Noncommutative martingale inequalities

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Overview

Classical martingale inequalities

Noncommutative martingale inequalities

Recent progress and problems

I. Classical Martingale Inequalities

Classical martingales

- $(\Omega, \mathcal{F}, \mathbb{P})$: a probability space.
  $L_0(\Omega)$: the set of all measurable functions.
  $L_p(\Omega)$: classical Lebesgue space.
  $(\mathcal{F}_n)_n$: $\uparrow$, sub-σ-algebras of $\mathcal{F}$, $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$.
- The conditional expectation $\mathbb{E}_n$ with respect to $\mathcal{F}_n$ is defined by

$$\int_A \mathbb{E}_n(f) = \int_A f, \quad \forall A \in \mathcal{F}_n.$$

- An adapted sequence $f = (f_n)_{n \geq 1}$ in $L_1(\Omega)$ is called a martingale if for any $n \geq 1$,

$$\mathbb{E}_n(f_{n+1}) = f_n.$$

  Example: $\Omega = [0, 1)$, $\mathcal{F}_n = \sigma(\{[k2^{-n}, (k+1)2^{-n}) : k \geq 0\})$, $\mathbb{P}$ is Lebesgue measure (dyadic martingales).

Basic operators in martingale theory

Let $f = (f_n)_n$ be a martingale and $df = (d_n f)_n = (f_n - f_{n-1})_n$ be the martingale difference sequence of f.

Doob's maximal operator:

$$Mf := \sup_n |f_n|.$$

Square function:

$$Sf := \Big(\sum_n |d_n f|^2\Big)^{1/2},$$

Conditioned square function:

$$sf := \Big(\sum_n \mathbb{E}_{n-1}|d_n f|^2\Big)^{1/2}$$

Classical martingale inequalities

Theorem (Doob's maximal inequality)

We have

$$\|Mf\|_{1,\infty} \lesssim \|f\|_1 \tag{1}$$

and

$$\|Mf\|_p \lesssim_p \|f\|_p, \quad 1 < p \leq \infty.$$

Theorem (Burkholder-Gundy inequality)

We have

$$\|Sf\|_{1,\infty} \lesssim \|f\|_1$$

and

$$\|f\|_p \approx_p \|Sf\|_p, \quad 1 < p < \infty.$$

Theorem (Burkholder inequality)

We have

$$\|f\|_p \approx_p \max\left\{\|sf\|_p,\ \Big(\sum_n \|df_n\|_p^p\Big)^{1/p}\right\}, \quad 2 \le p < \infty.$$

Theorem (Burkholder inequality with maximal diagonal)

In the above theorem, the diagonal part $\left(\sum_n \|d_n f\|_p^p\right)^{1/p}$ can be replaced with $\|\sup_n |d_n f|\|_p$; namely,

$$\|f\|_p \approx_p \max\left\{\|sf\|_p,\ \|\sup_n |d_n f|\|_p\right\}, \quad 2 \le p < \infty.$$

Remark

(i) The above are the most fundamental inequalities...

(ii) The proofs mainly depend on stopping times.

(iii) The above results also hold true in more general context (e.g., $L_{p,q}$, L_Φ, symmetric spaces E).

II. Noncommutative Martingale Inequalities

τ-measurable operators

- A noncommutative probability space $(\mathcal{M}, \tau)$: $\mathcal{M}$ is a finite von Neumann algebra equipped with a normal faithful trace τ and $\tau(1) = 1$.

  Example 1. $\mathcal{M} = L_\infty(\Omega, \mathbb{P})$, $\tau = \int_\Omega$; $\quad \tau(1) = \mathbb{P}(\Omega) = 1$
  $(\mathcal{M}, \tau)$: the classical probability space

  Example 2. $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$, $\tau = \frac{1}{n} Tr$
  $(\mathcal{M}, \tau)$: NC probability space.

- $L_0(\mathcal{M})$: the set of τ-measurable operators.

  Indeed, $L_0(\mathcal{M})$ consists of all the operators affiliated to $\mathcal{M}$ since $\mathcal{M}$ is finite.

