Sharp endpoint L_p estimate of Schrödinger groups under noncommutative algebraic framework

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This is a joint work with Guixiang Hong and Liang Wang

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This talk is based on

- Z. Fan, G. Hong and L. Wang, Sharp endpoint L_p estimate of Schrödinger groups under noncommutative algebraic framework. In progress.

Schrödinger equation on $\mathbb{R}^n$:

$$\begin{cases} i\partial_t u(x,t) - \Delta u(x,t) = 0 \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = f(x), \end{cases}$$

where $\Delta := -\sum_{i=1}^n \partial_{x_i}^2$. Then

$$u(x,t) = e^{it\Delta} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(\langle x,\xi\rangle + t|\xi|^2)} d\xi,$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx.$$

Schrödinger equation

- $e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^n)$ iff $p = 2$.
  - (Hormander, Acta. Math, 1960).
- If $p \neq 2$, then $e^{it\Delta}$ is bounded from $L^p_{2s}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where $s > n|1/2 - 1/p|$. Equivalently, $(I + \Delta)^{-s} e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^n)$.
  - (Lanconelli, Boll. Un. Mat. Ital, 1968);
  - (Sjöstrand, Ann. Scuola Norm. Sup. Pisa, 1970);
  - (Brenner, Ark. Mat, 1973).
- If $p \neq 2$, then $e^{it\Delta}$ is unbounded from $L^p_{2s}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where $s < n|1/2 - 1/p|$. Equivalently, $(I + \Delta)^{-s} e^{it\Delta}$ is unbounded on $L^p(\mathbb{R}^n)$.
  - (Sjöstrand, Ann. Scuola Norm. Sup. Pisa, 1970);
  - (Brenner, Ark. Mat, 1973).

Theorem 1 (Miyachi, 1980)

If $s = n|1/2 - 1/p|$, *then for any* $1 < p < \infty$,

$$\|(I+\Delta)^{-s} e^{it\Delta}\|_{L^p(\mathbb{R}^n)} \leq C(1+|t|)^s \|f\|_{L^p(\mathbb{R}^n)}.$$

Remark 1.1

This estimate is sharp:

- If $s < n|1/2 - 1/p|$, then $(I+\Delta)^{-s} e^{it\Delta}$ is unbounded on L^p;
- The upper bound on t is sharp.

Question: When Δ is replaced by L, it is possible to obtain corresponding L^p regularity estimate?;

- Balabane-Emami-Rad (1985): $P(D) + V(x)$;
- Thangavelu (1987): $-\Delta + V(x)$;
- Lohoué (1992); Alexopoulos (1994): Sub-Laplacian operator on Lie group or Riemannian manifold;
- Jensen-Nakamura (1994,1995): $e^{-itL}f(L)$, where $L = -\Delta + V(x)$ and $V(x) \geq 0$;

Let (X, d, μ) be measure space with distance d and measure μ, L is a non-negative self-adjoint operator on $L^2(X)$. Consider Schrödinger equation on X:

$$\begin{cases} i\partial_t u(x,t) + Lu(x,t) = 0, \quad x \in X, \quad t > 0, \\ u(x,0) = f(x), \end{cases}$$

Then for any $f \in L^2(X)$, by spectral decomposition,

$$e^{itL} f = \int_0^\infty e^{it\lambda} dE_L(\lambda) f.$$

where $E_L(\lambda)$ is the projection-valued measure supported on the spectrum of L. By spectral theorem, e^{itL} is bounded on $L^2(X)$.

Definition 2

If $\exists C > 0$ such that

$$V(x, 2r) \leq CV(x, r), \quad \forall r > 0, \; x \in X,$$

then we say that (X, d, μ) satisfies doubling property, where we denote $V(x, r) := \mu(B(x, r))$.

Note that the above inequality can deduce that: $\exists C, n > 0$ s.t.

$$V(x, \lambda r) \leq C\lambda^n V(x, r), \; \forall \lambda \geq 1, \; x \in X.$$

then we say that (X, d, μ) is a homogeneous space with dimension n.

Definition 3

Let $m \geq 2$ and L is a non-negative self-adjoint operator on $L^2(X)$, if integral kernel $p_t(x,y)$ of e^{-tL} satisfies:

$$|p_t(x,y)| \leq \frac{C}{\mu(B(x,t^{1/m}))} \exp\left(-c\left(\frac{d(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right), \quad (\mathrm{GE}_m)$$

then we say that the kernel $p_t(x,y)$ satisfies m-order Gaussian upper bound.

Example 4

- $-\Delta_M$ (Laplace-Beltrami operator on compact manifold);
- $-\Delta + V(x)$, $0 \leq V(x) \in L^1_{loc}(\mathbb{R}^n)$ (Schrödinger operator with non-negative potential)
- $-divA\nabla + V(x)$ (Divergence operator);
- $-\Delta_{\mathbb{H}}$ (Sub-Laplacian operator on homogeneous groups).

Theorem 5 (Chen–Duong–Li–Yan, 2020)

Let (X, d, μ) *be a homogeneous space with dimension n. Besides, if L is a non-negative self-adjoint operator, and semigroup* e^{-tL} *satisfies* $m(\geq 2)$*-order Gaussian-estimate, then for any* $1 < p < \infty$, $\exists C_{n,p} > 0$ *s.t.*

$$\|e^{itL}(I + L)^{-\sigma_p n} f\|_{L^p(X)} \leq C_{n,p}(1 + |t|)^{\sigma_p n}\|f\|_{L^p(X)},$$

where $\sigma_p = |1/2 - 1/p|$.

Reference

1. P. Chen, X.T. Duong, J. Li and L.X. Yan, Sharp endpoint L^p estimates for Schrödinger groups, *Math. Ann.* **378** (2020), 667–702.

Our task

In order to investigate L^p regularity estimate of non-commutative Schrödinger equation, we will study the L^p boundedness of the corresponding targeted operator

$$e^{itL}(I+L)^{-\sigma_p n}$$

on noncommutative measure space.

(1) Lack of pointwise estimates, how to formulate this question appropriately? In particular, how to define the dimension n? What conditions should be imposed on L?

(2) Lack of atomic decomposition of Hardy space on general VNA.

(3) One cannot apply sharp maximal function as effectively as in the classical setting.

Observation

Calderón-Zygmund singular integral theory

$\Rightarrow$ Hörmander-Mihlin multiplier theorem

$\Rightarrow$ Radial Hörmander-Mihlin multiplier theorem

$\Rightarrow$ Non-endpoint L^p boundedness of targeted operator

In the last "$\Rightarrow$", the oscillatory term $e^{-it\Delta}$ doesn't play any role.

