Campanato Spaces via Quantum Markov Semigroups: I. Finite case

Yuanyuan Jing
(joint with Guixiang Hong)

Wuhan University

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Outline

1. Introduction and Preliminary
2. Campanato spaces via Quantum Markov semigroup
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The classical definition

Let $\alpha \geq 0$ and $[\alpha]$ be the integer part of α, the classical Campanato norm of a function $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ can be defined as

$$\|f\|_{\mathcal{L}_\alpha(\mathbb{R}^n)} = \sup_B \frac{1}{|B|^{\frac{\alpha}{n}}} \left(\frac{1}{|B|} \int_B |f(x) - P_B f(x)|^2 dx \right)^{\frac{1}{2}} < \infty, \qquad (1.1)$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, and $P_B f$ is the unique polynomial degree at most $[\alpha]$ such that

$$\int_B [f(x) - P_B f(x)] x^\theta dx = 0$$

for all multi-indices $0 \leq |\theta| \leq [\alpha]$.

- J. Garcia-Cuerva, J. Rubio De Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.
  - $\alpha = 0$, $P_B f = \frac{1}{|B|}\int_B f = f_B$
    these spaces are variants of the classical BMO space introduced by John and Nirenberg.
  - $\alpha > 0$, the Campanato spaces $\mathcal{L}_\alpha(\mathbb{R}^n)$ are variants of the homogeneous Lipschitz spaces $\Lambda_\alpha(\mathbb{R}^n)$ which are duals of certain Hardy spaces (H^p, $0 < p < 1$, $\alpha = n(1/p - 1)$).

Some references: commutative case

$\alpha = 0$ in (1.1):
Let X be a locally compact with countable base and $B(X)$ be the space of bounded Borel functions on X.

$$\|f\|_{BMO} = \sup_{x \in X, y > 0} P_z|f - P_z f|$$

where $P_z f = P_y f(x)$, $\forall f \in B(X)$, $z = (x, y) \in X \times \mathbb{R}_+$.

- N. Varopoulos, Aspects of probabilistic Littlewood-Paley theory, *J. Funct. Anal.*, **38**(1), (1980), 25-60.

Some references: commutative case

$\alpha = 0$ in (1.1):

$$P_B f = f_B \to \varphi_{t_B} * f,$$

where φ_{t_B} is chosen to be the heat kernel or the Poisson kernel.

- X.T. Duong, L. Yan, New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications, *Commun. Pure Appl. Math.*, **58**, (2005), 1375-1420.

$$f_B \to e^{-t_B L} f.$$

- X.T. Duong, L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. *J. Am. Math. Soc.*, **18**, (2005), 943-973.

$$\alpha \geq 0 \text{ in } (1.1): \quad P_B f \to \varphi_{t_B} * f = [I - (I - h_{t_B})^{[\alpha]+1}] f,$$

where h_{t_B} is the heat kernel and t_B is scaled to the radius of the ball B.

- D. Deng, X.T. Duong and L. Yan, A characterization of the Morrey-Campanato spaces, *Math. Z.*, **250** (2005), 641-655.

- X.T. Duong, L. Yan, New Morrey-Campanato spaces associated with operators and applications. *East journal on approximations*, **15** (2009), 345-376.

Motivation: We have found few general theories of Campanato spaces defined intrinsically by the semigroups even in the commutative case.

Quantum Markov semigroups

Let $(\mathcal{M}, \tau)$ be a pair of a von Neumann algebra and a normal finite faithful trace. A semigroup of operators $(T_t)_{t\geq 0}$ is called a noncommutative Markov semigroup if $T_t T_s = T_{t+s}$, $T_0 = id$ and

(i) T_t is a normal completely positive map on $\mathcal{M}$ such that $T_t(1) = 1$;

(ii) T_t is self-adjoint with respect to the trace τ, i.e. $\tau(T_t(f^*)g) = \tau(f^* T_t(g))$ for any f, $g \in \mathcal{M} \cap L^2(\mathcal{M})$;

(iii) $T_t(f) \to f$ as $t \to 0^+$ in the weak-$*$ topology of $\mathcal{M}$ for any $f \in \mathcal{M}$.

