_{k_1, k_2 = 1}^{2d} (g_i^{-1})_{k_1, k_2} \otimes X_{k_1} X_{k_2}, \quad i \in \mathbb{I}.$$

Connes Trace Theorem for contact sub-Riemannian manifolds

Theorem

Let (X, G) be a compact contact sub-Riemannian manifold. For every $A \in \Pi_X$, we have

$$\varphi(A(1+\Delta_{G,\nu})^{-d-1}) = c_d \Lambda(\mathrm{sym}(A) \cdot e^{-q_X}).$$

Here, $\Delta_{G,\nu}$ is the sub-Laplace-Beltrami operator.

Thank you for your attention