Combine

Singular integral theory beyond Calderón-Zygmund framework

with

Digging out extra oscillatory information about e^{itL}

$\Rightarrow$ Endpoint L^p boundedness of targeted operator

[JMPX] first introduced a noncommutative form of Calderón-Zygmund theory under algebraic framework.

• Cover many concrete examples in both classical setting and noncommutative setting.

• Particularly interesting for measure spaces with poor geometric information.

Reference

1. M. Junge, T. Mei, J. Parcet and R. Xia, Algebraic Calderon-Zygmund theory, *Adv. Math.* **376** (2021), 107443.

We will follow the establishment of a general form of noncommutative SIO theory to establish non-commutative Schrödinger group theory under algebraic framework.

Reference

1. M. Junge, T. Mei, J. Parcet and R. Xia, Algebraic Calderon-Zygmund theory, *Adv. Math.* **376** (2021), 107443.

This will be divided into the following steps.

Step 1: Identify the approciate BMO spaces.

Step 2: Provide conditions on L which yield $L_\infty \to BMO$ boundedness of target operator.

Step 3: Verify these conditions for a large number of concrete examples.

(1) $\mathcal{M}$: semifinite von Neumann algebra;

(2) τ: a normal semifinite faithful trace on $\mathcal{M}$;

(3) $\mathcal{S} = (S_t)_{t\geq 0}$: Markovian semigroup over $\mathcal{M}$;

A semigroup $\mathcal{S}$ over $\mathcal{M}$ is said to be Markovian if

(i) Each S_t is a weak-$*$ continuous, contractive and completely positive map s.t. $S_t(1) = 1$;

(ii) Each S_t is self-adjoint in the sense that

$$\tau(S_t(f)g) = \tau(fS_t(g)), \quad \text{for all } f, g \in \mathcal{M} \cap L_1(\mathcal{M}). \tag{1}$$

(iii) The map $t \to S_t$ is a pointwise weak-$*$ continuous map.

By Stone's theorem, $\mathcal{S}$ admits an infinitesimal negative generator

$$Af = \lim_{t\to 0} \frac{S_t(f) - f}{t}$$

defined on $dom(A) = \cup_{1\leq p\leq\infty} dom_p(A)$, where $dom_p(A)$ is given by

$$dom_p(A) = \left\{ f \in L_p(\mathcal{M}) : \lim_{t\to 0} \frac{S_t(f) - f}{t} \text{ converges in } L_p(\mathcal{M}) \right\}.$$

Note that $L := -A$ is a non-negative self-adjoint operator acting on $L_2(\mathcal{M})$.

A P-Markovian metric associated to $(\mathcal{M}, \tau)$ and $\mathcal{S}$ is determined by a family

$$\mathcal{Q} = \{(R_{j,t}, \sigma_{j,t}) : j \in \mathbb{N}, t > 0\}.$$

Here each $R_{j,t} : \mathcal{M} \to \mathcal{M}$ is a normal completely positive unital map, and $\sigma_{j,t} \in \mathcal{M}$ s.t. the estimates below hold for some $m \geq 2$:

(i) *Semigroup majorization* :

$$S_t(|\xi|^2) \le \sum_{j\ge 0} \sigma_{j,t}^* R_{j,\sqrt[m]{t}}(|\xi|^2)\sigma_{j,t}, \forall\ t>0 \text{ and } \xi \in \mathcal{M};$$

(ii) *Average domination condition* :

$$\left\|R_{j,t}(|\xi|^2)\right\|_{\mathcal{M}} \le \left\|R_t(|\xi|^2)\right\|_{\mathcal{M}}, \forall t>0, j\ge 0 \text{ and } \xi \in \mathcal{M};$$

(iii) *Metric integrability condition* :

$$k_{\mathcal{Q}} := \sup_{t>0} \left\| \sum_{j\ge 0} |\sigma_{j,t}|^2 \right\|_{\mathcal{M}}^{1/2} < \infty.$$

Set $R_t(f) := R_{0,t}(f)$.

(i) *Semigroup majorization* :

$$S_t(|\xi|^2) \le \sum_{j\ge 0} \sigma_{j,t}^* R_{j,\sqrt[m]{t}}(|\xi|^2)\sigma_{j,t}, \forall\ t > 0 \text{ and } \xi \in \mathcal{M}.$$

The above inequality describes: if the heat kernel $p_t(x,y)$ associated with L satisfies m-order Gaussian upper bound, then

$$\int_X p_t(x,y)|\xi(y)|^2 d\mu(y)$$

$$\le C \sum_{j\ge 0} \frac{\mu(B(x,2^j t^{1/m}))}{\mu(B(x,t^{1/m}))} \exp\big(-c2^{\frac{jm}{m-1}}\big) \fint_{B(x,2^j t^{1/m})} |\xi(y)|^2 d\mu(y)$$

(ii) *Average domination condition* :

$$\left\|R_{j,t}(|\xi|^2)\right\|_{\mathcal{M}} \leq \left\|R_t(|\xi|^2)\right\|_{\mathcal{M}}, \forall t > 0, j \geq 0 \text{ and } \xi \in \mathcal{M};$$

The above inequality describes:

$$\left\|\fint_{B(x,2^j t^{1/m})} |\xi(y)|^2 d\mu(y)\right\|_{L_\infty(X)} \leq C \left\|\fint_{B(x,t^{1/m})} |\xi(y)|^2 d\mu(y)\right\|_{L_\infty(X)}.$$

$$(iii)\ \textit{Metric integrability condition}:$$
$$k_{\mathcal{Q}} := \sup_{t>0} \Big\| \sum_{j\geq 0} |\sigma_{j,t}|^2 \Big\|_{\mathcal{M}}^{1/2} < \infty.$$

The above inequality describes:

$$k_{\mathcal{Q}} = C \left\| \sum_{j\geq 0} \frac{\mu(B(x, 2^j t^{1/m}))}{\mu(B(x, t^{1/m}))} \exp\Big(- c 2^{\frac{jm}{m-1}} \Big) \right\|_{L_\infty(X)}^{1/2} < \infty.$$

