A Paley's inequality for non-abelian discrete group

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Paleys inequality in analytic H^p-space

Denote by $\mathbb{T}$ the unit circle.
Lacunary sequence (or, Lacunary set) $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{Z}$: there exists $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\frac{|n_{k+1}|}{|n_k|} > 1 + \delta.$$

Given $(c_k) \subseteq \mathbb{C}$, a classical Khintchine type inequality states that there exists $C_\delta < \infty$ such that

$$\left\| \sum_{k=1}^{\infty} c_k z^{n_k} \right\|_{L^1(\mathbb{T})} \leq \left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{2}} \leq C_\delta \left\| \sum_{k=1}^{\infty} c_k z^{n_k} \right\|_{L^1(\mathbb{T})}.$$

Paleys inequality in analytic H^p-space

Plancherel theorem

$$\int_0^{2\pi} |\sum_k c_k z^k|^2 d\theta = \sum_k |c_k|^2.$$

• $\ell_2(\mathbb{N}) \subseteq L^1(\mathbb{T})$. $(\sum_k |c_k|^2)^{\frac{1}{2}} \simeq \|f\|_{L^1(\mathbb{T})}$ provided $\widehat{f}(2^k) = c_k$ and $\widehat{f}(n) = 0$ otherwise.
• However, the map

$$P : f \mapsto \left(\hat{f}(n_k)\right)_{k\in\mathbb{N}}$$

does not extend to a bounded map from $L^1(\mathbb{T})$ to $\ell_2(\mathbb{N})$. e.g. $f(z) = \prod_{k=1}^N (1 + \frac{z^{2^k} + z^{-2^k}}{2})$ which have norm $\|f\|_{L^1(\mathbb{T})} = 1$ while $(\hat{f}(2^k))_{1\leq k\leq N}$ has norm $\frac{\sqrt{N}}{2}$ since $\hat{f}(2^k) = \frac{1}{2}$ for $k = 1, \cdots, N$.

Paleys inequality in analytic H^p-space

Let $H^1(\mathbb{T})$ be the real Hardy space on the unit circle:

$$H^1(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : \|f\|_{H^1(\mathbb{T})} = \|f\|_{L^1} + \|H(f)\|_{L^1} < \infty \right\},$$

with H the Hilbert transform of f. A classical theorem of Paley [1] asserts that

$$\Big(\sum_{j_k \in \Lambda} |\widehat{f}(n_k)|^2\Big)^{\frac{1}{2}} \leq C_\Lambda \|f\|_{H^1(\mathbb{T})},$$

where Λ is a lacunary sequence. Moreover,

$$\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \simeq \inf \left\{ \|f\|_{H^1(\mathbb{T})} : f \in H^1(\mathbb{T}), \hat{f}(n_k) = c_k \right\}$$

1. R. E. A. C. Paley, On the lacunary coefficients of power series, Ann. of Math, (2), 34(3):615616, 1933.

Paleys inequality in analytic H^p-space

- $\ell^2 \subseteq H^1(\mathbb{T})$. $P : H^1(\mathbb{T}) \to \ell_2$ is bounded.
- A subset $E \subset \mathbb{N}$ is called a Paley set([2]) if the above equivalence holds for all choices of $(c_k)_k \in \ell_2, n_k \in E$ with constants depending only on E.
- Rudin[3] proved that E is a Paley set only if

$$\sup_{n\in\mathbb{N}} \#E \cap [2^n, 2^{n+1}] < C$$

- Paley set E is a finite union of lacunary sequences.