These conditions imply that $\tau(T_t x) = \tau(x)$ for all x, and

$$\|T_t f\|_{L^p(\mathcal{M})} \leq \|f\|_{L^p(\mathcal{M})},\ 1 \leq p \leq \infty.$$

A Markov semigroup always admits an infinitesimal generator A such that

$$A = \lim_{t\to 0} \frac{id - T_t}{t} \text{ with } T_t = e^{-tA}.$$

The subordinated Poisson semigroup $\mathcal{P} = (P_t)_{t\geq 0}$ is defined by $P_t = e^{-t\sqrt{A}}$.

Subordination formula:

$$P_t = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u du. \tag{1.2}$$

Then

$$P_t f \leq \frac{t}{s} P_s f$$

holds for any positive f and $0 < s \leq t$.

- M. Junge, C. Le Merdy, Q. Xu, H^∞ Functional calculus and square functions on noncommutative L^p-spaces, *Astérisque*, **305**, (2006), vi+138 pp.
- E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, Princeton University Press, Princeton, (1970).

Some references: noncommutative case

Consider the subordinated Poisson semigroup $(P_t)_{t\geq 0}$ of a quantum Markov semigroup $(T_t)_{t\geq 0}$ acting on a finite von Neumann algebra $L^2(\mathcal{M})$.

$$\|f\|_{BMO^c(\mathcal{P})} = \sup_{t>0} \left\| P_t |f - P_t f|^2 \right\|_\infty^{\frac{1}{2}}, \quad f \in L^2(\mathcal{M}).$$

- T. Mei, Tent spaces associated with semigroups of operators, *J. Funct. Anal.*, **255**, (2008), 3356-3406.

Given a Markov semigroup of operators T_t on $\mathcal{M}$ and $f \in \mathcal{M} \cup L^2(\mathcal{M})$. Then

$$\|f\|_{bmo^c(\mathcal{T})} = \sup_{t>0} \left\| T_t|f|^2 - |T_t f|^2 \right\|_{\infty}^{\frac{1}{2}},$$

$$\|f\|_{BMO^c(\mathcal{T})} = \sup_{t>0} \left\| T_t|f - T_t f|^2 \right\|_{\infty}^{\frac{1}{2}}.$$

- M. Junge, T. Mei, BMO spaces associated with semigroups of operators, *Math. Ann.*, **352**, (2012), 691-743.
- T. Mei, An H^1-BMO duality theory for semigroups of operators, arXiv preprint arXiv:1204.5082 (2012).

Question 1: How to define the semigroup Campanato spaces in the noncommutative case?

1 Introduction and Preliminary

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Our definitions

For all $\alpha > 0$, the column Campanato space is given by

$$\mathcal{L}^c_\alpha(\mathcal{P}) = \left\{ f \in \mathcal{M} : \ \|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} < \infty \right\},$$

with

$$\|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^\alpha} \left\| P_t |(I - P_t)^{[\alpha]+1} f|^2 \right\|_\infty^{\frac{1}{2}}. \qquad (2.1)$$

- $\|\cdot\|_{\mathcal{L}^c_\alpha(\mathcal{P})}$ is a norm on $\mathcal{M}$.

The row space $\mathcal{L}^r_\alpha(\mathcal{P})$ is defined as the space of all f such that $f^* \in \mathcal{L}^c_\alpha(\mathcal{P})$, equipped with norm

$$\|f\|_{\mathcal{L}^r_\alpha(\mathcal{P})} = \|f^*\|_{\mathcal{L}^c_\alpha(\mathcal{P})} .$$

Then the mixture space $\mathcal{L}^{cr}_\alpha(\mathcal{P})$ is defined as the intersection $\mathcal{L}^c_\alpha(\mathcal{P}) \cap \mathcal{L}^r_\alpha(\mathcal{P})$ with the norm

$$\|f\|_{\mathcal{L}^{cr}_\alpha(\mathcal{P})} = \max\{\|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})}, \|f\|_{\mathcal{L}^r_\alpha(\mathcal{P})}\}.$$

- $\mathcal{L}^\dagger_\alpha(\mathcal{P})$ is a Banach space, where $\dagger = \{c, r, cr\}$.

