Marcinkiewicz interpolation for non-commutative maximal functions

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Motivation from ergodic theory

(X, μ) a measure space
T a measure preserving transformation of X

Ergodic averages

Let $f \in L_0(X)$ and $n \geq 0$, define

$$A_n(f) := \frac{1}{n+1} \sum_{k=0}^{n} f \circ T^k.$$

Maximal inequality

Let $f \in L_1(X)$ and $\lambda > 0$. Then

$$\mu\left(\left\{\sup_{n \geq 0} |A_n(f)| > \lambda\right\}\right) \lesssim \lambda^{-1} \|f\|_1 .$$

This is a crucial ingredient to prove the convergence of $A_n(f)$

Weak L_p-space

- Let $f \in L_0(X)$ and $p \in (0, \infty)$ then

$$\|f\|_{p,\infty} := \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\}).$$

- Let S be a (quasi-)linear operator. Then S is said to be of weak type (p, p) if there exists a constant $C > 0$ such that

$$\|S(f)\|_{p,\infty} \leq C \|f\|_p \quad \forall f \in L_p(X).$$

- Equivalently,

$$\forall f \in L_p(X) \; \forall \lambda > 0 \quad \mu(\{|S(f)| > \lambda\}) \leq C\lambda^{-1} \|f\|_p .$$

- The maximal inequality of the previous slide can be refrased as the weak $(1, 1)$ boundedness of the operator $S = \sup_{n \geq 0} A_n$.

From weak to strong inequalities

An operator S is said to be of strong type (p, p) if there is a constant $C > 0$ such that

$$\|S(f)\|_p \leq C \|f\|_p .$$

Theorem (Marcinkiewicz)

Let $p_0 < p < p_1 \in (0, \infty]$. Assume that S is of weak type (p_0, p_0) with constant C_0 and of weak type (p_1, p_1) with constant C_1. Then S is of strong type (p, p) with constant

$$C_p \leq C C_0^{1-\theta} C_1^{\theta} \left(\frac{1}{p} - \frac{1}{p_1}\right)^{-1} \left(\frac{1}{p_0} - \frac{1}{p}\right)^{-1}$$

where θ is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$ and C is universal.

- It applies once again to the supremum of ergodic averages

The framework

$(\mathcal{M}, \tau)$ - $\mathcal{M}$ von Neumann algebra and τ a n.s.f trace

- For $x \in L_p(\mathcal{M})$,

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

- For $x \in L_{p,\infty}(\mathcal{M})$,

$$\|x\|_{p,\infty} = \sup_{\lambda>0} \lambda \tau(\chi_{[\lambda,\infty)}(|x|))^{1/p}.$$

- $L_0(\mathcal{M})$ to be though of as the space of measurable functions
- For $x \in L_0(\mathcal{M})$ and $t > 0$

$$\mu(x,t) = \sup_{\tau(1-e)\leq t} \|exe\|_\infty,$$

  where e is a projection.
- $\|x\|_p = \|\mu(x)\|_{L_p(\mathbb{R})}$. True for any symmetric space.

Back to ergodic theory

T a positive trace preserving contraction on $\mathcal{M}$

Ergodic averages

Let $x \in L_0(\mathcal{M})$ and $n \geq 0$, define

$$A_n(f) := \frac{1}{n+1} \sum_{k=0}^{n} T^k(x).$$

Lance proposed a formulation in terms of measures or states

Maximal inequality (Yeadon)

Let $x \in L_1(\mathcal{M})$ and $\lambda > 0$, then there exists a projection $e \in \mathcal{M}$ such that

$$\lambda \tau(1-e) \leq \|x\|_1 \quad \text{and} \quad \|eA_n(x)e\|_\infty \leq \lambda \quad \forall n \geq 0.$$

e is to be thought of as $\left\{\sup_{n \geq 0} |A_n(x)| \leq \lambda\right\}$

Noncommutative weak type

Weak maximal norm

Let $p \in (0, \infty)$. Define

$$\|(x_n)_{n\geq 0}\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} = \sup_{\lambda>0} \inf_{e} \lambda(\tau(1-e))^{1/p}$$

where the sup is taken over all projections e such that $\|ex_ne\|_\infty \leq \lambda \ \forall n \geq 0$.