2. G. Pisier, Multipliers and lacunary sets in non-amenable groups. Amer. J. Math. 117 , no. 2, 337-376, (1995)

3. W. Rudin, Remarks on a theorem of Paley. *J. Lond. Math. Soc.*, 32(1957), 307-311.

Paleys inequality in analytic H^p-space

Theorem (Paley)

Let $0 < p < \infty$ and let (n_k) be a lacunary sequence of positive integers. Then $f(z) = \sum_{k=1}^{\infty} \widehat{f}(n_k) z^{n_k} \in H^p(\mathbb{T})$ if and only if $\sum_{k=1}^{\infty} |\widehat{f}(n_k)|^2 < \infty$. Moreover, for each such p there exists a constant C that depends only on p such that

$$C^{-1} \|(\widehat{f}(n_k))\|_{\ell^2} \leq \|f\|_{H^p(\mathbb{T})} \leq \|(\widehat{f}(n_k))\|_{\ell^2}.$$

• Rudin[5] has shown that Paley inequality holds for the case of compact connected abelian group with a total order.

4. Jevtic, M., Vukotic, D., Arsenovic, M., Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces, vol. 2. RSME Springer Series, New York (2016)

5. W. Rudin, Fourier Analysis on Groups. Wiley, New York, (1990).

Paleys inequality in analytic H^p-space

S^1: all trace class operators on ℓ^2,
$H^1(S^1) = \overline{H^1(\mathbb{T}) \otimes S^1} \subseteq L^1(\mathbb{T}; S^1)$.

Theorem (Lust-Piquard, Pisier; 1991)

Suppose Λ is a lacunary sequence, then there is a constant C such that for all functions $f = \sum_{k\in\Lambda} a_k e^{ikt} \in H^1(S^1), a_k \in S^1$,

$$|||(a_k)||| \leq C\|f\|_{H^1(S^1)},$$

where $|||(a_k)||| := \inf\{tr(\sum_k |a_k|^2)^{\frac{1}{2}} + tr(\sum_k |b_k^*|^2)^{\frac{1}{2}} : c_k = a_k + b_k\}$.

6. F. Lust-Piquard, G. Pisier, Noncommutative Khintchine and Paley inequalities. Ark. Mat. 29 , no. 2, 241-260, (1991)

Paleys inequality in BMO space

Fefferman-Stein: the dual space of $H^1(\mathbb{T})$ is $BMO(\mathbb{T})$

$$\|f\|_{BMO(\mathbb{T})} = \sup_I \frac{1}{|I|} \int_I |f - f_I| \, ds, f \in L^1(\mathbb{T})$$

with the supremum taking over all arcs $I \subseteq \mathbb{T}$. By Fefferman-Stein's H^1-BMO duality theory,

$$\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \simeq \inf \left\{ \|f\|_{H^1(\mathbb{T})} : f \in H^1(\mathbb{T}), \hat{f}(n_k) = c_k \right\}$$

has an equivalent formulation that, for any $(c_k) \in \ell_2$,

$$\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \simeq_\delta \left\| \sum_k c_k z^{n_k} \right\|_{BMO(\mathbb{T})} .$$

A w^* closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$(or $\mathcal{D}$), if

1. $\mathcal{A} + \mathcal{A}^*$ is w^* dense in $\mathcal{M}$;
2. $\mathcal{E}$ is multiplicative on $\mathcal{A}$, i.e., $\mathcal{E}(ab) = \mathcal{E}(a)\mathcal{E}(b)$ for all $a, b \in \mathcal{A}$;
3. The restriction of τ on $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ is semifinite.
4. $\tau(\mathcal{E}(x)) = \tau(x)$ for every positive operator $x \in \mathcal{M}$.

where $\mathcal{A}^* = \{x^* : x \in \mathcal{A}\}$.

Paley inequality on nc analytic hardy spaces

Let $(G, \leq)$ be a countable discrete group with a bi-invariant order:
$\bullet$let G^+ be a subsemigroup of G with the properties: $G^+ \cup G^- = G$ and $G^+ \cap G^- = \{e\}$.
$\bullet$ Define the relation $\leq$ in G by $x \leq y$ if and only if $x^{-1}y \in G^+$.
$\bullet$We write $x < y$ if $x^{-1}y \in G^+$ and $x^{-1}y \neq e$.
$\bullet x \leq y$ implies $zx \leq zy$ for every $z \in G$.
$\bullet$This order will be invariant under right multiplication if (and only if) G^+ is normal in the sense that $zG^+z^{-1} \subseteq G^+$, for every $z \in G$.

