Cesàro summability of double Fourier series on quantum tori

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Outline

Classical Fourier series: one dimensional setting

• Let $f \in L_1(\mathbb{T})$. The partial sum of the Fourier series of f is defined by

$$S_n(f)(x) = \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx}, \quad n \in \mathbb{N},$$

where $\widehat{f}(k)$ is the k-th Fourier coefficient of f:

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(y)e^{-iky}dy. \tag{1}$$

Theorem (Carleson 1966 Acta Math.; Hunt 1968)

$$\|\sup_n |S_n(f)|\|_{L_p(\mathbb{T})} \leq c_p \|f\|_{L_p(\mathbb{T})}, \quad 1 < p < \infty.$$

Consequently, $S_n(f) \to f$ a.e. provided $f \in L_p(\mathbb{T})$, answering Lusin's conjecture in 1915.

One dimensional setting

Theorem (Kolmogorov 1923)

There exists $g \in L_1(\mathbb{T})$ *such that* $S_n(g)$ *diverges almost everywhere.*

• For $L_1(\mathbb{T})$ functions, people consider instead the so-called Cesàro means (or Fejér means) which are defined as follows: for $f \in L_1(\mathbb{T})$ and $N \in \mathbb{N}$,

$$\sigma_N(f) := \frac{1}{N} \sum_{n=0}^{N-1} S_n(f).$$

Theorem (Fejér 1904; Lebesgue 1905)

Let $f \in L_1(\mathbb{T})$. *Then*

$$\sigma_N(f) \longrightarrow f, \quad a.e.$$

Two dimansional setting: partial sums

For $f \in L_1(\mathbb{T}^2)$, the rectangular partial sum $S_{m,n}(f)$ is defined by

$$S_{m,n}(f)(x,y) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} \widehat{f}(k,l) e^{i(kx+ly)}, \quad -\pi \leq x, y \leq \pi,$$

where

$$\widehat{f}(k,l) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-i(kx+ly)} dxdy.$$

Theorem (Tevzadze 1970)

$$S_{n,n}(f) \longrightarrow f, \quad a.e., \quad f \in L_2(\mathbb{T}^2).$$

Theorem (Fefferman 1971; Sjolin 1971)

$$S_{n,n}(f) \longrightarrow f, \quad a.e., \quad f \in L_p(\mathbb{T}^2), \quad 1 < p < \infty.$$

Cesàro means

Theorem (Fefferman 1971, BAMS)

There exists a continuous function f defined on $\mathbb{T}^2$ such that

$$\lim_{m,n\to\infty} S_{m,n}(f)(x) \quad \textit{exists nowhere.}$$

- For $f \in L_1(\mathbb{T}^2)$, the Cesàro means ($(C,1)$ means) are defined by

$$\begin{aligned}\sigma_{m,n}(f)(x,y) &:= \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} S_{k,l}(f)(x,y) \\ &= \sum_{k=-m}^{m} \sum_{l=-n}^{n} \left(1 - \frac{|k|}{m+1}\right)\left(1 - \frac{|l|}{n+1}\right) \widehat{f}(k,l) e^{i(kx+ly)}.\end{aligned}$$

Cesàro means

Theorem (Sake 1934, Fund. Math.)

There is a function $f \in L_1(\mathbb{T}^2)$ *such that*

$$\limsup_{m,n\to\infty} |\sigma_{m,n}(f)| \overset{a.e.}{=} \infty.$$

Theorem (Jessen-Marcinkiewicz-Zygmund 1935, Fund. Math.)

$$\sigma_{m,n}(f) \overset{a.e.}{\to} f, \quad \min\{m,n\} \to \infty, \quad f \in L \log L(\mathbb{T}^2).$$

- $L_p(\mathbb{T}^2) \subset L \log L(\mathbb{T}^2) \subset L_1(\mathbb{T}^2)$, $1 < p < \infty$.
- The space $L \log L(\mathbb{T}^2)$ is sharp in the sense of almost convergence; see Marcinkiewicz-Zygmund 1939.

Cesàro means

Theorem (Marcinkiewicz-Zygmund 1939, Fund. Math.)