Generalised singular value function

- For $x \in L_0(\mathcal{M})$, the distribution function of x is defined by

$$n_x(s) = \tau\left(\chi_{(s,\infty)}(x)\right), \quad -\infty < s < \infty.$$

- The generalised singular value function of x is defined by

$$\mu(t, x) = \inf\left\{s > 0 : n_{|x|}(s) \le t\right\}, \quad t > 0.$$

Example

If $\mathcal{M} = L_\infty(\Omega, P)$, then $n_f(s) = P(f > s)$ is the distribution function of f and $\mu(\cdot, f)$ is just the classical non-increasing rearrangement function of f. Moreover, we have

$$n_{|f|}(\lambda) = \mathbb{P}(\{\omega : |f(\omega)| > \lambda\}) = |\{t \in (0, 1] : \mu(t, f) > \lambda\}|.$$

Noncommutative symmetric spaces

- Symmetric space E: a Banach function space $(E, \|\cdot\|_E)$ on $(0,1]$ is called symmetric if for $g \in E$ and measurable f with $\mu(f) \le \mu(g)$, we have $f \in E$ and $\|f\|_E \le \|g\|_E$.
(examples: L_p, L_Φ, $L_{p,q}$, etc.)
- NC symmetric spaces $E(\mathcal{M}, \tau)$: given E and $(\mathcal{M}, \tau)$ as above, the corresponding NC symmetric space $E(\mathcal{M}, \tau)$ is defined by
$$E(\mathcal{M}, \tau) := \left\{x \in L_0(\mathcal{M}) : \mu(x) \in E\right\}$$
equipped with $\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$.
- Examples.
$$\text{NC Lp} : \|x\|_{L_p} := \left(\int_0^\infty \mu(t,x)^p dt\right)^{1/p}.$$
$$\text{NC weak Lp} : \|x\|_{L_{p,\infty}} := \sup_{t>0} t^{1/p}\mu(t,x).$$
$$\text{NC Lorentz} : \|x\|_{L_{p,q}} := \left(\int_0^\infty t^{q/p-1}\mu(t,x)^q dt\right)^{1/q} \ldots\ldots$$

Noncommutative martingales

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space.

- $(\mathcal{M}_n)_n$ is an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\overline{\bigcup_n \mathcal{M}_n}^{weak} = \mathcal{M}$.
- $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ is a trace preserving conditional expectation.
- An adapted sequence $x = (x_n)_n$ in $L_1(\mathcal{M})$ is called a noncommutative martingale with respect to $(\mathcal{M}_n)_n$ if

$$\mathcal{E}_n(x_{n+1}) = x_n.$$

Examples. Noncommutative dyadic martingales...

A NC version of Doob's maximal inequality

Theorem (Cuculescu, J. Multivariate Anal., 1971)

Let $x = (x_n)_n$ be a nonnegative martingale. For any $\lambda > 0$, there exists a projection q_λ satisfying

$$q_\lambda x_n q_\lambda \leq \lambda q_\lambda, \quad \text{for all } n,$$

and such that

$$\lambda \tau(1 - q_\lambda) \lesssim \|x\|_1.$$

Remark

The above result can be regarded as a NC version of (1). Indeed,

$$1 - q_\lambda \sim \{\sup_n |f_n| > \lambda\} = \{Mf > \lambda\};$$

$$\tau(1 - q_\lambda) \sim \mathbb{P}\Big(\sup_n |f_n| > \lambda\Big) = \mathbb{P}\big(Mf > \lambda\big).$$

However, no more results for NC martingales until 1997!

Main difficulties

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- How to define the square function? Do we have

$$\Big\|\Big(\sum_n |x_n|^2\Big)^{1/2}\Big\|_p \approx \Big\|\Big(\sum_n |x_n^*|^2\Big)^{1/2}\|_p \quad ?$$

Answer: No!

Example. Let $(\mathcal{M},\tau) = (M_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr})$. Set $x_k = e_{k,0}$. It is immediate that

$$\Big\|\Big(\sum_{k=0}^{n-1} |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{M})} = n^{1/2-1/p}, \quad \Big\|\Big(\sum_{k=0}^{n-1} |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{M})} = 1.$$

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- Stopping times are not available...