Paley inequality on nc analytic hardy spaces

$\mathcal{L}(G) = \{\lambda_g : g \in G\}''$; τ is the trace on $\mathcal{L}(G)$.

Put $\mathcal{A}_G = \overline{\{\sum c_g \lambda_g : g \geq e\}}^{w^*}$ and $\mathcal{D}_G := \mathcal{A}_G \cap \mathcal{A}_G^* = \{\lambda 1 : \lambda \in \mathbb{C}\}$.

Lemma

Let $\mathcal{N} := \mathcal{L}(G)\overline{\otimes}B(\mathcal{H})$. *Then*

$$\mathcal{A}_{\mathcal{N}} := \mathcal{A}_G \overline{\otimes} B(\mathcal{H}) = \overline{\{x = \sum_{g \in G} \lambda_g \otimes c_g \in B(\mathcal{H}) : g \geq e, c_g \in \mathcal{M}\}}^{w^*}.$$

is a maximal semifinite subdiagonal subalgebra of $\mathcal{N}$ *with respect to* $\mathcal{E} \otimes 1$.

Paley inequality on nc analytic hardy spaces

For $0 < p < \infty$, we define the noncommutative Hardy spaces $H^p(\mathcal{N})$ by

$$H^p(\mathcal{N}) = \overline{(\mathcal{A}_G \overline{\otimes} \mathcal{M}) \cap L^p(\mathcal{N})}^{\|\cdot\|_p}$$

7. W. B. Arveson, Analyticity in operator algebras, *Amer. J Math.*,89(1967), 578-642.

8. M. Marsalli, G. West, Noncommutative H_p spaces, *Journal of Operator Theory*, 40(1998), 339-355.

Paley inequality on nc analytic hardy spaces

Let $0 < p \leq \infty$. We define the space $S^p(\ell^2_{rc})$ as follows:

1. If $0 < p < 2$,
$$S^p(\ell^2_{rc}) = S^p(\ell^2_c) + S^p(\ell^2_r)$$
equipped with the intersection norm:
$$\|(a_k)_{n\geq 0}\|_{S^p(\ell^2_{rc})} = \inf_{a_k = b_k + c_k} \{\|(b_k)_{n\geq 0}\|_{S^p(\ell^2_r)} + \|(c_k)_{n\geq 0}\|_{S^p(\ell^2_c)}\}$$
2. If $p \geq 2$,
$$S^p(\ell^2_{rc}) = S^p(\ell^2_c) \cap S^p(\ell^2_r)$$
equipped with the intersection norm:
$$\|(a_k)_{n\geq 0}\|_{S^p(\ell^2_{rc})} = \max\{\|(a_k)_{n\geq 0}\|_{S^p(\ell^2_r)}, \|(a_k)_{n\geq 0}\|_{S^p(\ell^2_c)}\}.$$

9. M. Junge, C. Le Merdy, Q. Xu, H^∞ functional calculus and square functions on noncommutative L_p-spaces, *Astérisque* 305, vi+138 pp(2006).

Paley inequality on nc analytic hardy spaces

For each $g \in G_+$, let $L_g = \{h : g \leq h \leq g^2\}$. For $E \subset G_+$, let $N(E, g) = \#(L_g \cap E)$. We say $E \subset G_+$ is lacunary, if

$$N(E) = \sup_{g \in G_+} N(E, g) < \infty.$$

For a general subset $E \subset G$, let $E_+ = E \cap G_+, E_- = E - E_+$. We say E is lacunary if $N(E) = N(E_+) + N((E_-)^{-1}) < \infty$.