For fixed $\beta > 0$, *define*

$$\Sigma_\beta = \{(m,n) \in \mathbb{N}^2 : m/n \le \beta, n/m \le \beta\}. \qquad (2)$$

There exists a positive constant c_β *depending only on* β *such that, for any* $f \in L_1(\mathbb{T}^2)$,

$$\| \sup_{(m,n)\in\Sigma_\beta} |\sigma_{m,n}(f)| \|_{L_{1,\infty}(\mathbb{T}^2)} \le c_\beta \|f\|_{L_1(\mathbb{T}^2)}.$$

As a consequence, for each $f \in L_1(\mathbb{T}^2)$, *the Cesàro means* $\sigma_{m,n}(f)$ *converges almost everywhere to* f *whenever* $\min\{m,n\} \to \infty$ *with* $(m,n) \in \Sigma_\beta$.

- In 1996, Weisz proved that the operator $\sup_{(m,n)\in\Sigma_\beta} |\sigma_{m,n}|$ is bounded from the Hardy space $H_{p,q}(\mathbb{T}^2)$ to $L_{p,q}(\mathbb{T}^2)$ provided $3/4 < p < \infty$ and $0 < q \le \infty$.

Noncommutative trigonometric Fourier series

Our aim: Can we obtain the Marcinkiewicz-Zygmund theorem for noncommutative trigonometric Fourier series in quantum tori setting?

Z. Chen, Q. Xu, Z. Yin, Harmonic analysis on quantum tori. Comm. Math. Phys. 322 (2013), 755–805.

Noncommutative trigonometric Fourier series

- Transference theorem:

operator-valued $\rightarrow$ quantum tori setting

Z. Chen, Q. Xu, Z. Yin, Harmonic analysis on quantum tori. Comm. Math. Phys. 322 (2013), 755–805.

Fourier series: operator-valued setting

• In the sequel, always let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra.
• Noncommutative Lebesgue spaces: $L_p(\mathcal{M})$, $1 \le p \le \infty$

$$L_\infty(\mathcal{M}) = \mathcal{M}, \quad \|x\|_p = [\tau(|x|^p)]^{1/p}.$$

Examples and notes:

(i) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $\mathcal{M} = L_\infty(\Omega)$, $\tau = \int_\Omega$

(ii) $\mathcal{M} = B(\ell_2)$, $\tau = \mathrm{Tr}$, the usual trace for matrices.

(iii) The tensor von Neumann algebra $(\mathcal{N}, \tau)$:

$$\mathcal{N} = L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M}, \quad \varphi = \tau \otimes \int_{\mathbb{T}^2}.$$

(iv) $L_p(\mathcal{N}) = L_p(\mathbb{T}^2, L_p(\mathcal{M}))$, $1 \le p < \infty$.

• For $f \in L_1(L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M})$, we mean

$$f : \mathbb{T}^2 \to \mathcal{M}.$$

• In the case $f \in L_1(L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M})$,

$$\widehat{f}(k,l) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-i(kx+ly)}dxdy \in \mathcal{M}.$$

• In the case $f \in L_1(L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M})$, $S_{m,n}(f)$ and $\sigma_{m,n}(f)$ can be similarly defined:

$$S_{m,n}(f)(x,y) = \sum_{k=-m}^{m}\sum_{l=-n}^{n}\widehat{f}(k,l)e^{i(kx+ly)}$$

Fourier series: operator-valued setting

- In the sequel, we always let

$$\mathcal{N} = L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M}.$$

Theorem (Chen-Xu-Yin 2013)

Let $f \in L_1(\mathcal{N})$. Then

$$\|(\sigma_{n,n}(f))_n\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c\|f\|_{L_1(\mathcal{N})}.$$

Consequently, $\sigma_{n,n}(f) \to f$ bilaterally almost uniformly provided $f \in L_1(\mathcal{N})$.

Note: the case for $\sigma_{m,n}(f)$ is unknown.