Breakthrough I: NC Burkholder-Gundy inequality

NC square functions:

$$S_c(x) := \Big(\sum_n |d_n x|^2\Big)^{1/2}, \quad S_r(x) := \Big(\sum_n |d_n x^*|^2\Big)^{1/2}.$$

NC BG inequality (Pisier-Xu, CMP, 1997)

For $2 \leq p < \infty$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \max\Big\{\|S_c(x)\|_{L_p(\mathcal{M})}, \|S_r(x)\|_{L_p(\mathcal{M})}\Big\}.$$

For $1 < p < 2$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \inf_{x=y+z}\Big\{\|S_c(y)\|_{L_p(\mathcal{M})} + \|S_r(z)\|_{L_p(\mathcal{M})}\Big\}.$$

Key: iteration method and duality.

Breakthrough II: NC Doob's maximal inequality

NC maximal function space:

Definition (Junge, J. Reine Angew. Math., 2002)

The space $L_p(\mathcal{M};\ell_\infty)$ is defined as the set of all sequences $x=(x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ for which there exist $a,b\in L_{2p}(\mathcal{M})$, $y=(y_n)_n\subset L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad n\geq 1. \tag{2}$$

For $x\in L_p(\mathcal{M};\ell_\infty)$, we define

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\left\{\|a\|_{L_{2p}(\mathcal{M})}\sup_n \|y_n\|_{L_\infty(\mathcal{M})}\|b\|_{L_{2p}(\mathcal{M})}\right\}.$$

Remark

If we consider a sequence of positive operators $x=(x_k)$, then it can be seen that $x\in L_p(\mathcal{M};\ell_\infty)$ iff there is $a\in L_p^+(\mathcal{M})$ s.t. $x_n\leq a$ for all n; moreover,

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\left\{\|a\|_{L_p(\mathcal{M})} : a\in L_p^+(\mathcal{M}),\ x_n\leq a,\ \forall n\right\}.$$

Obviously, in the classical case, the above goes back to $\|\sup_n|f_n|\|_p$.

NC Dualised Doob inequality (Junge, 2002)

$$\Big\| \sum_k \mathcal{E}_k a_k \Big\|_p \lesssim_p \Big\| \sum_k a_k \Big\|_p, \quad 1 \le p < \infty, \ a_k \ge 0.$$

NC Doob's maximal inequality (Junge, 2002)

$$\|(\mathcal{E}_n x)_n\|_{L_p(\mathcal{M};\ell_\infty)} \lesssim_p \|x\|_p, \quad 1 < p \le \infty.$$

Remark

(i) The proof is quite complicated and is totally different from the classical case.

(ii) An alternative proof can be found in [Junge-Xu, JAMS, 2007].

NC Burkholder inequality

NC conditioned square function:

$$s_c(x) := \Big(\sum_n \mathcal{E}_{n-1}|d_n x|^2\Big)^{1/2}, \quad s_r(x) := \Big(\sum_n \mathcal{E}_{n-1}|d_n x^*|^2\Big)^{1/2}.$$

NC Burkholder inequality (Junge-Xu, AOP, 2003)

For $2 \le p < \infty$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \max\Big\{\|s_c(x)\|_{L_p(\mathcal{M})}, \|s_r(x)\|_{L_p(\mathcal{M})}, \Big(\sum_n \|d_n x\|^p_{L_p(\mathcal{M})}\Big)^{1/p}\Big\}.$$

For $1 < p < 2$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \inf_{x=y+z+w}\Big\{\|s_c(y)\|_{L_p(\mathcal{M})} + \|s_r(z)\|_{L_p(\mathcal{M})} + \Big(\sum_n \|d_n w\|^p_{L_p(\mathcal{M})}\Big)^{1/p}\Big\}.$$

NC Burkholder inequality with maximal diagonal (Junge-Xu, Isreal J. Math., 2008)

For $2 < p < \infty$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \max\left\{\|s_c(x)\|_{L_p(\mathcal{M})}, \|s_r(x)\|_{L_p(\mathcal{M})}, \|(d_n x)_n\|_{L_p(\mathcal{M};\ell_\infty)}\right\}.$$