Theorem (CHLM2020)

Assume that E is a lacunary subset of G_+. Then, for any sequence $(c_k)_k \subset S^1$, and any sequence $(g_k)_{k=1}^{\infty} \subseteq E$, we have

$$\|(c_k)_{k=1}^{\infty}\|_{S^1(\ell^2_{cr})} \simeq \inf\left\{(tr \otimes \tau)(|f|) : f \in L^1(\mathcal{N}), \hat{f}(g_k) = c_k, \hat{f}(g) = 0, \forall g < e\right\}$$

Sketch for the proof of Theorem

• By the convexity of $|\cdot|^2$ and the complete positivity of τ_G, we have that, for any finite sequence $g_k \in G$,

$$\tau_G|\sum_k a_k\lambda_{g_k}| \leq (\tau_G|\sum_k a_k\lambda_{g_k}|^2)^{\frac{1}{2}} = (\sum_k |a_k|^2)^{\frac{1}{2}}.$$

• Writing $c_k = a_k + b_k$, we get,

$$\|\sum_k c_k\lambda_{g_k}\|_1 \leq \|(c_k)_{k=1}^{\infty}\|_{S^1(\ell^2_{cr})}. \tag{2.1}$$

• For $f \in H^1(\mathcal{N})$ and $\varepsilon > 0$, by Riesz factorization theorem, there exist $y, z \in H^2(\mathcal{N})$ such that $f = yz$ and $\|y\|_2\|z\|_2 \leq \|f\|_1 + \varepsilon$.

Sketch for the proof of Theorem

Given an element $g_i \in E$ with $\widehat{f}(g_i) \neq 0$. Recall that $\hat{f}(g) = \tau_G(f\lambda_g^*)$, we have

$$\widehat{f}(g_i) = \sum_{e \leq h \leq g_i} \widehat{y}(h)\widehat{z}(h^{-1}g_i) = A_i + B_i,$$

where

$$\begin{aligned} A_i : &= (\tau_G \otimes 1)\left[yZ_i\left(\lambda_{g_i^{-1}} \otimes 1\right)\right] \\ B_i : &= (\tau_G \otimes 1)\left[\left(\lambda_{g_i^{-1}} \otimes 1\right)Y_i z\right], \end{aligned}$$

where

$$Z_i = \sum_{e \leq h \leq h^2 < g_i} \lambda_h \otimes \widehat{z}(h),$$

$$Y_i = \sum_{e \leq h \leq h^2 \leq g_i} \lambda_h \otimes \widehat{y}(h).$$

Sketch for the proof of Theorem

- Since $N(E,g) \leq K$, we get

$$\|(A_i)_{i=1}^n\|_{S^1(\ell_r^2)} \leq K^{\frac{1}{2}}(\|f\|_{L^1(\mathcal{N})} + \varepsilon).$$

and

$$\|(B_i)_1^n\|_{S^1(\ell_c^2)} \leq K^{\frac{1}{2}}(\|f\|_{L^1(\mathcal{N})} + \varepsilon).$$

Therefore,

$$\begin{aligned} \|(\widehat{f}(g_i))_{i=1}^n\|_{L^1(\mathcal{M},\ell_{cr}^2)} &\leq \|(B_i)_{i=1}^n\|_{L^1(\mathcal{M},\ell_c^2)} + \|(A_i)_{i=1}^n\|_{L^1(\mathcal{M},\ell_r^2)} \\ &\leq 2K^{\frac{1}{2}}(\|f\|_{L^1(\mathcal{N})} + \varepsilon), \end{aligned}$$

This completes the proof by letting $\varepsilon \to 0$.

Paley inequality on nc analytic hardy spaces

Let H be the linear map on $L^2(\mathcal{L}G)$ such that

$$H(\sum_g c_g \otimes \lambda_g) = -i(\sum_{g \geq e} c_g \otimes \lambda_g - \sum_{g \leq e} c_g \otimes \lambda_g). \qquad (2.2)$$

For $f = \sum_g c_g \otimes \lambda_g \in L^2(\mathcal{L}G)$, set

$$\|f\|_{BMO(\mathcal{L}G)} = \inf\{\|u\|_{L^\infty(\mathcal{L}G)} + \|v\|_{L^\infty(\mathcal{N})} : f = u + Hv\}$$

where the infimum is taken over all $u, v \in L^\infty(\mathcal{N})$.
Let $BMOA(\mathcal{L}G)$ be the space of all $f \in H^2(\mathcal{L}G)$ with finite $\|\cdot\|_{BMO(\mathcal{L}G)}$-norms.