Noncommutative maximal functions

- Let $x = (x_n)_{n \in B}$

$$\|x\|_{\Lambda_{1,\infty}(\mathcal{M},\ell_\infty))} = \sup_{t>0} \inf_{e \in \mathcal{P}(\mathcal{M})} \{t\tau(\mathbf{1} - e) : \sup_n \|ex_n e\|_{L_\infty(\mathcal{M})} \leq t\}$$

- Noncommutative maximal function "makes no sense";

see Junge and Xu 2007, JAMS

- In the classical setting,

$$e = \chi_{\{\sup_n |x_n| > \lambda\}}$$

Theorem (classical result)

Let $f \in L_1(\mathbb{T}^2)$. Then

$$\| \sup_n |\sigma_{n,n}(f)| \|_{L_{1,\infty}(\mathbb{T}^2)} \leq c \|f\|_{L_1(\mathbb{T}^2)}.$$

M. Junge and Q. Xu, Noncommutative maximal ergodic theorems. J. Amer. Math. Soc. 20 (2007), no. 2, 385–439.

Noncommutative pointwise convergence

- Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. Consider $(x_k)_{k\geq 1} \subset \mathcal{M}$ and $x \in \mathcal{M}$. We say $(x_k)_{k\geq 1}$ converges to x bilaterally almost uniformly (b.a.u. in short) if for any $\varepsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(1-e) < \varepsilon, \quad \lim_{k\to\infty} \|e(x_k - x)e\|_{L_\infty(\mathcal{M})} = 0.$$

- almost uniform convergence:

$$\tau(1-e) < \varepsilon, \quad \lim_{k\to\infty} \|(x_k - x)e\|_{L_\infty(\mathcal{M})} = 0.$$

- By Egorov's theorem, in the commutative setting with $\tau(1) < \infty$, the above convergence is just almost everywhere convergence.

M. Junge and Q. Xu, Noncommutative maximal ergodic theorems. J. Amer. Math. Soc. 20 (2007), no. 2, 385–439.

Cesàro means $L\log^2 L(\mathcal{N})$

Theorem (Hong-Sun 2018, JFA)

$$\|(\sigma_{m,n}(f))\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c\|f\|_{L\log^2 L(\mathcal{N})}.$$

Consequently,

$$\sigma_{m,n}(f) \xrightarrow{b.a.u.} f, \quad m,n\to\infty.$$

1. Let $\log_+(t) = \max\{0, \log(t)\}$, $t > 0$. Considering the Orlicz function

$$\Phi_p(t) = t(1+\log_+ t)^p, \quad 0 < p < \infty,$$

   the corresponding noncommutative Orlicz space is denoted by $L\log^p L(\mathcal{M})$.
2. $L\log^2 L \subset L\log L$
3. Conde-Alonso, González-Pérez and Parcet 2020, Forum Math. Sigma

Why is $L\log^2 L(\mathcal{N})$

1. for $p > 1$, $\|(\sigma_n(f))\|_{L_p(L_\infty(\mathbb{T})\bar{\otimes}\mathcal{M},\ell_\infty)} \leq c_p\|f\|_{L_p(L_\infty(\mathbb{T})\bar{\otimes}\mathcal{M})}$,
$$c_p = cp^2/(p-1)^2, \quad \text{this is a sharp order as } p \to 1.$$
2. $\|(\sigma_n(f))\|_{L_1(L_\infty(\mathbb{T})\bar{\otimes}\mathcal{M},\ell_\infty)} \leq c\|f\|_{L\log^2 L(L_\infty(\mathbb{T})\bar{\otimes}\mathcal{M})}$.
3. combining the last two items, we can get the above theorem.

M. Junge and Q. Xu, Noncommutative maximal ergodic theorems. J. Amer. Math. Soc. 20 (2007), no. 2, 385–439.

Y. Hu, Noncommutative extrapolation theorems and applications. Illinois J. Math. 53 (2009), no. 2, 463–482.

G. Hong and M. Sun, *Noncommutative multi-parameter Wiener-Wintner type ergodic theorem*, J. Funct. Anal. **275** (2018), no. 5, 1100–1137.

More comments related to $L \log L$ type spaces

Finding the largest r.i. subspace Y of $L_1(\mathbb{T})$ such that

$$S_n(f), \quad \text{or} \quad S_{n_k}(f) \to f, \quad f \in Y.$$

- Lie, Victor The pointwise convergence of Fourier series (II). Strong L1 case for the lacunary Carleson operator. Adv. Math. 357 (2019), 106831, 84 pp
- Lie, Victor Pointwise convergence of Fourier series (I). On a conjecture of Konyagin. J. Eur. Math. Soc. (JEMS) 19 (2017), no. 6, 1655–1728.