Given $M \in \mathbb{Z}_+$, we define the semigroup $BMO_{\mathcal{S},M}(\mathcal{M})$ semi-norm for $f \in \mathcal{M}$ as

$$\|f\|_{\mathrm{BMO}_{\mathcal{S},M}(\mathcal{M})} := \max\left\{\|f\|_{\mathrm{BMO}^r_{\mathcal{S},M}(\mathcal{M})}, \|f\|_{\mathrm{BMO}^c_{\mathcal{S},M}(\mathcal{M})}\right\},$$

where the column BMO semi-norm is given by

$$\|f\|_{\mathrm{BMO}^c_{\mathcal{S},M}(\mathcal{M})} := \sup_{t>0} \left\|e^{-tL}|(I - e^{-tL})^{M} f|^2\right\|_{\mathcal{M}}^{1/2}$$

For any $M \in \mathbb{Z}_+$, we define the semigroup $BMO_{\mathcal{Q},M}(\mathcal{M})$ semi-norm for $f \in \mathcal{M}$ as

$$\|f\|_{BMO_{\mathcal{Q},M}(\mathcal{M})} = \max\left\{\|f\|_{BMO^c_{\mathcal{Q},M}(\mathcal{M})}, \|f\|_{BMO^r_{\mathcal{Q},M}(\mathcal{M})}\right\},$$

where the column BMO semi-norm is given by

$$\|f\|_{BMO^c_{\mathcal{Q},M}(\mathcal{M})} := \sup_{t>0} \left\| R_{\sqrt[m]{t}} |(I - e^{-tL})^M f|^2 \right\|_{\mathcal{M}}^{1/2}$$

Lemma 6 (Z. Fan, G. Hong and L. Wang)

Let $(\mathcal{M}, \tau)$ be a noncommutative measure space equipped with a markovian semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ and a P-markovian metric $\mathcal{Q} = \{(R_{j,t}, \sigma_{j,t}) : j \in \mathbb{N}, t > 0\}$. Then for any $M \in \mathbb{Z}_+$,

$$\|f\|_{\mathrm{BMO}^c_{\mathcal{S},M}} \lesssim k_{\mathcal{Q}} \|f\|_{\mathrm{BMO}^c_{\mathcal{Q},M}}.$$

Notation:

(1) $\rho_1, \rho_2 : \mathcal{M} \to \mathcal{N}_\rho$ are injective $*$-homomorphisms into certain von Neumann algebra $\mathcal{N}_\rho$;

(2) q_r's are projections in $\mathcal{N}_\rho$ satisfying $q_r \nearrow 1$ as $r \to +\infty$.

(3) $E_\rho : \mathcal{N}_\rho \to \rho_1(\mathcal{M})$ is an operator-valued weight;

Special model:

(1) $\mathcal{N}_\rho = L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\rho_1 f(x, y) = f(x)$, $\rho_2 f(x, y) = f(y)$.

(2) $q_r(x, y) = \chi_{\mathrm{B}_r(x)}(y) = \chi_{\mathrm{B}_r(y)}(x) = \chi_{|x-y|<r}$.

(3) E_ρ is the integral in $\mathbb{R}^n$ with respect to the variable y.

Reference

1. U. Haagerup, Operator-valued weights in von Neumann algebras I, *J. Function. Anal.* **32** (1979), 175–206.

Take

$$L_\infty^c(\mathcal{N}_\rho; E_\rho) = \Big\{ f \in \mathcal{N}_\rho : \big\| E_\rho(f^* f) \big\|_{\mathcal{M}} < \infty \Big\}.$$

The simplest nontrivial model:

$$E_\rho = \mathrm{tr}_{\mathcal{A}} \otimes id_{\mathcal{M}} \text{ with } \mathcal{N}_\rho = \mathcal{A} \bar{\otimes} \mathcal{M}$$

and $\mathcal{A}$ a semifinite non-finite von Neumann algebra.

Notation

$\mathcal{A}_{\mathcal{M}}$: weak-$*$ dense subalgebra of $\mathcal{M}$ and dense in $L_p(\mathcal{M})$ s.t. for any C^∞ differential function F on $\mathbb{R}$ satisfying

$$\left|\frac{d^\alpha}{dx^\alpha}F(x)\right| \leq C_\alpha, \text{for any } \alpha,$$

we have

$$F(L) : \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{M}}.$$

Assume that cpu map R_r from the P-markovian metric $\mathcal{Q}$ is of the following form:

$$\mathcal{M} \xrightarrow{\rho_j} \mathcal{N}_\rho \xrightarrow{E_\rho} \rho_1(\mathcal{M}) \simeq \mathcal{M}$$

$$R_r f = E_\rho(q_r)^{-\frac{1}{2}} E_\rho(q_r \rho_2(f) q_r) E_\rho(q_r)^{-\frac{1}{2}}, \tag{2}$$

Denote $L(X,Y) = \{\text{all linear operators from } X \text{ to } Y\}$.

Assume that there is a homomorphism

$$\pi : L(\mathcal{A}_{\mathcal{M}}, \mathcal{A}_{\mathcal{M}}) \to L(\mathcal{A}_{\mathcal{N}_\rho}, \mathcal{N}_\rho)$$

satisfying:
(1) continuous in certain suitable topology;
(2) for any $T \in L(\mathcal{A}_{\mathcal{M}}, \mathcal{A}_{\mathcal{M}})$,

$$\pi(T) \circ \rho_2 = \rho_2 \circ T \text{ on } \mathcal{A}_{\mathcal{M}}. \tag{3}$$

Algebraic conditions

(ALi) $\mathcal{Q}$-monotonicity of E_ρ

$$E_\rho(b_{k,r}|\xi|^2 b_{k,r}) \leq E_\rho(|\xi|^2)$$

for any $k \geq 1$, $\xi \in \mathcal{N}_\rho$ and every projection q_r, where

$$b_{k,r} = \begin{cases} q_{2r}, & \text{if } k = 1, \\ q_{2^{k+1}r} - q_{2^k r}, & \text{if } k \geq 2, \end{cases}$$

The above condition desribes:

$$\int_{U_k(B(x,r))} |\xi(x,y)|^2 d\mu(y) \leq \int_{\mathbb{R}^n} |\xi(x,y)|^2 d\mu(y),$$

where $U_k(B(x,r)) := B(x, 2^{k+1}r) - B(x, 2^k r)$ for $k \geq 2$ and $U_1(B(x,r)) = B(x, 2r)$.

Algebraic conditions

(ALii) Right $\mathcal{B}$-modularity of $\pi(F(L))$

$$\pi(F(L))(\eta b) = \pi(F(L))(\eta)b$$

for all $\eta \in \mathcal{A}_{\mathcal{N}_\rho}$, all F being Borel measurable function and all b lying in some von Neumann subalgebra $\mathcal{B}$ of $\rho_1(\mathcal{M})$ which includes $E_\rho(q_r)$ for every projection q_r.