- $H^1(\mathcal{L}G)^* = BMOA(\mathcal{L}G)$.

10. M. Marsalli, G. West, The dual of noncommutative H^1. Indiana Univ. Math. J. 47, no. 2, 489-500, (1998)

Paley inequality on nc analytic hardy spaces

Let $\mathcal{N} = \mathcal{M}\overline{\otimes}\mathcal{L}(G)$ with the trace $tr \otimes \tau_G$. For $1 \leq p \leq \infty$, let $H^p(\mathcal{N})$ be the norm (respectively weak operator) closure in $L^p(\mathcal{N})$ of the collection of all finite sums $\sum_{g \geq e} c_g \otimes \lambda_g$ with $c_g \in L^p(\mathcal{M})$. In this case, $H^1(\mathcal{N})$ coincides with the projective tensor product $L^1(\mathcal{M})\hat{\otimes}H^1(\mathcal{L}(G))$, and its dual is isomorphic to BMOA$(\mathcal{N}) = \mathcal{M}\bar{\otimes}BMOA(\mathcal{L}(G))$ the injective tensor product. The Hilbert transform $id \otimes H$ extends to a bounded map on $L^p(\mathcal{N})$ for all $1 < p < \infty$. So, for $1 < p < \infty$, $H^p(\mathcal{N})$ is a complemented subspace of $L^p(\mathcal{N})$, and we have the following equivalence for $f = \sum_g c_g \otimes \lambda_g \in L^p(\mathcal{N})$,

$$\|f\|_p \simeq \|\sum_{g \geq e} c_g \otimes \lambda_g\| + \|\sum_{g < e} c_g \otimes \lambda_g\|_p.$$

Paley inequality on nc analytic hardy spaces

Corollary (CHLM2020)

Assume that E is a lacunary subset of G_+. Then, for any sequence $(c_k)_k \subset S^p$, and any sequence $(g_k)_{k=1}^{\infty} \subseteq E$, we have

$$\begin{aligned}
\|(c_k)_{k=1}^{\infty}\|_{S^\infty(\ell^2_{cr})} &\simeq \|\sum_{k=1}^{\infty} c_k \otimes \lambda_{g_k}\|_{BMO(\mathcal{N})}, \\
\|(c_k)_{k=1}^{\infty}\|_{S^1(\ell^2_{cr})} &\simeq \|\sum_{k=1}^{\infty} c_k \otimes \lambda_{g_k}\|_{S^1(\mathcal{N})}, \\
\|(c_k)_{k=1}^{\infty}\|_{S^p(\ell^2_{cr})} &\simeq \|\sum_{k=1}^{\infty} c_k \otimes \lambda_{g_k}\|_{S^p(\mathcal{N})}, 1 < p < \infty
\end{aligned}$$

Paley inequality on nc analytic hardy spaces

Corollary (CHLM2020)

For any sequence $(g_i)_{i=1}^{\infty}$ *in a lacunary subset* $E \in G$

$$\left\| \sum_{i=1}^{\infty} \lambda_{g_i} \otimes c_{g_i} \right\|_{L^p(\mathcal{N})} \simeq \|(c_{g_i})_{i=1}^{\infty}\|_{S^p(\ell^2_{cr})}, \quad 0 < p < \infty,$$

$$\|(c_{g_i})_{i=1}^{\infty}\|_{S^p(\ell^2_{cr})} \simeq \inf \left\{ \|f\|_{L^p(\mathcal{N})} : f \in L^p(\mathcal{N}), \hat{f}(g_i) = c_{g_i} \right\}, 1 < p < \infty.$$

Free group $\mathbb{F}_2$

A conditionally negative definite length ψ on G. By that, we mean ψ is a $\mathbb{R}_+$-valued function on G satisfying $\psi(g) = 0$ if and only if $g = e$, $\psi(g) = \psi(g^{-1})$, and

$$\sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \leq 0 \tag{3.1}$$

for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$.