Cesàro means in L_p, $p > 1$

Theorem (Hong-Wang-Wang, 2019)

$$\|(\sigma_{m,n}(f))_{m,n\geq 1}\|_{L_p(\mathcal{N},\ell_\infty)} \leq c_p \|f\|_{L_p(\mathcal{N})}, \quad 1 < p < \infty.$$

G. Hong, S. Wang, and X. Wang, Pointwise convergence of noncommutative Fourier series, arxiv, 2019.

Noncommutative Fourier series: recent results

- X. Lai, Sharp estimates of noncommutative Bochner-Riesz means on two-dimensional quantum tori. Comm. Math. Phys. 390 (2022), 193–230.
- S. Wang, Lacunary Fourier series for compact quantum groups. Comm. Math. Phys. 349 (2017), no. 3, 895–945.
- Conde-Alonso, Gonzalez-Perez and Parcet Noncommutative strong maximals and almost uniform convergence in several directions. Forum Math. Sigma 8 (2020)
- C. Pang, M. Wang, and B. Xu, Noncommutative pointwise convergence of orthogonal expansions of several variables. Math. Nachr. 294 (2021), no. 8, 1559–1577.
- G. Hong, S. Wang, and X. Wang, Pointwise convergence of noncommutative Fourier series, arxiv, 2019.
- . . .

Main result 1

Conclusion:

1. $\sigma_{m,n}$ in L_1, only for $\sigma_{n,n}$;
2. $\sigma_{m,n}$ in $L \log^2 L$, done;
3. $\sigma_{m,n}$ in L_p, done.

Main result 1

Theorem (Jiao-Zhou 2022)

For fixed $\beta > 0$, define

$$\Sigma_\beta = \{(m, n) \in \mathbb{N}^2 : m/n \leq \beta, n/m \leq \beta\}. \tag{3}$$

There exists an absolute constant $c_\beta > 0$ such that for any $f \in L_1(\mathcal{N})$,

$$\|(\sigma_{m,n}(f))_{(m,n)\in\Sigma_\beta}\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c_\beta \|f\|_{L_1(\mathcal{N})}.$$

Comments for Possion means

Let $0 < r, s < 1$. For $f \in L_1(\mathcal{N})$ where $\mathcal{N} = L_\infty(\mathbb{T}^2)\bar{\otimes}\mathcal{M}$, define the (Abel-) Poisson mean as follows:

$$P_{r,s}(f)(x,y) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \widehat{f}(k,l) r^{|k|} s^{|l|} e^{i(kx+ly)}, \quad (x,y) \in [-\pi,\pi]^2.$$

Theorem (Jiao-Zhou 2022)

Let $\beta > 0$. Then there exists a constant $c_\beta > 0$ depending only on β such that, for any $f \in L_1(\mathcal{N})$,

$$\|(P_{r,s}(f))_{(r,s)\in\widehat{\Sigma}_\beta}\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c_\beta \|f\|_{L_1(\mathcal{A})},$$

where

$$\widehat{\Sigma}_\beta = \Big\{(r,s) \in [\frac{1}{2},1) \times [\frac{1}{2},1) : \frac{1-r}{1-s}, \frac{1-s}{1-r} \leq \beta\Big\}, \quad \beta > 0.$$

Main result 2

For $f \in L_1(\mathbb{T}^2)$, let (Marcinkiewicz 1939)

$$F_n(f) = \frac{1}{(n+1)} \sum_{k=0}^{n} S_{k,k}(f), \quad n \in \mathbb{N}.$$

Theorem (Marcinkiewicz 1939)

$$F_n(f) \longrightarrow f, \quad a.e. \quad f \in L \log L(\mathbb{T}^2)$$

Theorem (Grünwald 1941; Herriot 1942)

$$F_n(f) \longrightarrow f, \quad a.e. \quad f \in L_1(\mathbb{T}^2).$$

- In 1968, Zhizhiashvili also provided a proof for this result.