- We say that $S = (S_n)_{n\geq 0}$ a sequence of linear maps is of weak type (p, p) if it sends $L_p(\mathcal{M})$ to $\Lambda_{p,\infty}(\mathcal{M}, \ell_\infty)$.
- Can be generalized to any symmetric space

$$\mu_\Lambda((x_n)_{n\geq 0}, t) = \inf_{\tau(1-e)\leq t} \sup_{n\geq 0} \|ex_ne\|_\infty .$$

By definition, $\|(x_n)_{n\geq 0}\|_{\Lambda_{p,\infty}(\mathcal{M},\ell_\infty)} = \|\mu_\Lambda((x_n)_{n\geq 0}\|_{p,\infty}$.

Noncommutative weak type

The weak type inequality offers a control over a diagonal of the x_n's. Assume that $\|(x_n)_{n\geq 0}\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty)} \leq C$.

- Let $k \in \mathbb{Z}$ and choose a projection e_k such that

$$\tau(1-e_k) \leq 2^{-k} \cdot 2C \quad \text{and} \quad \|e_k x_n e_k\|_\infty \leq 2^k \quad \forall n \geq 0.$$

- Considering $e'_k = \bigwedge_{i\geq k} e_i$, the e_k's may be assumed to be increasing, losing a factor 2 in the previous estimate.
- Set $q_k = e_{k+1} - e_k$ then the q_k's are disjoint and for any $k \in \mathbb{Z}$,

$$\tau(q_k) \leq 2^{-k} \cdot 4C \quad \text{and} \quad \|q_k x_n q_k\|_\infty \leq 2 \cdot 2^k \quad \forall n \geq 0.$$

  And for all $n \geq 0$

$$\left\| \sum_{k\in\mathbb{Z}} q_k x_n q_k \right\|_{1,\infty} \leq 4C.$$

Noncommutative strong type

Definition due to Junge and Pisier:

Maximal p-norm

Let $p \in (0, \infty)$,

$$\|(x_n)_{n\geq 0}\|_{L_p(\mathcal{M},\ell_\infty)} := \inf_{x_n = a y_n b} \|a\|_{2p} \|b\|_{2p} \sup_{n\geq 0} \|y_n\|_\infty .$$

- The corresponding space is denoted by $L_p(\mathcal{M}, \ell_\infty)$ to be thought of as the space of sequences (x_n) such that $\|\sup_n |x_n|\|_p < \infty$.
- If for all n, $x_n \geq 0$ then

$$\|(x_n)_{n\geq 0}\|_{L_p(\mathcal{M},\ell_\infty)} = \inf \left\{ \|a\|_p : x_n \leq a \quad \forall n \geq 0 \right\}.$$

- If $p \in [1, \infty)$, the infimum above is achieved uniquely but on an element depending on p!
- Many pathologies outside of positive sequences: $(x_n)_{n\geq 0} \in L_p(\mathcal{M}, \ell_\infty)$ does not imply that $(|x_n|)_{n\geq 0} \in L_p(\mathcal{M}, \ell_\infty)$.

Noncommutative strong type

Let $p \in (0, \infty)$.

Definition

A family of maps $S_n : L_1(\mathcal{M}) + L_\infty(\mathcal{M}) \to L_0(\mathcal{M})$ is of strong (p, p)-type if for all $x \in L_p(\mathcal{M})$

$$\|(T_n(f))\|_{L_p(\mathcal{M}, \ell_\infty)} \leq C\|f\|_p.$$

- If T_n is defined only on the positive cone, we relax the definition for positive f only.