Free group $\mathbb{F}_2$

For $g \in \mathbb{F}_2$ in the form of $g = a^{j_1} b^{k_1} \cdots a^{j_N} b^{k_N}$, let

$$|g|_z = \left| \sum_{i=1}^{N} j_i \right|^2 + \left| \sum_{i=1}^{N} k_i \right|^2 .$$

Then

$$\psi_z : g \mapsto |g|_z$$

is a conditionally negative definite function on $\mathbb{F}_2$, and the unbounded linear operator $L_z : \lambda_g \mapsto \psi_z \lambda_g$ generates a symmetric Markov semigroup on the free group von Neumann algebra $\mathcal{L}(\mathbb{F}_2)$.

12. C. Berg, J. Christensen, P. Ressel, Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions. *Graduate Text in Mathematics, Springer-Verlag*, (1984)

Free group $\mathbb{F}_2$

For $(z_1, z_2) \in \mathbb{T}^2$, let π_z be the $*$-homomorphism on $\mathcal{L}(\mathbb{F}_2)$ such that

$$\pi_z(\lambda_a) = z_1\lambda_a, \quad \pi_z(\lambda_b) = z_2\lambda_b.$$

Given $f \in \mathcal{L}(\mathbb{F}_2)$, viewing $\pi_z(f)$ as an operator valued function on $\mathbb{T}^2$, one can see that

$$\pi_z^{-1}(\Delta \otimes id)\pi_z(f) = L_z(f)$$

with Δ being the Laplacian on $\mathbb{T}^2$.
• $L_z \longleftrightarrow$ Laplacian. • subgroup $ker(\psi_z)$.

Free group $\mathbb{F}_2$

A bi-invariant order on free groups $\mathbb{F}_2 : \langle a, b\rangle \longleftrightarrow \mathbb{Z}[A, B]$:

$$\mu(a) = 1 + A,\ \mu(a^{-1}) = 1 - A + A^2 - A^3 + \cdots ,$$
$$\mu(b) = 1 + B,\ \mu(b^{-1}) = 1 - B + B^2 - B^3 + \cdots .$$

Denote by "$\leq$" the dictionary order on $\mathbb{Z}[A, B]$ assuming $0 \leq B \leq A$. We then formally define the ordering on the free group $\mathbb{F}_2$ by setting

$$g \leq h \text{ in } \mathbb{F}_2 \quad \text{if} \quad \mu(g) \leq \mu(h) \text{ in } \Lambda.$$

For any word X of A, B, denote by $J_X(g)$ the coefficient of the X term in $\mu(g)$.

12. A. A. Vinogradov, On the free product of ordered groups. (Russian) Mat. Sbornik N.S. 25(67), 163-168, (1949)

Free group $\mathbb{F}_2$

Let

$$\mathbb{F}_2^0 \quad = \quad \mathsf{ker}(\psi_z) = \{g \in \mathbb{F}_2 : J_A(g) = J_B(g) = 0\}$$

For $g \in \mathbb{F}_2^0$, $g > e$ if $J_{AB}(g) > 0$ since $J_{AA}(g) = 0$.
• Given a sequence $g_n \in \mathbb{F}_2$, then $E = \{g_n : n \in \mathbb{N}\}$ is a lacunary subset of $\mathbb{F}_2$ if any of the following holds:

- The sequence $J_A(g_n) \in \mathbb{Z}$ is lacunary.
- $J_A(g_n) = 0$ for all n and the sequence $J_B(g_n) \in \mathbb{Z}$ is lacunary.
- $J_A(g_n) = J_B(g_n) = 0$ for all n, and $J_{AB}(g_n)$ is lacunary.