Main result 2

Theorem (Jiao-Zhou 2022)

There exists an absolute constant $c > 0$ *such that for any* $f \in L_1(\mathbb{T}_\theta^2)$,

$$\|(F_n(f))_{n\geq 1}\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c\|f\|_{L_1(\mathcal{N})}.$$

Note: (1) Main results 1 and 2 can be transferred to quantum tori setting. (2) Both the noncommutative maximal inequalities imply the b.a.u.

The first key lemma: a maximal inequality with respect to rectangles

For $f \in L_1(\mathcal{N})$ and $h, s > 0$, set

$$I_{h,s}(f)(x,y) = \frac{1}{4hs}\int_{-h}^{h}\int_{-s}^{s} f(x+u, y+v)dudv, \quad (x,y) \in \mathbb{R}^2.$$

Lemma

If $h, s > 0$, $h = 2^{\alpha}s$ or $s = 2^{\alpha}h$ for some fixed $\alpha \in \mathbb{N}$, then there exists an absolute constant $c > 0$ (which is independent of α) such that, for any $f \in L_1(\mathcal{N})$,

$$\|(I_{h,s}(f))_s\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c\|f\|_{L_1(\mathcal{N})}.$$

- The constant c above is independent of α.

1. In the case $h = s$, this is just the noncommutative Hardy-Littlewood maximal inequality, which is proved by Mei 2007, Mem. AMS via nc martingales.
2. The proof of Chen-Xu-Yin theorem for $\sigma_{n,n}$ is reduced to noncommutative Hardy-Littlewood maximal inequality.
3. In the classical setting ($\mathcal{M} = \mathbb{C}$), this lemma can be proved by covering lemma.
4. In the noncommutative setting, covering lemma seems do not work; see also Hong-Liao-Wang.
5. The proof of last lemma can be proved by Mei's lemma.

Mei, *Operator valued Hardy spaces*, Mem. Amer. Math. Soc. **188** (2007), no. 881, vi+64.

G. Hong, B. Liao, and S. Wang, *Noncommutative maximal ergodic inequalities associated with doubling conditions*, Duke Math. J. **170** (2021), no. 2, 205–246.

Proof of main result 1

Theorem 1

$$\|(\sigma_{m,n}(f))_{(m,n)\in\Sigma_\beta}\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c_\beta \|f\|_{L_1(\mathcal{N})}.$$

- Take positive $f \in L_1(\mathcal{N})$.
- Take $k \in \mathbb{N}$ such that $2^k \leq m < 2^{k+1}$, and decomposition $\sigma_{m,n}(f)$ into four parts:

$$\sigma_{m,n}(f) = P_{m,n,k}(f) + Q_{m,n,k}(f) + R_{m,n,k}(f) + S_{m,n,k}(f).$$

- Control every part by $I_{h,s}(f)$.

The second key lemma: a maximal inequality with respect to parallelograms

For $f \in L_1(\mathcal{N})$, $\alpha \in \mathbb{Z}$ and $h > 0$, define

$$A_{h,\alpha}(f)(x,y) = \frac{1}{2^{\alpha+2}h^2} \int_{-h}^{h} \int_{t-2^\alpha h}^{t+2^\alpha h} f(x+s, y+t)\,dsdt.$$

Lemma

Let $\alpha \in \mathbb{Z}$. There exists an absolute constant $c > 0$ such that, for any $f \in L_1(\mathbb{N})$,

$$\|(A_{h,\alpha}(f))_{h>0}\|_{\Lambda_{1,\infty}(\mathcal{N},\ell_\infty)} \leq c\|f\|_{L_1(\mathcal{N})}.$$

- The constant c above is independent of α.
- This result actually belongs to Cadilhac and Wang.