Asymmetric norms

- More recently, people consider asymmetric versions $0 \leq \theta \leq 1$:
  $L_p(\mathcal{M}, \ell_\infty^\theta) =$
  $$\{(x_n)_{n\geq 0} | \exists a \in L_{p/\theta}(\mathcal{M}), b \in L_{p/1-\theta}, u_n \in \mathcal{M}, \|u_n\| \leq 1,\ x_n = a u_n b\}$$
  and
  $$\|(x_n)\|_{L_p(\ell_\infty^\theta)} = \inf \|a\|_{p/\theta} \|b\|_{p/1-\theta}$$
- These are Banach spaces if $p \geq 2 \max\{\theta, 1-\theta\}$.
- When $\theta = 0$ this is $L_p(\ell_\infty^c)$ and when $\theta = 1$, $L_p(\ell_\infty^r)$.
  $$L_p(\mathcal{M}, \ell_\infty^c) = \{(x_n) | b \in L_p, u_n \in \mathcal{M}, \|u_n\| \leq 1,\ x_n = u_n b\}$$
- One can also define θ-asymmetric strong (p,p)-type
  $$\|(S_n(x))\|_{L_p(\mathcal{M}, \ell_\infty^\theta)} \leq C \|x\|_p.$$

Maximal spaces and interpolation

Fundamental result by Junge

$(L_p(\mathcal{M}, \ell_\infty^\theta))_{p \geq 2\max\{\theta, 1-\theta\}}$ form an interpolation scale for the complex method.

They are good for complex interpolation, but not for real interpolation:

Bad news from Junge and Xu

$$(L_1(\mathcal{M}, \ell_\infty), L_\infty(\mathcal{M}, \ell_\infty))_{\frac{1}{p}, p} \neq L_p$$

The nc Marcinkiewicz theorem

Junge and Xu

Let $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)$ be a sequence of maps from $L_{p_0}^+ + L_{p_1}^+$ into L_0^+. Assume that S is subadditive in the sense that $S_n(x+y) \leq S_n(x) + S_n(y)$ for all $n \geq 0$. If S is of weak type (p_0, p_0) with constant C_0 and of type (p_1, p_1) type (p_1, p_1) with constant C_1, then for any $p_0 < p < p_1$, S is of strong type (p, p) with constant C_p satisfying

$$C_p \leq C C_0^{1-\theta} C_1^{\theta} \left(\frac{1}{p_0} - \frac{1}{p}\right)^{-2} \left(\frac{1}{p} - \frac{1}{p_1}\right)^{-1},$$

where θ is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$ and C is universal.

Applications

Corollary : The Maximal Doob inequality (Junge)

Consider a filtration $\mathcal{M}_0 \subset ... \subset \mathcal{M}_k \subset ...$ of $\mathcal{M}$ with conditional expectations $\mathbb{E}_n$, for $1 < p < \infty$, there is C_p such that for any $x \in L_p^+$, there exists $z \in L_p$ with

$$0 \leq \mathbb{E}_n x \leq z, \qquad \|z\|_p \leq c_p \|x\|_p$$

- It is also known from another Junge and Xu work that c_p behaves like $1/(p-1)^2$ when p is close to 1. Thus the constant in the nc Marcinkiewicz theorem is optimal.
- Also applies to ergodic averages!

Many generalizations

- Extensions to more general symmetric function spaces by Bekjan, Chen and Osekowski.
  Still the restriction to strong (p_1, p_1)-type.
- An almost optimal form by Dirksen
  He got the weak (p_1, p_1)-type hypothesis and the result for all interpolation spaces.

The nc Marcinkiewicz theorem

After Dirksen, simple statement

Let $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)$ be a sequence of maps from $L_{p_0}^+ + L_{p_1}^+$ into L_0^+. Assume that S is subadditive in the sense that $S_n(x+y) \leq S_n(x) + S_n(y)$ for all $n \geq 0$. If S is of weak type (p_0, p_0) with constant C_0 and of weak type (p_1, p_1) with constant C_1, then for any $p_0 < p < p_1$, S is of strong type (p, p) with constant C_p satisfying

$$C_p \leq C C_0^{1-\theta} C_1^{\theta} \Big(\frac{1}{p_0} - \frac{1}{p} \Big)^{-2} \Big(\frac{1}{p} - \frac{1}{p_1} \Big)^{-2},$$

where θ is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$ and C is universal.