The above condition desribes:

$$F(L \otimes id_{\mathcal{M}})(f\mu(B(x,r))) = F(L \otimes id_{\mathcal{M}})(f)\mu(B(x,r))$$

(ANi) Doubling condition
For any $r_1 \geq r_2$, $\exists C, n > 0$ s.t.

$$E_\rho(q_{r_2})^{-\frac{1}{2}} E_\rho(q_{r_1}) E_\rho(q_{r_2})^{-\frac{1}{2}} \leq C \left(\frac{r_1}{r_2}\right)^n.$$

We call n the dimension of $\mathcal{M}$.

The above condition desribes: if $r_2 \geq r_1 > 0$, then

$$\frac{\mu(B(x,r_2))}{\mu(B(x,r_1))} \leq C\Big(\frac{r_2}{r_1}\Big)^n.$$

(ANii) Covering inequalities
$\exists C > 0,\ D_1 \geq 0$, s.t. for any $r_1 \geq r_2 \geq r_3$ and $f \in \mathcal{M}$,

$$\left\| E_\rho\left(\left| \rho_2(f) q_{r_1} E_\rho(q_{r_3})^{-\frac{1}{2}} \right|^2 \right) \right\|_{\mathcal{M}} \leq C \left(\frac{r_1}{r_2} \right)^{D_1} \left(\frac{r_1}{r_3} \right)^{n} \left\| E_\rho\left(\left| \rho_2(f) q_{r_2} E_\rho(q_{r_2})^{-\frac{1}{2}} \right|^2 \right) \right\|_{\mathcal{M}}.$$

The above condition desribes:

$$\left\| \frac{1}{\mu(B(x,r_3))} \int_{B(x,r_1)} |f(y)|^2 d\mu(y) \right\|_{L_\infty(X)\bar{\otimes}\mathcal{M}} \leq C \left(\frac{r_1}{r_2} \right)^{D_1} \left(\frac{r_1}{r_3} \right)^{n} \left\| \frac{1}{\mu(B(x,r_2))} \int_{B(x,r_2)} |f(y)|^2 d\mu(y) \right\|_{L_\infty(X)\bar{\otimes}\mathcal{M}},$$

Operator conditions
(Oi) Boundedness condition

$$\|\pi(F(L))\|_{L^c_\infty(\mathcal{N}_\rho;E_\rho)\to L^c_\infty(\mathcal{N}_\rho;E_\rho)} \le \|F\|_{L_\infty(\mathbb{R})},$$

for all bounded Borel measurable function F.

The above condition desribes:

$$\left\|\left(\int_X |F(L\otimes id_{\mathcal{M}})f(x,y)|^2 d\mu(y)\right)^{1/2}\right\|_{\mathcal{N}} \le \|F\|_\infty \left\|\left(\int_X |f(x,y)|^2 d\mu(y)\right)^{1/2}\right\|_{\mathcal{N}},$$

Operator condition

(Oii) L_2-Gaussian estimate

$\exists C, c > 0$, $m \geq 2$ s.t. for any $t > 0$, $k \geq 1$, $0 < r_1 \leq r_2$ and $f \in \mathcal{N}_\rho$,

$$\left\| \pi(e^{-tL})(f b_{k,r_2}) q_{r_1} E_\rho(q_{r_1})^{-\frac{1}{2}} \right\|_{L_\infty^c(\mathcal{N}_\rho; E_\rho)} \leq C E_k \left(-c \left(\frac{r_2}{t^{1/m}} \right)^{\frac{m}{m-1}} \right) \left\| f b_{k,r_2} E_\rho(q_{\max\{r_1, t^{1/m}\}})^{-\frac{1}{2}} \right\|_{L_\infty^c(\mathcal{N}_\rho; E_\rho)},$$

where

$$E_k(x) = \begin{cases} 1, & \text{if } k = 1, \\ \exp(2^{\frac{mk}{m-1}} x), & \text{if } k \geq 2. \end{cases}$$

The above condition desribes:

$$\left\| \left(\fint_{B(x,r_1)} |e^{-t(-\Delta \otimes id_{\mathcal{M}})}(\chi_{U_k(B(x,r_2))} f(x,\cdot))(y)|^2 d\mu(y) \right)^{\frac{1}{2}} \right\|_{L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}}$$

$$\leq CE_k \left(-c \left(\frac{r_2}{t^{1/m}} \right)^{\frac{m}{m-1}} \right) \times$$

$$\times \left\| \left(\frac{1}{\mu(B(x, \max\{r_1, t^{1/m}\}))} \int_{U_k(B(x,r_2))} |f(x,y)|^2 d\mu(y) \right)^{\frac{1}{2}} \right\|_{L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}}$$

Theorem 7 (Z. Fan, G. Hong and L. Wang)

Let $(\mathcal{M},\tau)$ be a semifinite VNA equipped with a markovian semigroup $\mathcal{S}=(e^{-tL})_{t\geq 0}$ with associated P-markovian metric $\mathcal{Q}$ satisfying our algebraic and analytic assumptions. Then $\exists C=C(n)>0$ s.t. for any $f\in\mathcal{A}_{\mathcal{M}}$,

$$\left\|e^{itL}(I+L)^{-s}f\right\|_{\mathrm{BMO}_{\mathcal{S}}(\mathcal{M})}\leq C(1+|t|)^{n/2}\|f\|_{\mathcal{M}},\ \text{for } s\geq\frac{n}{2}.$$

If in addition $\mathcal{S}$ admits a Markov dilation, then we have

$$\left\|e^{itL}(I+L)^{-s}f\right\|_{L_p(\mathcal{M})}\leq C(1+|t|)^{\sigma_p}\|f\|_{L_p(\mathcal{M})}$$

for $s\geq\sigma_p=n\left|\frac{1}{2}-\frac{1}{p}\right|$.

Key Lemma:

Lemma 8 (Z. Fan, G. Hong and L. Wang)

Under our our algebraic and analytic assumptions, for any $M \in \mathbb{Z}_+$, the spaces $\mathrm{BMO}^c_{\mathcal{Q},1}(\mathcal{M})$ and $\mathrm{BMO}^c_{\mathcal{Q},M}(\mathcal{M})$ coincide, and their norms are equivalent.