Comments for key lemma 2

Hong-Xu 2021

$$\|(\sum_{k\in\mathbb{Z}}|I_{2^k,2^k}(f)-\mathbb{E}_k(f)|^2)^{1/2}\|_{L_{1,\infty}}\leq c\|f\|_1,$$

consequently,

$$\|\sup_{k\in\mathbb{Z}} I_{2^k,2^k}(f)\|_{L_{1,\infty}}\leq c\|f\|_1,$$

- G. Hong and B. Xu, *A noncommutative weak type (1,1) estimate for a square function from ergodic theory*, J. Funct. Anal. **280** (2021), no. 9, Paper No. 108959, 29 pp.
- L. Cadilhac and S. Wang, *Noncommutative maximal ergodic inequalities for amenable groups*, manuscript (2022).
- Jones et al. 1998

Proof for key lemma 2: main ideas

- Step 1: By Mei's lemma, reduce the problem into lacunary case.
- Step 2: using noncommutative CZ decomposition to decompose f into three parts.
- Step 3: estimate every part.

Mei's Lemma

Mei's Lemma For any interval $I \subset \mathbb{R}$, there exist $n_I, j_I \in \mathbb{Z}$ such that $I \subset D_{n_I}^{[1],j_I} \in D_{n_I}^{[1]}$ and $|D_{n_I}^{[1],j_I}| \leq 6|I|$, or $I \subset D_{n_I}^{[2],j_I} \in D_{n_I}^{[2]}$ and $|D_{n_I}^{[2],j_I}| \leq 6|I|$. The number n_I is actually the unique element in $\mathbb{Z}$ such that

$$\frac{2^{-n_I-1}}{3} \leq |I| < \frac{2^{-n_I}}{3}.$$

- Let $(I_k)_{k\in\mathbb{Z}} \subset \cup_{m\in\mathbb{Z}}\mathcal{F}_m^{[i]}$ $(i \in \{1,2\})$ be a decreasing dyadic intervals. For $f \in L_1(\mathcal{N})$ and $\alpha \in \mathbb{Z}$, let $(I_k := (a_k, b_k]$ with $b_k - a_k = 2^{-k})$

$$\widetilde{A}_{I_k,\alpha}(f)(x,y) = \frac{1}{2^{\alpha}2^{-2k}} \int_{a_k}^{b_k} \int_{t+2^{\alpha}a_k}^{t+2^{\alpha}b_k} f(x+s, y+t)dsdt.$$

- For positive $f \in L_1(\mathcal{N})$,

$$A_{h,\alpha}(f) \leq 144\widetilde{A}_{I_k,\alpha}(f) + \text{similar formula}.$$

Cuculescu projections

Lemma

Take $f \in L_1(\mathcal{M})$ such that $f \geq 0$. Consider the martingale $(f_n)_{n\geq 1}$ with $f_n = \mathcal{E}_n(f)$, $n \geq 1$. For any fixed $\lambda > 0$, there exists a sequence of decreasing projections $(q_n^{(\lambda)})_{n\geq 1}$ in $\mathcal{M}$ satisfying the following properties:

(i) *for every $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$;*

(ii) *for every $n \geq 1$, $q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)} f_n q_{n-1}^{(\lambda)}$;*

(iii) *for every $n \geq 1$, $q_n^{(\lambda)} f_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$;*

(iv) *if we set $q^{(\lambda)} = \wedge_{n=1}^{\infty} q_n^{(\lambda)}$, then $q^{(\lambda)} f q^{(\lambda)} \leq \lambda q^{(\lambda)}$ and*

$$\lambda\tau(1 - q^{(\lambda)}) \leq \tau\big((1 - q^{(\lambda)})f\big) \leq \|f\|_{L_1(\mathcal{M})}.$$

- $q_0 = 1$, $q_n = q_{n-1}\chi_{(0,\lambda)}(q_{n-1} f_n q_{n-1})$.

New CZ decomposition

Theorem (Cadilhac et al. 2021)

Let $f \in L_1(\mathcal{N})$, $f \geq 0$ *and* $\lambda > 0$. *Then there is a decomposition of* f *such that*

$$f = g + b_d + b_{off} \tag{4}$$

where

(i) $g = qfq + \sum_{k \geq 1} p_k f_k p_k$,
$\|g\|_{L_1(\mathcal{N})} \leq \|f\|_{L_1(\mathcal{N})}$ *and* $\|g\|_{L_\infty(\mathcal{N})} \leq R_{\mathrm{reg}}\lambda$;

(ii) $b_d = \sum_{k \geq 1} p_k(f - f_k)p_k$ *and*

$$\sum_{k \geq 1} \|p_k(f - f_k)p_k\|_{L_1(\mathcal{N})} \leq 2\tau(f(1-q)) \leq 2\|f\|_{L_1(\mathcal{N})};$$

(iii) $b_{off} = \sum_{k \geq 1} p_k(f - f_k)q_k + q_k(f - f_k)p_k$.