• For instance, $\{a^{2^i} b^{k_i} \in \mathbb{F}_2 : i, k_i \in \mathbb{N}^+\}$ and $\{a^{2^k} b^{2^k} a^{-2^k} b^{-2^k} : k \in \mathbb{N}\}$ are lacunary subsets of $\mathbb{F}_2$.

Free group $\mathbb{F}_2$

Corollary (CHLM2020)

Suppose $(g_k)_k \in \mathbb{F}_2^0$ *is a sequence with* $(J_{AB}(g_k))_k \in \mathbb{Z}$ *lacunary. Then for any* $(c_k)_k$ *with elements in* $S^p(H)$*, we have*

$$\|(c_k)\|^p_{S^p(\ell^2_{cr})} \simeq (\tau \otimes tr)\left(\left|\sum_k c_k \otimes \lambda_{g_k}\right|^p\right)$$

for all $0 < p < \infty$*. Moreover, for* $p = 1$*, we have*

$$\|(c_k)\|_{S^1(\ell^2_{cr})} \simeq \inf\left\{(\tau \otimes tr)\left(\left|\sum_{J_{AB}(g)\geq 0} \hat{f}(g) \otimes \lambda_g\right| + \left|\sum_{J_{AB}(g)<0} \hat{f}(g) \otimes \lambda_g\right|\right)\right\}$$

Here, the infimum runs over all $f \in L^1(\mathcal{L}(\mathbb{F}_2)) \otimes S^1(H)$ *with* $\hat{f}(g_k) = c_k$.

Paley's inequality in the semigroup language

For each $t > 0$, let $P_t(e^{ik\theta}) = e^{-|k|t}e^{ik\theta}$. For $f \in L^1(\mathbb{T})$,

$$\|f\|_{BMO(\mathbb{T})} \simeq \sup_{t>0} \left\| P_t \left[|f - P_t(f)|^2 \right] \right\|_{L^\infty(\mathbb{T})}^{\frac{1}{2}}$$

$$\|f\|_{H^1(\mathbb{T})} \simeq \left\| \left(\int_0^\infty \left| \frac{\partial}{\partial t} P_t f \right|^2 t dt \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{T})}.$$

Garnett, Garsia, Stein, Mcintosh, Duong/Yan, Hoffman/Pascal...

Paley's inequality in semigroup language

For a conditionally negative definite length ψ on G, we say a sequence $(h_k)_{k\in\mathbb{N}}$ of elements of G is ψ-lacunary if there exists a constant $\delta > 0$ such that

$$\begin{aligned} \psi(h_k) &\geq (1+\delta)\psi(h_j) \\ \psi(h_j^{-1}h_k) &\geq \delta\psi(h_k). \end{aligned}$$

for any $k > j$.

Paley's inequality in semigroup language

Let

$$T_t : \lambda_g \mapsto e^{-t\psi(g)} \lambda_g$$

be the semigroup of operators on the group von Neumann algebra $\mathcal{L}(G)$ associated with ψ. Let

$$\|f\|_{H_c^1(\psi)} = \tau \left[\left(\int_0^\infty \left| \frac{\partial}{\partial s} T_s(f) \right|^2 s \, ds \right)^{\frac{1}{2}} \right]$$

$$\|f\|_{\mathrm{BMO}_c(\psi)} = \sup_{s>0} \left\| T_s \left[|f - T_s(f)|^2 \right] \right\|^{\frac{1}{2}} .$$

- $H_c^1(\psi)^* = \mathrm{BMO}_c(\psi)$?
- $\mathcal{M} \bar{\otimes} \mathcal{L}(G) \longleftrightarrow id \otimes T_t$

13. M. Junge, T. Mei, BMO spaces associated with semigroups of operators, Math. Ann. 352, no. 3, 691-743, (2012)

Paley's inequality in semigroup language

Lemma (M)

Let $f=\sum_k c_k\otimes\lambda_{h_k}\in L^2(B(H)\bar{\otimes}\mathcal{L}(G))$, *we have*

$$\frac{1}{2}\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\geq\tau\left[\left(\int_0^\infty\left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right)^{\frac{1}{2}}\right]$$