The main novelty

We can recover Dirksen statement quite easily with variants.
For simplicity, we state the following theorem for linear maps.

Theorem (Ricard, C.)

Let $1 \leq p_0 < p_1 \leq \infty$. Let $S = (S_n)_{n\geq 0}$ be a sequence of linear positive maps from $L_{p_0} + L_{p_1}$ into L_0. Assume that S of weak type (p_0, p_0) with constant C_0 and of weak type (p_1, p_1) with constant C_1. Then there is are constants C_{θ,p_0,p_1} such that for any $x \in L_{p_0} \cap L_{p_1}$, there exist $a \in (L_1 + L_\infty)^+$ and contractions $u_n \in M$ indep of θ with

$$S_n(x) = a u_n a, \qquad \|a\|_{2p}^2 \leq C_{\theta,p_0,p_1} \|x\|_p.$$

Moreover, for any $\varepsilon > 0$,

$$C_{\theta,p_0,p_1} \leq C_\varepsilon C_0^{1-\theta} C_1^{\theta} \Big(\frac{1}{p_0} - \frac{1}{p}\Big)^{-(2+\varepsilon)} \Big(\frac{1}{p} - \frac{1}{p_1}\Big)^{-(2+\varepsilon)}.$$

Application to martingale theory

Corollary : a more explicit Maximal Doob inequality

Consider a filtration $\mathcal{M}_0 \subset ... \subset \mathcal{M}_k \subset ...$ of $\mathcal{M}$ (finite) with conditional expectations $\mathbb{E}_n$. There are constants C_p, $1 < p < \infty$, such that for any let $x \in L_\infty^+$, there exists $z \in L_1^+$ with

$$0 \leq \mathbb{E}_n x \leq z, \qquad \forall\, 1 < p < \infty,\ \|z\|_p \leq C_p \|x\|_p$$

- This means that one can construct an element z playing the role of $\sup_{n\geq 0} \mathbb{E}_n(x)$.
- The construction becomes explicit using Cuculescu's projections.
- Unfortunately, there is a price to pay on the constants.

The asymmetric version

Theorem

Let $1 \leq p_0 < p_1 \leq \infty$ and $0 < \gamma < 1$. Let $S = (S_n)$ be a sequence of linear positive maps from $L_{p_0} + L_{p_1}$ into $L_1 + L_\infty$. Assume that S of weak type (p_0, p_0) with constant C_0 and of weak type (p_1, p_1) with constant C_1. Then S is of strong γ-asymmetric type (p, p) for $p_0 < p < p_1$ and $p > 2\max\{\gamma, 1-\gamma\}$. Moreover, there are constants C_{θ,p_0,p_1} such that for any $x \in L_{p_0} \cap L_{p_1}$ and $a \in (L_1 + L_\infty)^+$ and contractions $u_n \in M$ indep of θ with

$$S_n(x) = a^\gamma u_n a^{1-\gamma}, \qquad \|a\|_p \leq C_{\theta,p_0,p_1}\|x\|_p.$$

This is false if $p = 2\max\{\gamma, 1-\gamma\}$ in general.