For $0 < \alpha < 1$, we can denote by $\|f\|_{\ell^c_\alpha(\mathcal{P})}$ the column little Campanato norm as follows:

$$\|f\|_{\ell^c_\alpha(\mathcal{P})} = \|f\|_\infty + \sup_t \frac{1}{t^\alpha} \left\| P_t|f|^2 - |P_t f|^2 \right\|_\infty^{\frac{1}{2}}, \qquad (2.2)$$

Similarly, we can define the row norm as $\|f\|_{\ell^r_\alpha(\mathcal{P})} = \|f^*\|_{\ell^c_\alpha(\mathcal{P})}$ and the mixture norm as $\|f\|_{\ell^{cr}_\alpha(\mathcal{P})} = \max\left\{\|f\|_{\ell^c_\alpha(\mathcal{P})}, \|f\|_{\ell^r_\alpha(\mathcal{P})}\right\}$.

- The notations $\mathcal{L}^\dagger_\alpha(\mathcal{T})$ and $\|f\|_{\ell^\dagger_\alpha(\mathcal{T})}$ are used when $(P_t)_{t\geq 0}$ is replaced by the general Markov semigroup $(T_t)_{t\geq 0}$, where $\dagger = \{r, c, cr\}$.

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Lipschitz spaces

Let $\alpha > 0$. Then

$$\Lambda_\alpha(\mathbb{R}^n) = \left\{ f \in L^\infty(\mathbb{R}^n) : \ \|f\|_{\Lambda_\alpha(\mathbb{R}^n)} < \infty \right\},$$

with

$$\|f\|_{\Lambda_\alpha(\mathbb{R}^n)} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^{\alpha-([\alpha]+1)}} \left\| \frac{\partial^{[\alpha]+1} u(t,\cdot)}{\partial t^{[\alpha]+1}} \right\|_\infty,$$

where $u(t,\cdot) = \int_{\mathbb{R}^n} p_t(y) f(\cdot - y) dy$ and $p_t(y) = \frac{c_n y}{(|t|^2+y^2)^{\frac{n+1}{2}}}$.

- E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, (1970).

For all $\alpha > 0$, we define

$$\Lambda_\alpha(\mathcal{P}) = \left\{ f \in \mathcal{M} : \ \|f\|_{\Lambda_\alpha(\mathcal{P})} < \infty \right\},$$

with

$$\|f\|_{\Lambda_\alpha(\mathcal{P})} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^{\alpha-([\alpha]+1)}} \left\| \frac{\partial^{[\alpha]+1} P_t f}{\partial t^{[\alpha]+1}} \right\|_\infty .$$

It is easy to see that $\|f^*\|_{\Lambda_\alpha(\mathcal{P})} = \|f\|_{\Lambda_\alpha(\mathcal{P})}$ from the positivity of the subordinated Poisson semigroup.

$0 < \alpha < 1$ (2019)

- A. González-Pérez, Hölder classes via semigroups and Riesz transforms, arXiv:1904.10332.

Lemma 1 (Stein)

Suppose $f \in L^\infty(\mathbb{R}^n)$, and $\alpha > 0$. Let k, l be two integers both greater than α. Then the two conditions

$$\frac{1}{t^{\alpha-k}} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_\infty \leq A_k \text{ and } \frac{1}{t^{\alpha-l}} \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_\infty \leq A_l$$

are equivalent. Moreover, the smallest A_k and A_l holding in the above inequalities are comparable.

Let k be an integer such that $k > \alpha > 0$, we define $\Lambda_{\alpha,k}(\mathcal{P})$ as

$$\Lambda_{\alpha,k}(\mathcal{P}) = \left\{ f \in \mathcal{M} : \ \|f\|_{\Lambda_{\alpha,k}(\mathcal{P})} < \infty \right\},$$

with

$$\|f\|_{\Lambda_{\alpha,k}(\mathcal{P})} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^{\alpha-k}} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{\mathcal{M}}.$$

The following proposition tells that, for $\alpha > 0$ fixed, the spaces $\Lambda_{\alpha,k}(\mathcal{P})$ are all the same when $k > \alpha$.