Key steps:

$$\begin{aligned}
&\left\| e^{itL}(I+L)^{-s} f \right\|_{\mathrm{BMO}_{\mathcal{S}}(\mathcal{M})} \\
&\le C \left\| e^{itL}(I+L)^{-s} f \right\|_{\mathrm{BMO}_{\mathcal{Q}}(\mathcal{M})} \\
&\le C \left\| e^{itL}(I+L)^{-s} f \right\|_{\mathrm{BMO}_{\mathcal{Q},M}(\mathcal{M})} \\
&\le C_M (1+|t|)^{n/2} \|f\|_{\mathcal{M}}.
\end{aligned}$$

Role: The term $(I - e^{-\lambda L})^M$ endows function higher cancellation property in the low frequency.

Rough observation:
Let ϕ be a non-negative C_c^∞ function on $\mathbb{R}$ s.t. $\mathrm{supp}\phi \subset (1/4, 1)$. Let $\phi_\ell(u) = \phi(2^{-\ell}u)$. Then

$$\sup_{u\geq 0} |(I - e^{-\lambda u})^M \phi_\ell(u)| \leq C \min\{1, (2^\ell\lambda)^M\}.$$

We need:

$$\sum_{\ell\in\mathbb{Z}} (2^\ell\lambda)^{-\frac{n}{2m}} \min\{1, (2^\ell\lambda)^M\} < +\infty,$$

which requires $M > \frac{n}{2m}$.

Applications

By choosing different values of each parameter, the above theorem can be applied to obtain Schrödinger groups theory in the below concrete settings:

(1) Classical doubling metric space associated with non-negative self-adjoint operator with heat kernel bound;

(2) Operator-valued setting associated with non-negative self-adjoint operator with non-negative heat kernel bound;

(3) Matrix algebras;

(4) Quantum Euclidean space;

(5) Group VNA;

(6) ......

Notation:
(X, d, μ): n-dimensional doubling metric space X with distance d and measure μ;
$\mathcal{N} := L_\infty(X)\bar{\otimes}\mathcal{M}$.

Theorem 9 (Z. Fan, G. Hong and L. Wang)

Suppose that L is a non-negative self-adjoint operator on $L_2(X)$ and its associated heat kernel $p_t(x, y)$ satisfies the Gaussian upper bound (GE_m) and $p_t(x, y) \geq 0$ for any $x, y \in X$. Then $\exists C = C(n, m) > 0$ s.t. for any $f \in \mathcal{A}_\mathcal{N}$,

$$\left\|(id_\mathcal{N} + L \otimes id_\mathcal{M})^{-s} e^{it(L\otimes id_\mathcal{M})} f\right\|_{BMO_S(\mathcal{N})} \leq C(1 + |t|)^{n/2} \|f\|_\mathcal{N}$$

for $s \geq \frac{n}{2}$.

Examples:

1. Schrödinger operators: $-\Delta + V(x)$, where $V(x) \in L^1_{loc}(\mathbb{R}^n)$ s.t. $V(x) \geq 0$.
2. Bessel operator;
3. Sub-Laplacian operator on homogeneous groups;
4. Neumann operator on half-plane.
5. $(-\Delta)^m$, m is a positive integer.

6.......

Remark 1.2

In the case of $\mathcal{M} = L_\infty(\mathbb{C})$, the positivity assumption on $p_t(x, y)$ can be removed. This result goes back to the result due to Chen–Duong–Li–Yan.

Reference:

1. P. Chen, X.T. Duong, J. Li and L.X. Yan, Sharp endpoint L^p estimates for Schrödinger groups, *Math. Ann.* **378** (2020), 667–702.

Θ: anti-symmetric $\mathbb{R}$-valued $n \times n$ matrix;
$\mathcal{R}_\Theta$: quantum Euclidean space associated with Θ;
Δ_Θ: quantum Laplacian operator.

Theorem 10 (Z. Fan, G. Hong and L. Wang)

$\exists C = C(n) > 0$ *s.t.*

$$\left\| (id_{\mathcal{R}_\Theta} - \Delta_\Theta)^{-s} e^{-it\Delta_\Theta} f \right\|_{BMO^c_{\mathcal{S}}(\mathcal{R}_\Theta)} \leq C(1+|t|)^{n/2} \|f\|_{\mathcal{R}_\Theta}, \text{ for } s \geq \frac{n}{2}.$$

Furthermore, for any $1 < p < \infty$, $\exists C = C(n,p) > 0$ *s.t.*

$$\left\| (id_{\mathcal{R}_\Theta} - \Delta_\Theta)^{-s} e^{-it\Delta_\Theta} f \right\|_{L_p(\mathcal{R}_\Theta)} \leq C(1+|t|)^{n\left|\frac{1}{2}-\frac{1}{p}\right|} \|f\|_{L_p(\mathcal{R}_\Theta)}$$

for $s \geq n\left|\frac{1}{2} - \frac{1}{p}\right|$.

Given $A = \sum_{m,k} a_{m,k} e_{m,k} \in B(\ell_2)$, define

$$S_t(A) = \sum_{m,k} e^{-t|m-k|^2} a_{m,k} e_{m,k}.$$

By Stone's theorem, S_t admits an infinitesimal non-negative generator

$$L_{B(\ell_2)} A = -\lim_{t \to 0} \frac{S_t(A) - A}{t} = \sum_{m,k} |m-k|^2 a_{m,k} e_{m,k}.$$

Theorem 11 (Z. Fan, G. Hong and L. Wang)

$\exists C > 0$ *s.t.*

$$\left\|\left(id_{B(\ell_2)} + L_{B(\ell_2)}\right)^{-s} e^{itL_{B(\ell_2)}} A\right\|_{BMO^c_{\mathcal{S}}(B(\ell_2))} \leq C(1+|t|)^{1/2}\|A\|_{B(\ell_2)}$$

for $s \geq \frac{1}{2}$. *Furthermore, for any* $1 < p < \infty$, $\exists C_p > 0$ *s.t.*

$$\left\|\left(id_{B(\ell_2)} + L_{B(\ell_2)}\right)^{-s} e^{itL_{B(\ell_2)}} A\right\|_{S_p} \leq C(1+|t|)^s\|A\|_{S_p}$$

for $s \geq \left|\frac{1}{2} - \frac{1}{p}\right|$.

Notation:

G: discrete group;
$\lambda : G \to B(\ell_2(G))$ left regular representation;
$\mathcal{L}(G)$:group von Neumann algebra;
τ_G: standard normalized trace on $\mathcal{L}(G)$.