For $1 < p < 2$,

$$\|x\|_{L_p(\mathcal{M})} \approx_p \inf_{x=y+z+w}\left\{\|s_c(y)\|_{L_p(\mathcal{M})} + \|s_r(z)\|_{L_p(\mathcal{M})} + \|(d_n w)_n\|_{L_p(\mathcal{M};\ell_1)}\right\}.$$

Key:
1) NC BG inequality, duality
2) NC Burkholder with normal diagonal, interpolation, duality

Weak type inequalities for NC martingales

(Randrianantoanina, PLMS, 2005) There are two martingales a and b such that $x = a + b$

$$\|S_c(a)\|_{L_{1,\infty}(\mathcal{M})} + \|S_r(b)\|_{L_{1,\infty}(\mathcal{M})} \lesssim \|x\|_{L_1(\mathcal{M})}.$$

(Randrianantoanina, AOP, 2007) There are three adapted sequences $\eta = (\eta_n)_{n\geq 1}$, $\zeta = (\zeta_n)_{n\geq 1}$, and $\xi = (\xi_n)_{n\geq 1}$ such that $d_n y = \eta_n + \zeta_n + \xi_n$ and satisfy the weak-type estimate:

$$\big\|\eta\big\|_{L_{1,\infty}(\mathcal{M}\overline{\otimes}\ell_\infty)} + \big\|s_c(\zeta)\big\|_{L_{1,\infty}(\mathcal{M})} + \big\|s_r(\xi)\big\|_{L_{1,\infty}(\mathcal{M})} \lesssim \big\|x\big\|_1.$$

Key: Cuculescu projection $\leftrightarrow$ stopping time
or NC Gundy's decomposition

Problem: Whether we can find three martingales such that the last inequality holds true is unknown. This is open even for classical martingales.

Generalizations: $L_p \to L_{p,q}$, L_Φ, E...

NC Burkholder-Gundy inequality ($E(\mathcal{M})$ or Φ-moment):

- T. N. Bekjan, Z. Chen, *Interpolation and Φ-moment inequalities of noncommutative martingales*, Probab. Theory Related Fields. 152 (2012).
- S. Dirksen *Noncommutative Boyd interpolation theorems*, Trans. Amer. Math. Soc., 367 (2015), no 6, 4079–4110.

NC Doob's maximal inequality ($E(\mathcal{M})$ or Φ-moment):

- S. Dirksen, *Weak-type interpolation for noncommutative maximal operators*, J. Operator Theory 73 (2015) no. 2, 515-532.
- T. N. Bekjan, Z. Chen, A. Osekowski, *Noncommutative maximal inequalities associated with convex functions*, Trans. Amer. Math. Soc. 369 (2017), no. 1, 409-427.

NC Burkholder inequality ($E(\mathcal{M})$ or Φ-moment):

- N. Randrianantoanina, L. Wu, *Martingale inequalities in noncommutative symmetric spaces*, J. Funct. Anal. 269 (2015), 2222-2253.
- N. Randrianantoanina and L. Wu, *Noncommutative Burkholder/Rosenthal inequalities associated with convex functions*, Ann. Poincaré Probab. Statist. (2017).
- N. Randrianantoanina, L. Wu and Q. Xu, *Noncommutative Davis type decompositions and applications*, J. Lond. Math. Soc. (2019).
- Y. Jiao, D. Zanin and D. Zhou, *Noncommutative Burkholder/Rosenthal inequalities with maximal diagonal*, submitted.

Key: most of the above results depend on interpolations.