Proposition 2

Let $\alpha > 0$ and k be an integer such that $k \geq [\alpha] + 1$. Then, for $f \in \mathcal{M}$, we have

$$\|f\|_{\Lambda_\alpha(\mathcal{P})} \simeq_{\alpha,k} \|f\|_{\Lambda_{\alpha,k}(\mathcal{P})} . \tag{3.1}$$

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Main results 1

Theorem 3

Let $(T_t)_{t\geq 0}$ be a Markov semigroup and $(P_t)_{t\geq 0}$ be its associated Poisson semigroup. Then,

(i) *for any $0<\alpha<2$, we have*

$$\mathcal{L}^c_\alpha(\mathcal{P})=\mathcal{L}^r_\alpha(\mathcal{P})=\mathcal{L}^{cr}_\alpha(\mathcal{P});$$

(ii) $\ell^c_\alpha(\mathcal{P})=\ell^r_\alpha(\mathcal{P})=\ell^{cr}_\alpha(\mathcal{P})$ *when* $\Gamma^2\geq 0$ *and* $0<\alpha<\frac{1}{2}$.

Assume that there exists a $*$-algebra $\mathcal{A}$ which is weak-$*$ dense in $\mathcal{M}$ such that $T_s(\mathcal{A}) \subset \mathcal{A} \subset \mathrm{dom}_\infty(A)$.

■ P.A Meyer's gradient form Γ (also called "Carré du Champ") associated with T_t is defined as

$$2\Gamma(f,g) = (A\left(f^*\right)g) + f^*(A(g)) - A\left(f^*g\right) \tag{4.1}$$

■ Bakry-Émery's iterated gradient form Γ^2 associated with T_t is defined as

$$2\Gamma^2(f,g) = \Gamma\left(Af,g\right) + \Gamma\left(f,Ag\right) - A\Gamma\left(f,g\right) \tag{4.2}$$

for $f,g \in \mathcal{A}$.
example: $A = -\Delta = \frac{\partial^2}{\partial^2 x}$, $\Gamma(f,g) = \nabla f^* \nabla g$ and $\Gamma^2(f,g) = \frac{\partial^2 f^*}{\partial^2 x}\frac{\partial^2 g}{\partial^2 x}$.

■ $\Gamma^2 \geq 0 \Leftrightarrow \Gamma\left(T_t f, T_t g\right) \leq T_t\Gamma\left(f,g\right)$.

Sketch for the proof of Theorem

Theorem 4

Suppose $f \in \mathcal{M}$. Then for $\alpha > 0$,

$$\|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} \lesssim_\alpha \|f\|_{\Lambda_\alpha(\mathcal{P})} . \tag{4.3}$$

Furthermore, if $0 < \alpha < 2$, we have

$$\|f\|_{\Lambda_\alpha(\mathcal{P})} \lesssim_\alpha \|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} , \tag{4.4}$$

i.e. $\mathcal{L}^c_\alpha(\mathcal{P}) = \Lambda_\alpha(\mathcal{P})$ with equivalent norms when $0 < \alpha < 2$.

- J. Garcia-Cuerva, J. Rubio De Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.
- S. Janson, M.H. Taibleson and G. Weiss, Elementary characterizations of the Morrey-Campanato spaces, *Lecture Notes in Math*, **992**, (1983), 101-114.
- M.H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Société Mathématique de France, (1980).
- H. Greenwald, On the theory of homogeneous Lipschitz spaces and Campanato spaces, *Pacific Journal of Mathematics*, (1983), **106**, 87-93.