Any $f \in \mathcal{L}(G)$ admits a Fourier series expansion

$$\sum_{g \in G} \hat{f}(g)\lambda(g), \text{ with } \hat{f}(g) = \tau_G(f\lambda(g^{-1})) \; s.t. \; \tau_G(f) = \hat{f}(e),$$

where e denotes the identity of G.

For any $m : G \to \mathbb{R}$, a Fourier multiplier on $\mathcal{L}(G)$ is defined by

$$T_m : \sum_{g \in G} \hat{f}(g)\lambda(g) \mapsto \sum_{g \in G} m(g)\hat{f}(g)\lambda(g).$$

Notation:
ψ: length function on G;
$\mathcal{H}_\psi$: representation space associated with ψ;
$m_s(\psi(g)) := e^{it\psi(g)}(1+\psi(g))^{-s}$.

Theorem 12 (Z. Fan, G. Hong and L. Wang)

Assume $\dim\mathcal{H}_\psi = n < \infty$, *then* $\exists C = C(n) > 0$ *s.t.*

$$\|T_{m_s}(f)\|_{BMO_{S_\psi}(\mathcal{L}(G))} \leq C(1+|t|)^{n/2}\|f\|_{\mathcal{L}(G)}, \text{ for } s \geq \frac{n}{2}.$$

Furthermore, for any $1 < p < \infty$, $\exists C = C(n,p) > 0$ *s.t.*

$$\|T_{m_s}(f)\|_{L_p(\mathcal{L}(G))} \leq C(1+|t|)^{\sigma_p}\|f\|_{L_p(\mathcal{L}(G))}, \text{ for } s \geq n\Big|\frac{1}{2}-\frac{1}{p}\Big|.$$

Thank you!

The establishment of a general form of noncommutative SIO theory needs the following steps.

Step 1: Identify the approciate BMO spaces.

Step 2: Prove the expected interpolation results with L_p spaces.

Step 3: Provide conditions on CZO's which yield $L_\infty \to BMO$ boundedness.

Step 1: Identify the approciate BMO spaces.

Given a standard Markovian semigroup of operators of $\mathcal{S} = \{T_t\}$ on $\mathcal{M}$ and $f \in \mathcal{M} \cup L_2(\mathcal{M})$, we define

$$\|f\|_{bmo_{\mathcal{S}}^c} = \sup_t \|T_t|f|^2 - |T_t f|^2\|_{\mathcal{M}}^{1/2}.$$

$$\|f\|_{BMO_{\mathcal{S}}^c} = \sup_t \|T_t|f - T_t f|^2\|_{\mathcal{M}}^{1/2}.$$

Fact: If in addition (T_t) satisfies the $\Gamma_2 \geq 0$ condition, then

$$\|f\|_{BMO_{\mathcal{S}}^c} \simeq \|f\|_{bmo_{\mathcal{S}}^c} + \sup_t \|T_t f - T_{2t} f\|_{\mathcal{M}}.$$

Reference

1. M. Junge and T. Mei, BMO spaces associated with semigroups of operators, *Math. Ann.* **352** (2012), 691–743.

Step 2: Prove the expected interpolation results with L_p spaces.

Define

$$L_p^\circ(\mathcal{M}) = \Big\{ f \in L_p(\mathcal{M}) \, : \, \lim_{t\to\infty} S_t f = 0 \Big\}.$$

Theorem 13

If $\mathcal{S} = (S_t)_{t\geq 0}$ is regular on $(\mathcal{M}, \tau)$, then

$$\big[BMO_{\mathcal{S}}, L_p^\circ(\mathcal{M}) \big]_{p/q} \simeq L_q^\circ(\mathcal{M}) \quad \text{for all} \quad 1 \leq p < q < \infty.$$

Reference

1. M. Junge and T. Mei, BMO spaces associated with semigroups of operators, *Math. Ann.* **352** (2012), 691–743.

Step 3: Provide conditions on CZO's which yield $L_\infty \to BMO$ boundedness.

Step 3.1: Construct a 'metric' governing the Markov process

Reference

1. M. Junge, T. Mei, J. Parcet and R. Xia, Algebraic Calderon-Zygmund theory, *Adv. Math*. **376** (2021), 107443.

A Markov metric associated to $(\mathcal{M}, \tau)$ and $\mathcal{S}$ is determined by a family

$$\mathcal{Q} = \Big\{ \big(R_{j,t}, \sigma_{j,t}, \gamma_{j,t} \big) : \ (j,t) \in \mathbb{Z}_+ \times \mathbb{R}_+ \Big\}$$

where $R_{j,t} : \mathcal{M} \to \mathcal{M}$ are cpu maps and $\sigma_{j,t}, \gamma_{j,t}$ are elements of $\mathcal{M}$ with $\gamma_{j,t} \geq 1_{\mathcal{M}}$, s.t. the estimates below hold:

i) Hilbert module majorization:
$\big\langle \xi, \xi \big\rangle_{S_t} \leq \sum_{j \geq 1} \sigma_{j,t}^* \big\langle \xi, \xi \big\rangle_{R_{j,t}} \sigma_{j,t}$,

ii) Metric integrability condition:
$$k_{\mathcal{Q}} = \sup_{t>0} \Big\| \sum_{j \geq 1} \sigma_{j,t}^* \gamma_{j,t}^2 \sigma_{j,t} \Big\|_{\mathcal{M}}^{\frac{1}{2}} < \infty.$$

Let $\mathcal{S} = (S_t)_{t \geq 0}$ be a Markov semigroup on (Ω, μ) with associated kernels $s_t(x, y)$ satisfying 2-order Gaussian upper bound. Given $\xi : \Omega \times \Omega \to \mathbb{C}$ essentially bounded, we have

$$\begin{aligned}\langle \xi, \xi \rangle_{S_t} &= \int_\Omega s_t(x, y) |\xi(x, y)|^2 \, d\mu(y) \\ &\leq \sum_{j=1}^{\infty} \frac{|\sigma_{j,t}(x)|^2}{\mu(\Sigma_{j,t}(x))} \int_{\Sigma_{j,t}(x)} |\xi(x, y)|^2 \, d\mu(y).\end{aligned}$$

This means that $R_{j,t} f(x)$ is the average of f over the set $\Sigma_{j,t}(x)$.

Step 3.2: Define ‘metric BMO’ spaces which still interpolate with the L_p scale.