(Junge, J. Reine Angew. Math., 2002)

$$\|(\mathcal{E}_n x)_n\|_{L_p(\mathcal{M};\ell_\infty^\theta)} \leq \|x\|_p, \quad 2 < p < \infty,\ 0 \leq \theta \leq 1.$$

(Hong-Junge-Parcet, JFA, 2016)

$$\|(\mathcal{E}_n x)_n\|_{L_p(\mathcal{M};\ell_\infty^\theta)} \leq \|S_c(x)\|_p, \quad 1 \leq p \leq 2,\ 1 - p/2 < \theta < 1.$$

(Randrianantoanina-W-Zhou, JFA, 2021) If $E \in \mathrm{Int}[L_p, L_q]$ for $1 < p \leq q < 2$, then we have

$$\|(\mathcal{E}_n x)_n\|_{E(\mathcal{M};\ell_\infty^\theta)} \leq \|S_c(x)\|_E, \quad 1 - p/2 < \theta < 1.$$

Remark

(i) Whether the last estimate holds true for $E \in \mathrm{Int}[L_p, L_q]$ with $1 \leq p \leq q \leq 2$?

(ii) Asymmetric versions of Burkholder inequality and Davis inequality ?

III. Recent Progress and Problems

(i) NC atomic decomposition

Open Question:

Atomic decomposition of $\mathrm{h}_p^c \longrightarrow$ dual of h_p^c for $0 < p \leq 1$, new martingale inequalities, real interpolation... etc.

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- T. N. Bekjan, Z. Chen, M. Perrin and Z. Yin, *Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales*, JFA (2010).
- G. Hong and T. Mei, *John-Nirenberg inequality and atomic decomposition for noncommutative martingales*, JFA (2012).
- Y. Jiao, L. Wu, D. Zanin, and D. Zhou, *Noncommutative dyadic martingales and Walsh–Fourier series*, J. Lond. Math. Soc. (2018).

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- Y. Jiao, L. Wu, D. Zanin, and D. Zhou, *Noncommutative dyadic martingales and Walsh–Fourier series*, J. Lond. Math. Soc. (2018).

A Real Breakthrough:

- Z. Chen, N. Randrianantoanina and Q. Xu, *Atomic decompositions for noncommutative martingales*, arXiv: 2001.08775, 2020.

Algebraic atoms

Definition

Let $0 < p < 2$. An operator $x \in L_p(\mathcal{M})$ is called an algebraic h_p^c-atom if $x = \sum_{n\geq 1} y_n b_n$ and for $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$:

(i) $\mathcal{E}_n(y_n) = 0$ and $b_n \in L_q(\mathcal{M}_n)$ for all $n \geq 1$;

(ii) $\sum_{n\geq 1} \|y_n\|_2^2 \leq 1$ and $\|(\sum_{n\geq 1} |b_n|^2)^{1/2}\|_q \leq 1$.

Definition

For $0 < p < 2$, we say that $x \in L_p(\mathcal{M})$ admits an algebraic h_p^c-atomic decomposition if $x = \sum_k \lambda_k a_k$, where for each k, a_k is an algebraic h_p^c-atom or an element of the unit ball of $L_p(\mathcal{M}_1)$, and $\lambda_k \in C$ satisfying $\sum_k |\lambda_k|^p < \infty$ for $0 < p \leq 1$ and $\sum_k |\lambda_k| < \infty$ for $1 < p < 2$.

Algebraic atomic decompositions

Definition

The algebraic atomic column martingale Hardy space $\mathsf{h}^c_{p,\mathrm{aa}}(\mathcal{M})$ is defined to be the space of all x which admit a algebraic h^c_p -atomic decomposition and is equipped with

$$\|x\|_{\mathsf{h}^c_{p,\mathrm{aa}}} = \inf \Big(\sum_k |\lambda_k|^p \Big)^{1/p} \text{ for } 0 < p \leq 1;$$

$$\|x\|_{\mathsf{h}^c_{p,\mathrm{aa}}} = \inf sum_k |\lambda_k| \text{ for } 1 < p < 2.$$

Theorem (Chen-Randrianantoanina-Xu,2020)

Let $0 < p < 2$. Then

$$\mathsf{h}^c_p(\mathcal{M}) = \mathsf{h}^c_{p,\mathrm{aa}}(\mathcal{M})$$

with equivalent (quasi) norms.

Future Problems:

- Following Chen-Randrianantoanina-Xu's method, we can also construct the atomic decompositions for NC martingale Hardy-Orlicz spaces h_Φ^c. However, nothing is known for the NC martingale Hardy-Lorentz spaces $\mathrm{h}_{p,q}^c$.
- Chen-Randrianantoanina-Xu's atomic decomposition has certain distance from the one constructed in classical case. Constructing NC atomic decompositions which exactly corresponding to the classical one remains open.