The good Lemma

How to majorize positive elements by diagonal type ones

Let $N \geq 0$ and $q_0, \ldots, q_N$ be disjoint projections in $\mathcal{M}$, set $q = \sum_{i=0}^{N} q_i$. Let $d_0, \ldots, d_N > 0$. Then for any $x \in L_0(\mathcal{M})^+$,

$$qxq \leq \left(\sum_{k=0}^{N} 1/d_k \right) \sum_{k=0}^{N} d_k q_k x q_k .$$

The proof is simple. Assume $\sum_{k=0}^{N} 1/d_k = 1$:

- Note that the matrix $A = \sum_{k=0}^{n} d_k e_{k,k} + \sum_{k,l=0}^{N} e_{k,l}$ is positive
- Then A admits a square root $B = (b_{i,j})_{1 \leq i,j \leq N}$ (B real symmetric)
- Note that

$$\sum_{k=0}^{N} d_k q_k x q_k - qxq = \sum_{l=0}^{N} \left(\sum_{i=0}^{N} b_{i,l} q_i \right) x \left(\sum_{j=0}^{N} b_{j,l} q_j \right) \geq 0.$$

Some ideas case $p_0 = 1$, $p_1 = \infty$

Assume that $x = p$ is a projection with $\tau(p) = t$

- Recall that we can find disjoint projections q_k, $k \geq 0$ such that

$$\tau(q_k) \leq C2^k t \quad \text{and} \quad \|q_k S_n(p) q_k\|_\infty \leq 2^{-k} \quad \forall n \geq 0.$$

- By the good lemma, for any sequence $(d_k)_{k\geq 0}$ of positive numbers, we have

$$S_n(x) \leq \left(\sum_{k=0}^{\infty} 1/d_k\right) \sum_{k=0}^{\infty} d_k q_k x q_k.$$

- It remains only to choose d_k wisely. Set for example $d_k = 1 + k^2$ and $z = \sum_{k=0}^{\infty} d_k q_k x q_k$. Note that

$$\|z\|_p^p \lesssim \sum_{k=0}^{\infty} (1 + k^2) 2^{kp} t^p 2^{-k} = t^p \sum_{k=0}^{\infty} (1 + k^2) 2^{k(1-p)}.$$

Ideas of proof

If $x \geq 0$ is no longer a projection then $x = \sum_{j\in\mathbb{Z}} 2^j p_j$ with p_j projections.

- $\mu(x) \approx \sum_{j\in\mathbb{Z}} 2^j \mu(p_j)$
- construct z_j thanks to the previous slide and $z = \sum_{j\in\mathbb{Z}} 2^j z_j$
- $S_n(x) \leq z$ for all $n \geq 0$
- $\mu(z)$ is dominated by $\sum_{j\in Zb} 2^j \mu(z_j)$ which is dominated by

$$\sum_{j\in Zb}\sum_{k\geq 0} 2^j 2^{-k} d_k \mu q_{j,k} \leq \sum_{k\geq 0} 2^{-k} \sum_{j\in\mathbb{Z}} d_k D_{2^k}(2^j \mu(p_j)) \leq \sum_{k\geq 0} d_k 2^{-k} D_{2^k}(\mu(x))$$

where D_{2^k} is the dilation by a factor 2^k.

Beyond positivity ?

The Marcinkiewicz theorem is simply false if S_n are not assumed to be positive !

Natural Conjecture

If S is of strong type (∞, ∞) and weak $(1,1)$, then S is bounded from L_p to $L_p(\ell_\infty^c) + L_p(\ell_\infty^r)$ for $2 \leq p < \infty$?
i.e.

$$S_n(x) = zu_n + v_n z$$

for some $z \in L_p$ with $\|z\|_p \leq C_p \|x\|_p$ and contractions u_n, v_n

Ok if x is a projection !

Beyond positivity?

One can try to use interpolation techniques but

Big problem

The spaces $(L_p(\mathcal{M}, \ell_\infty^c))_{p\geq 2}$ don't form a scale for the real interpolation method !

Junge and Xu have a partial result for positive cones which is at the heart of their proof of the Marcinkiewicz theorem.

Best try at the conjecture

Let $1 < p < q < \infty$. If S is of strong type (∞, ∞) and weak $(1,1)$, then

$$S_n(x) = zu_n + v_n z$$

for some $z \in L_p$ with $\|z\|_p \leq C_{p,q} \|x\|_q^{q/p} \|x\|_\infty^{1-q/p}$ and contractions u_n, v_n.

Thank you!