Define $\|f\|_{\mathrm{BMO}_{\mathcal{Q}}} = \max\{\|f\|_{\mathrm{BMO}_{\mathcal{Q}}^c}, \|f^*\|_{\mathrm{BMO}_{\mathcal{Q}}^c}\}$, where the column BMO-norm is given by

$$\sup_{t>0} \inf_{M_t \,\mathrm{cpu}} \sup_{j\geq 1} \Big\| \Big(\gamma_{j,t}^{-1} \big[R_{j,t}|f|^2 - |R_{j,t}f|^2 + |R_{j,t}f - M_t f|^2 \big] \gamma_{j,t}^{-1} \Big)^{\frac{1}{2}} \Big\|_{\mathcal{M}}$$

and the infimum runs over cpu maps $M_t : \mathcal{M} \to \mathcal{M}$.

Theorem 14 (Junge–Mei–Parcet–Xia, 2021)

Let $(\mathcal{M}, \tau)$ be a noncommutative measure space equipped with a Markov semigroup $\mathcal{S} = (S_t)_{t \geq 0}$. Let us consider a Markov metric $\mathcal{Q}$ associated to $\mathcal{S} = (S_t)_{t \geq 0}$. Then, we find

$$\|f\|_{\mathrm{BMO}_{\mathcal{S}}} \lesssim \mathrm{k}_{\mathcal{Q}} \|f\|_{\mathrm{BMO}_{\mathcal{Q}}}.$$

In particular, we see that $L_\infty(\mathcal{M}) \subset \mathrm{BMO}_{\mathcal{Q}} \subset \mathrm{BMO}_{\mathcal{S}}$ and

$$\big[\mathrm{BMO}_{\mathcal{Q}}, L_p^\circ(\mathcal{M})\big]_{p/q} \simeq L_q^\circ(\mathcal{M}) \quad \text{for all} \quad 1 \leq p < q < \infty$$

for any Markov metric $\mathcal{Q}$ associated to a regular semigroup $\mathcal{S} = (S_t)_{t \geq 0}$ on $(\mathcal{M}, \tau)$.

Step 3.3: Provide CZ conditions giving $L_\infty \to BMO$ boundedness for metric BMO's.

Let $E_\mathcal{M}$ be operator-valued weight from $\mathcal{N}$ to $\mathcal{M}$. And we take

$$L_\infty^c(\mathcal{N}; E_\mathcal{M}) = \Big\{ f \in \mathcal{N} : \big\| E_\mathcal{M}(f^* f) \big\|_\mathcal{M} < \infty \Big\}.$$

The simplest nontrivial model:

$$E_\mathcal{M} = \mathrm{tr}_\mathcal{A} \otimes id_\mathcal{M} \text{ with } \mathcal{N} = \mathcal{A} \bar{\otimes} \mathcal{M}$$

and $\mathcal{A}$ a semifinite non-finite von Neumann algebra.

Reference

1. M. Junge, T. Mei, J. Parcet and R. Xia, Algebraic Calderon-Zygmund theory, *Adv. Math.* **376** (2021), 107443.

Assume the cpu maps $R_{j,t}$ from $\mathcal{Q}$ are of the following form

$$\begin{aligned} \mathcal{M} \xrightarrow{\rho_j} \mathcal{N}_\rho \xrightarrow{E_\rho} \rho_1(\mathcal{M}) \simeq \mathcal{M}, \\ R_{j,t} f \ = \ E_\rho(q_{j,t})^{-\frac{1}{2}} E_\rho\big(q_{j,t}\rho_2(f)q_{j,t}\big) E_\rho(q_{j,t})^{-\frac{1}{2}}, \end{aligned} \tag{4}$$

Notation:
(1) $\rho_1, \rho_2 : \mathcal{M} \to \mathcal{N}_\rho$ are $*$-homomorphisms into certain von Neumann algebra $\mathcal{N}_\rho$;

(2) $E_\rho : \mathcal{N}_\rho \to \rho_1(\mathcal{M})$ is an operator-valued weight;

(3) $q_{j,t}$'s are projections in $\mathcal{N}_\rho$.

Notation:

(1) $\rho_1, \rho_2 : \mathcal{M} \to \mathcal{N}_\rho$ are $*$-homomorphisms into certain von Neumann algebra $\mathcal{N}_\rho$;

(2) $E_\rho : \mathcal{N}_\rho \to \rho_1(\mathcal{M})$ is an operator-valued weight;

(3) $q_{j,t}$'s are projections in $\mathcal{N}_\rho$.

Special model:

(1) $\mathcal{N}_\rho = L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\rho_1 f(x,y) = f(x)$, $\rho_2 f(x,y) = f(y)$.

(2) E_ρ is the integral in $\mathbb{R}^n$ with respect to the variable y.

(3) $q_{j,t}(x,y) = \chi_{\mathrm{B}_{\sqrt{4jt}}(x)}(y) = \chi_{\mathrm{B}_{\sqrt{4jt}}(y)}(x) = \chi_{|x-y|<\sqrt{4jt}}$.

Let T s.t. $Tf \in \mathcal{M}$ for all f in a weak-$*$ dense subalgebra $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$. Consider $*$-homomorphisms $\pi_1, \pi_2 : \mathcal{M} \to \mathcal{N}_\pi$ and an operator-valued weight $E_\pi : \mathcal{N}_\pi \to \pi_1(\mathcal{M})$. Assume there exists a (densely defined) map

$$\begin{aligned} \widehat{T} : \mathcal{A}_{\mathcal{N}_\pi} \subset \mathcal{N}_\pi \to \mathcal{N}_\rho \\ \text{satisfying} \quad \widehat{T} \circ \pi_2 = \rho_2 \circ T \quad \text{on} \quad \mathcal{A}_{\mathcal{M}}. \end{aligned} \tag{5}$$

Algebraic conditions:

i) *$\mathcal{Q}$-monotonicity of E_ρ*

$$E_\rho(q_{j,t}|\xi|^2 q_{j,t}) \leq E_\rho(|\xi|^2)$$

for all $\xi \in \mathcal{N}_\rho$ and every projection $q_{j,t}$ determined by $\mathcal{Q}$ via the identity. Similarly, we assume the same inequality holds when we replace the $q_{j,t}$'s by the q_t's.

ii) *Right $\mathcal{B}$-modularity of $\widehat{T}$*

$$\widehat{T}\left(\eta\, \pi_1 \rho_1^{-1}(b)\right) \;=\; \widehat{T}(\eta) b$$

for all $\eta \in \mathcal{A}_{\mathcal{N}_\pi}$ and all b lying in some von Neumann subalgebra $\mathcal{B}$ of $\rho_1(\mathcal{M})$ which includes $E_\rho(q_t)$, $E_\rho(q_{j,t})$ and $\rho_1(\gamma_{j,t})$ for every projection q_t and $q_{j,t}$ determined by $\mathcal{Q}$.