Sketch of the Proof of Theorem

Let

$$\mathcal{M}^{\circ} = \{f \in \mathcal{M},\ \lim_{t\to\infty} T_t f = 0\}$$

$$ker(A_\infty) = \{f \in \mathcal{M} : Af = 0\} = \left\{f \in \mathcal{M} : \lim_{t\to 0} T_t f = f\right\}.$$

Lemma 5

If the trace τ is finite, T_t induces a canonical spiltting on $\mathcal{M}$ such that $\mathcal{M} = \mathcal{M}^{\circ} \oplus ker(A_\infty)$. This implies that, for $f \in \mathcal{M}$, we have

$$f = f_0 + f_1,\ \ f_0 \in \mathcal{M}^{\circ},\ \ f_1 \in ker(A_\infty).$$

▶ M. Junge, Q. Xu, Noncommutative maximal ergodic theorems, *Journal of the American Mathematical Society*, (2007), 20, 385-439.

Note that $Af = 0$ implies $T_t f = f$, by (1.2), we deduce that

$$P_t f = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} T_u f du = \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-\frac{t^2}{4u}} u^{-\frac{3}{2}} f du = f.$$

It follows that $ker(A_\infty) = ker(A_\infty^{\frac{1}{2}}) = \{f \in \mathcal{M}, P_t f = f\}$.

$$\Rightarrow \quad \mathcal{M} = \mathcal{M}^\circ \oplus ker(A_\infty^{\frac{1}{2}}).$$

- M. Junge, T. Mei, BMO spaces associated with semigroups of operators, *Math. Ann.*, **352**, (2012), 691-743.

Proposition 6

Let $f \in \mathcal{M}$. For all $j \in \mathbb{N}$ and $t > 0$, we have

$$\left| \frac{\partial^j P_t f}{\partial t^j} \right|^2 \leq \frac{8^j 2^{j(j+1)}}{t^{2j}} P_t |f|^2.$$

Proposition 7

Let $(P_t)_{t\geq 0}$ be the Markov semigroup associated to $(T_t)_{t\geq 0}$ and $f \in \mathcal{M}$. Then

(i) *for $0 < \alpha < 1$, we have*

$$\|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} \leq (2^\alpha + 1)\|f\|_{\ell^c_\alpha(\mathcal{P})} + \sup_{t>0} \frac{1}{t^\alpha} \|P_t f - P_{2t} f\|_\infty ;$$

(ii) *if in addition $\Gamma^2 \geq 0$, we get for $0 < \alpha < \frac{1}{2}$,*

$$\|f\|_{\ell^c_\alpha(\mathcal{P})} + \sup_{t>0} \frac{1}{t^\alpha} \|P_t f - P_{2t} f\|_\infty \leq \left(\frac{\sqrt{2}}{1 - 2^{\alpha - \frac{1}{2}}} + 1 \right) \|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})}.$$

we need to introduce the following norm:

$$\|f\|_{\mathcal{L}_\alpha^c(\partial)} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^\alpha} \left\| P_t \int_0^t \left| \frac{\partial P_s f}{\partial s} \right|^2 s ds \right\|_\infty^{\frac{1}{2}}.$$

Lemma 8

Let $(T_t)_{t\geq 0}$ be a Markov semigroup, $(P_t)_{t\geq 0}$ its associated Poisson semigroup and $f \in \mathcal{M}$. Then, for $0 < \alpha < 1$,

$$\sup_{t>0} \frac{1}{t^\alpha} \|P_t f - P_{2t} f\|_\infty \lesssim_\alpha \|f\|_{\mathcal{L}_\alpha^c(\partial)} \lesssim \|f\|_{\ell_\alpha^c(\mathcal{P})}.$$

Question 2: As in the classic case, for α fixed, we would like to explore the question whether the indicator $[\alpha]+1$ in the definition of the norm of $\mathcal{L}_{\alpha}^{c}(\mathcal{P})$ can be substituted for any integer greater than α.