(ii) NC good-λ inequalities

Open Question: NC good-λ inequality.

"On the other hand, the noncommutative analogue of good-λ inequality seems open. Then, in order to prove the noncommutative Φ-moment inequalities we need new ideas."

T. N. Bekjan, Z. Chen, *Interpolation and Φ-moment inequalities of noncommutative martingales*, Probab. Theory Related Fields. 152 (2012).

Main difficulty:
finding an appropriate form of good-λ inequality which is transferrable to NC setting.

Solution:
Y. Jiao, A. Osękowski, L. Wu, *Noncommutative good-λ inequalities*, arXiv: 1805.07057v2, 2018.

Definition (Good-λ testing condition)

Let $A, B \in L_2(\mathcal{M})$ be self-adjoint operators. (A, B) is said to satisfy the good-λ testing conditions if we have

$$\mathcal{E}_k(|A - \mathcal{E}_{k-1}A|^2) \leq \mathcal{E}_k(B^2), \quad k \geq 0.$$

Theorem (Jiao et al., 2022)

Let $E \in \mathrm{Int}[L_p, L_q]$ for $2 < p < q < \infty$. Let (A, B) satisfy good-λ testing condition with $B \in E(\mathcal{M})$. We have

$$\|A\|_E \leq c_E \|B\|_E.$$

Remark

(i) Φ-moment version holds true as well.

(ii) Searching more applications of NC good-λ inequalities...

(ii) Asymmetric martingale inequalities

Asymmetric maximal function space:

Definition (Junge, J. Reine Angew. Math., 2002)

Let $0 \leq \theta \leq 1$. The space $L_p(\mathcal{M}; \ell_\infty^\theta)$ is defined as the set of all sequences $x = (x_n)_{n \geq 1}$ in $L_p(\mathcal{M})$ for which there exist $a \in L_{\frac{p}{1-\theta}}(\mathcal{M})$, $b \in L_{\frac{p}{\theta}}(\mathcal{M})$, and $y = (y_n)_n \subset L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad n \geq 1. \tag{3}$$

For $x \in L_p(\mathcal{M}; \ell_\infty^\theta)$, we define

$$\|x\|_{L_p(\mathcal{M}; \ell_\infty^\theta)} = \inf \left\{ \|a\|_{L_{\frac{p}{1-\theta}}(\mathcal{M})} \sup_n \|y_n\|_{L_\infty(\mathcal{M})} \|b\|_{L_{\frac{p}{\theta}}(\mathcal{M})} \right\}.$$

Remark

(i) If $\theta = 1/2$, then the above goes back to $L_p(\mathcal{M}; \ell_\infty)$.

(ii) Given symmetric space E, one may define $E(\mathcal{M}; \ell_\infty)$ similarly.

(Junge, J. Reine Angew. Math., 2002)

$$\|(\mathcal{E}_n x)_n\|_{L_p(\mathcal{M};\ell_\infty^\theta)} \leq \|x\|_p, \quad 2 < p < \infty,\, 0 \leq \theta \leq 1.$$

(Hong-Junge-Parcet, JFA, 2016)

$$\|(\mathcal{E}_n x)_n\|_{L_p(\mathcal{M};\ell_\infty^\theta)} \leq \|S_c(x)\|_p, \quad 1 \leq p \leq 2,\, 1 - p/2 < \theta < 1.$$

(Randrianantoanina-W-Zhou, JFA, 2021) If $E \in \mathrm{Int}[L_p, L_q]$ for $1 < p \leq q < 2$, then we have

$$\|(\mathcal{E}_n x)_n\|_{E(\mathcal{M};\ell_\infty^\theta)} \leq \|S_c(x)\|_E, \quad 1 - p/2 < \theta < 1.$$

Remark

(i) Whether the last estimate holds true for $E \in \mathrm{Int}[L_p, L_q]$ with $1 \leq p \leq q \leq 2$?

(ii) Asymmetric versions of Burkholder inequality and Davis inequality ?