Comments for Cuculescu projections and CZ decomposition

- I. Cuculescu, *Martingales on von Neumann algebras*, J. Multivariate Anal. **1** (1971), no. 1, 17–27.
- N. Randrianantoanina, *Non-commutative martingale transforms*, J. Funct. Anal. **194** (2002), no. 1, 181–212.
- J. Parcet, *Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory*, J. Funct. Anal. **256** (2009), no. 2, 509–593.
- G. Hong, X. Lai, B. Xu, Maximal singular integral operators acting on noncommutative L_p-spaces, arxiv, to appear in Math. Ann.
- L. Cadilhac, J. Conde-Alonso, and J. Parcet, Spectral multipliers in group algebras and noncommutative Calderón-Zygmund theory, arXiv:2105.05036 (2021), to appear in J. Math. Pure Appl.

Sketch of the proof: New CZ decomposition

- q_n's are the so-called Cuculescu projections,

$$\lambda\varphi(1-q) \leq \|f\|_{L_1(\mathcal{N})}.$$

Since we are in dyadic case, $q_n = \sum_{Q\in D(\mathcal{F}_n)} q_Q \chi_Q$,

$$p_n = q_{n-1} - q_n = \sum_{Q\in D(\mathcal{F}_n)} q_Q \chi_Q,$$

$$b_d = \sum_{k\geq 1} \sum_{Q\in D(\mathcal{F}_k)} p_Q(f - f_Q)p_Q\chi_Q$$

and

$$b_{off} = \sum_{k\geq 1} \sum_{Q\in D(\mathcal{F}_k)} p_Q(f - f_Q)q_Q\chi_Q + q_Q(f - f_Q)p_Q\chi_Q.$$

Three estimates

- Easy! Estimate 1 There exists a projection $e_1 \in \mathcal{N}$ such that

$$\sup_n \|e_1 \sigma_n(g) e_1\|_\infty \leq \lambda, \quad \lambda\varphi(1 - e_1) \leq c\|f\|_1.$$

- Estimate 2 There exists a projection $e_2 \in \mathcal{N}$ such that

$$\sup_n \|e_2 \sigma_n(b_d) e_2\|_\infty \leq \lambda, \quad \lambda\varphi(1 - e_2) \leq c\|f\|_1.$$

- Estimate 3 There exists a projection $e_3 \in \mathcal{N}$ such that

$$\sup_n \|e_3 \sigma_n(b_{off}) e_3\|_\infty \leq \lambda, \quad \lambda\varphi(1 - e_3) \leq c\|f\|_1.$$

Lemma

Suppose that b_{d} and b_{off} are from (4) associated with the given positive $f \in L_1(\mathcal{N})$ and $\lambda > 0$. For $k < n$, there exists an absolute constant c such that

$$\|\widetilde{A}_{2^{-k},\alpha}(b_{\mathrm{d},n})\|_{L_1(\mathcal{N})} \leq c2^{k-n}\|b_{\mathrm{d},n}\|_{L_1(\mathcal{N})}.$$

- This estimate is movatied by G. Hong and B. Xu 2021, JFA.

Comments for $\sigma_{m,n}$ and F_n in Hardy spaces

- In 1996, Weisz proved that the operator $\sup_{(m,n)\in\Sigma_\beta} |\sigma_{m,n}|$ is bounded from the Hardy space $H_{p,q}(\mathbb{T}^2)$ to $L_{p,q}(\mathbb{T}^2)$ provided $3/4 < p < \infty$ and $0 < q \leq \infty$.

- Atomic decomposition!

- Z. Chen, N. Randrianantoanina, and Q. Xu, Atomic decompositions for noncommutative martingales, arXiv, 2020.

Thanks for your attentions!