This means the left hand subtract the right hand of above is a nonnegative self-adjoint element of $B(H)$.
Moreover, if we assume (h_k) *is a* ψ*-lacunary sequence, then*

$$\left\|\int_0^\infty\left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right\| \;\leq\; \left(1+\frac{2}{\delta}\right)\left\|\sum_k |c_k|^2\right\|.$$

Sketch for the proof of Theorem

- $$\int_0^\infty \left|\frac{\partial}{\partial s} T_s f\right|^2 s ds = \sum_{k,j} a_{k,j} (c_k \lambda_{h_k})^* c_j \lambda_{h_j},$$

with

$$a_{k,j} = \frac{\psi(h_k)\psi(h_j)}{(\psi(h_k) + \psi(h_j))^2} \geq 0$$

- First inequality: Cauchy-Schwartz inequality.
- Second inequality: ψ-lacunary property and $\sum_k a_{k,j} \leq 1 + \frac{2}{\delta}$..

Paley's inequality in semigroup language

Theorem (M)

Suppose that $(h_k)_k \subseteq G$ *is a* ψ*-lacunary sequence. Then, for any* $N \in \mathbb{N}$ *and* $f = \sum_{k=1}^N c_k \otimes \lambda_{h_k}$ *with* $c_k \in B(H)$*, we have*

$$\|f\|^2_{BMO_c(\psi)} \simeq_\delta \left\| \sum_{k, h_k \neq e} |c_k|^2 \right\|.$$

At the other end, we have, for any $(c_k) \in S^1(\ell^2_c)$*,*

$$tr \left[\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \right]$$

$$\simeq_\delta \inf \left\{ (tr \otimes \tau) \left[\left(\int_0^\infty \left| \frac{\partial}{\partial s} T_s f \right|^2 s \; ds \right)^{\frac{1}{2}} \right] : \tau(f \lambda^*_{h_k}) = c_k \right\}.$$

where the infimum runs over all $f \in L^1(B(H)\bar{\otimes}\mathcal{L}(G))$

Sketch for the proof of Theorem

- $$T_t\left[|f-T_t(f)|^2\right]=\sum_{k,j}a_{k,j}(c_k\lambda_{h_k})^*c_j\lambda_{h_j},$$

with

$$a_{k,j}=e^{-t\psi(h_k^{-1}h_j)}(1-e^{-t\psi(h_k^{-1})})(1-e^{-t\psi(h_j)})\geq 0.$$

- By ψ-lacunary property:

$$\sup_j\sum_k a_{k,j}\leq 1+\delta^{-1}+\delta^{-2}=:c_\delta,\sup_k\sum_j a_{k,j}\leq c_\delta.$$

- The BMO estimate follows from

$$\left\|T_t\left[|f-T_t(f)|^2\right]\right\|\leq c_\delta\left\|\sum_k|c_k|^2\right\|$$

and

$$\|T_t\left[|f-T_t(f)|^2\right]\|\geq\left\|\sum_{k,h_k\neq e}\left|\left[1-e^{-t\psi(h_k)}\right]c_k\right|^2\right\|.$$

Sketch for the proof of Theorem

- the H^1-estimate: By duality, we may choose b_k such that $\|\sum |b_k|^2\| = 1$ and

$$\mathrm{tr}\left[\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\right] = \sup_{f,\varphi}(tr \otimes \tau)(f^*\varphi),$$

where the superum runs over all finite sum $\varphi = \sum_{k=1}^N b_k \lambda_{h_k}, f = \sum_{k=1}^N c_k \lambda_{h_k}$. Combining the Hölder inequality with the second inequality of above Lemma we obtain

$$\mathrm{tr}\left[\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\right] \leq 4\left(1+\frac{2}{\delta}\right)^{\frac{1}{2}}(\tau \otimes \mathrm{tr})\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right)^{\frac{1}{2}}\right].$$

The other direction follows by taking tr on the both sides of the first inequality of above Lemma.

Thanks for your attention!