A high-order version of the Campanato norms

Suppose $\alpha > 0$. Let k be an integer greater than α. We define the $\mathcal{L}^c_{\alpha,k}(\mathcal{P})$ column norm as

$$\|f\|_{\mathcal{L}^c_{\alpha,k}(\mathcal{P})} = \|f\|_\infty + \sup_{t>0} \frac{1}{t^\alpha} \left\| P_t |(I-P_t)^k f|^2 \right\|_\infty^{\frac{1}{2}}, \tag{4.5}$$

and the row norm as $\|f\|_{\mathcal{L}^r_{\alpha,k}(\mathcal{P})} = \|f^*\|_{\mathcal{L}^c_{\alpha,k}(\mathcal{P})}$. Similarly, the symmetric norm is given by

$$\|f\|_{\mathcal{L}^{cr}_{\alpha,k}(\mathcal{P})} = \max\{\|f\|_{\mathcal{L}^c_{\alpha,k}(\mathcal{P})}, \|f\|_{\mathcal{L}^r_{\alpha,k}(\mathcal{P})}\}.$$

- S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, *Math. Ann.*, **344**, (2009), 37-116.
- S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, L.X. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, *Mem. Amer. Math. Soc.*, **214**, (2011).

- meanings in the classic setting???

Main results 2

Theorem 9

Let $0 < \alpha < \alpha_0$, where α_0 verifies $2^{\alpha_0 - 2} + 3^{\alpha_0 - 3} = 1$. Let k be an integer such that $k > \alpha > 0$. Then, the spaces $\mathcal{L}^c_\alpha(\mathcal{P})$ and $\mathcal{L}^c_{\alpha,k}(\mathcal{P})$ coincide, and their norms are equivalent.

Corollary 10

Let $0 < \alpha < \alpha_0$ *where* $2^{\alpha_0 - 2} + 3^{\alpha_0 - 3} = 1$, *and* k *be an integer such that* $k > \alpha$. *Then,*

$$\mathcal{L}^c_{\alpha,k}(\mathcal{P}) = \mathcal{L}^r_{\alpha,k}(\mathcal{P}) \quad \text{with equivalent norms.}$$

Main results 3

Theorem 11

For any $0 < \alpha < 1$ *and* $f \in \mathcal{M}$, *then*

(i) *if* $(T_t)_{t\geq 0}$ *is quasi-monotone, we have* $\|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} \lesssim \|f\|_{\mathcal{L}^c_{\frac{\alpha}{2}}(\mathcal{T})}$;

(ii) *if* $(T_t)_{t\geq 0}$ *is quasi-monotone, we have* $\|f\|_{\Lambda_\alpha(\mathcal{P})} \lesssim_\alpha \|f\|_{\Lambda_{\frac{\alpha}{2}}(\mathcal{T})}$;

(iii) *moreover, if* $(T_t)_{t\geq 0}$ *has the* $\Gamma^2 \geq 0$ *property, we then have* $\|f\|_{\ell^c_{\frac{\alpha}{2}}(\mathcal{T})} \lesssim \|f\|_{\ell^c_\alpha(\mathcal{P})}$.

- We say the semigroup $(T_t)_{t\geq 0}$ is quasi-monotone if $\frac{T_t}{t^\beta}$ is either decreasing or increasing for some constant $\beta \geq 0$.

(i) T. Mei, 2012, (math. Ann.): $\|f\|_{BMO^c(\mathcal{P})} \lesssim \|f\|_{BMO^c(\mathcal{T})}$

(ii) $\|f\|_{\Lambda_\alpha(\mathcal{P})} \lesssim_\alpha \|f\|_{\mathcal{L}^c_\alpha(\mathcal{P})} \lesssim \|f\|_{\mathcal{L}^c_{\frac{\alpha}{2}}(\mathcal{T})} \lesssim \|f\|_{\Lambda_{\frac{\alpha}{2}}(\mathcal{T})}$

(iii) T. Ferguson, T. Mei, B. Simanek, 2019, (Adv. Math.): $0 < \gamma < \delta \leq 1$, $\|f\|_{bmo^c(L^\delta)} \lesssim \|f\|_{bmo^c(L^\gamma)}$

Question: $\mathcal{L}^c_\alpha(\mathcal{T}) = \mathcal{L}^r_\alpha(\mathcal{T})$?

Thanks for your attention!