(iv) NC differential subordinate martingale inequalities

Backgrounds

• The classical differential subordination of martingales was introduced by Burkholder in the eighties.

• Let $f=(f_n)_n$, $g=(g_n)_n$ be two martingales. We say that g is differentially subordinate to f if for any n, we have

$$|d_n g| \leq |d_n f|.$$

Theorem (Burkholder, AOP, 1984)

Suppose that g is differentially subordinate to f. Then

$$\begin{aligned} ||g||_{1,\infty} &\leq 2||f||_1; \\ ||g||_p &\leq (p^*-1)||f||_p, \quad 1<p<\infty, \end{aligned}$$

where $p^* = \max\{p, p/(p-1)\}$. The constants are both sharp.

Natural question: a NC version of the above theorem?

Main difficulty: an appropriate definition

- Let y, x be two self-adjoint martingales. If $|d_n y|^2 \leq |d_n x|^2$, then $S_c(y) = S_r(y) \leq S_r(x) = S_c(x)$. NC Burkholder-Gundy inequality yields that

$$\|y\|_p \lesssim_p \|S_c(y)\|_p \leq \|S_c(x)\|_p \lesssim_p \|x\|_p, \quad 2 < p < \infty.$$

  Moreover, obviously, $|d_n y|^2 \leq |d_n x|^2$ goes back to the classical definition. This means that $|d_n y|^2 \leq |d_n x|^2$ is a possible candidate (at least for $2 < p < \infty$).

- On the other hand, it is not hard to construct martingales y, x satisfies $|d_n y|^2 \leq |d_n x|^2$; while the weak-type $(1, 1)$ and strong-type (p, p) estimate for $1 < p < 2$ fails. Therefore, we need a 'stronger' (compared with $|d_n y|^2 \leq |d_n x|^2$) definition for the case $1 \leq p < 2$.

Definition (Jiao-Osękowski-W, Adv. Math., 2018)

We say that y is *differentially subordinate* to x, if for any n and any projection $R \in \mathcal{M}_{n-1}$, we have

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR. \tag{DS}$$

We say that y is *weakly differentially subordinate* to x if for any n

$$|dy_n|^2 \leq |dx_n|^2. \tag{WDS}$$

Remark

(i) In the commutative case, the above definitions are identical!

(ii) However, in the NC case, differential subordination $\Longrightarrow$ weak differential subordination.

Main results

Let x, y be two self-adjoint martingales.

Theorem 1 (Jiao-Osękowski-W, Adv. Math., 2018)

Suppose that y, x satisfy (DS). Then we have

$$\|y\|_{1,\infty} \leq 36||x||_1;$$

$$\|y\|_p \leq c_p\|x\|_p, \quad 1 < p < 2.$$

Theorem 2 (Jiao-Osękowski-W, Adv. Math., 2018)

Suppose that y, x satisfy (WDS). Then

$$||y||_p \leq c_p||x||_p, \quad 2 \leq p < \infty.$$

Remark

(i) The constant c_p in Theorem 1 is of order $O((p-1)^{-1})$ as $p \to 1_+$. The constant in Theorem 2 is of order $O(p)$ as $p \to \infty$. These are already optimal in the commutative setting.

(ii) The proof of Theorem 1 depends on new Gundy's decomposition, which is of independent interest; while, Theorem 2 relies on an idea of NC good-λ method.

(iii) Based on the above results, we further considered strong differential subordination for NC submartingales and square function estimate for NC differential subordinate martingales.

Y. Jiao, A. Osekowski and L. Wu, *Strong differential subordinates for noncommutative submartingales*, Ann. Probab., (2019).

Y. Jiao, N. Randrianantoanina, L. Wu and D. Zhou, *Square Functions for Noncommutative Differentially Subordinate Martingales* , Comm. Math. Phys., (2020).

(v) Two future topics

The following two topics are blank or almost blank.

- NC A_p weights and weighted martingale inequalities.
  $\rightarrow$ nothing is known at all!
- NC continuous-time martingale theory and NC stochastic integral theory
  $\rightarrow$ not sufficient at all!

Thank You !