Consider derivations $\delta : \mathcal{N}_\rho \to \mathcal{N}_\sigma$ given by the difference $\delta = \sigma_1 - \sigma_2$ of two $*$-homomorphisms, s.t. $\delta(ab) = \sigma_1(a)\sigma_1(b) - \sigma_2(a)\sigma_2(b) = \delta(a)\sigma_1(b) + \sigma_2(a)\delta(b)$.

Define

$$\widehat{R}_{j,t} : \mathcal{N}_\rho \ni \xi \mapsto E_\rho(q_{j,t})^{-\frac{1}{2}} E_\rho(q_{j,t}\xi q_{j,t}) E_\rho(q_{j,t})^{-\frac{1}{2}} \in \rho_1(\mathcal{M}),$$

$$\widehat{M}_t : \mathcal{N}_\rho \ni \xi \mapsto E_\rho(q_t)^{-\frac{1}{2}} E_\rho(q_t \xi q_t) E_\rho(q_t)^{-\frac{1}{2}} \in \rho_1(\mathcal{M}).$$

Analytic conditions:

i) *Mean differences conditions*

- $\widehat{R}_{j,t}(\xi^*\xi) - \widehat{R}_{j,t}(\xi)^*\widehat{R}_{j,t}(\xi) \le \Phi_{j,t}\big(\delta(\xi)^*\delta(\xi)\big)$,
- $\big[\widehat{R}_{j,t}(\xi) - \widehat{M}_t(\xi)\big]^*\big[\widehat{R}_{j,t}(\xi) - \widehat{M}_t(\xi)\big] \le \Psi_{j,t}\big(\delta(\xi)^*\delta(\xi)\big)$,

for some derivation $\delta : \mathcal{N}_\rho \to \mathcal{N}_\sigma$ and cpu maps $\Phi_{j,t}, \Psi_{j,t} : \mathcal{N}_\sigma \to \rho_1(\mathcal{M})$.

Remark: the above inequality is a replacement of a couple of Jensen's inequality, i.e.

$$\begin{aligned} ⨍_{\mathrm{B}_1} |f|^2 d\mu - \Big| ⨍_{\mathrm{B}_1} f d\mu \Big|^2 &\le ⨍_{\mathrm{B}_1\times \mathrm{B}_1} \big|f(y) - f(z)\big|^2 d\mu(y) d\mu(z), \\ \Big| ⨍_{\mathrm{B}_1} f d\mu - ⨍_{\mathrm{B}_2} f d\mu \Big|^2 &\le ⨍_{\mathrm{B}_1\times \mathrm{B}_2} \big|f(y) - f(z)\big|^2 d\mu(y) d\mu(z). \end{aligned}$$

Analytic conditions:

ii) *Metric/measure growth conditions*

- $\mathbf{1} \leq \pi_1\rho_1^{-1}E_\rho(q_t)^{-\frac{1}{2}}E_\pi(a_t^* a_t)\pi_1\rho_1^{-1}E_\rho(q_t)^{-\frac{1}{2}} \lesssim \pi_1\rho_1^{-1}(\gamma_{j,t}^2)$,
- $\mathbf{1} \leq \pi_1\rho_1^{-1}E_\rho(q_{j,t})^{-\frac{1}{2}}E_\pi(a_{j,t}^* a_{j,t})\pi_1\rho_1^{-1}E_\rho(q_{j,t})^{-\frac{1}{2}} \lesssim \pi_1\rho_1^{-1}(\gamma_{j,t}^2)$,

for some family of operators $a_t, a_{j,t} \in \mathcal{N}_\pi$ to be determined later on.

Calderón-Zygmund type conditions:

i) *Boundedness condition*

$$\widehat{T} : L_\infty^c(\mathcal{N}_\pi; E_\pi) \to L_\infty^c(\mathcal{N}_\rho; E_\rho).$$

Remark: The above inequality is a replacement of L_2 boundedness of T.

Calderón-Zygmund type conditions:

ii) *Size "kernel" condition*

- $\widehat{M}_t\left(\left|\widehat{T}(\pi_2(f)(A_{j,t}-a_t))\right|^2\right) \lesssim \gamma_{j,t}^2\|f\|_\infty^2$,
- $\widehat{R}_{j,t}\left(\left|\widehat{T}(\pi_2(f)(A_{j,t}-a_{j,t}))\right|^2\right) \lesssim \gamma_{j,t}^2\|f\|_\infty^2$,

for a family of operators $A_{j,t} \in \mathcal{N}_\pi$ with $A_{j,t} \geq a_{j,t}, a_t$ to be determined.

The above inequality is a weaker replacement of pointwise inequality:

$$|K(x,y)| \leq \frac{C}{|x-y|}.$$

iii) *Smoothness "kernel" condition*

- $\Phi_{j,t}\Big(\big|\delta\big(\widehat{T}(\pi_2(f)(\mathbf{1} - a_{j,t}))\big)\big|^2\Big) \lesssim \gamma_{j,t}^2 \|f\|_\infty^2,$
- $\Psi_{j,t}\Big(\big|\delta\big(\widehat{T}(\pi_2(f)(\mathbf{1} - A_{j,t}))\big)\big|^2\Big) \lesssim \gamma_{j,t}^2 \|f\|_\infty^2.$

The above inequality is a weaker replacement of pointwise inequality:

$$\sup_{y_1,y_2\in\mathbb{R}^n} \int_{|y_1-z|\geq 2|y_1-y_2|} |K(y_1,z) - K(y_2,z)|dz < +\infty.$$

Theorem 15 (Junge–Mei–Parcet–Xia, 2021)

Let $(\mathcal{M}, \tau)$ be a noncommutative measure space equipped with a Markov semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ with associated Markov metric $\mathcal{Q}$ which fulfills our algebraic and analytic assumptions. Then, any algebraic column CZO T defines a bounded operator

$$T : \mathcal{A}_{\mathcal{M}} \to \mathrm{BMO}^c_{\mathcal{Q}}.$$

By choosing different values of each parameter, the above theorem can be applied to obtain CZO theory in the above settings:

(1) Classical doubling metric space;

(2) Ornstein-Uhlenbeck semigroup on the Euclidean space equipped with Gaussian measure;

(3) Operator-valued setting;

(4) Matrix algebras;

(5) Quantum Euclidean space;

(6) ......

Thank you!