Quantitative mean ergodic inequalities I: power bounded operators acting on one noncommutative L_p space

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Classical ergodic theory

Given a Hilbert space $L_2(X, \mathcal{F}, \mu)$. Let T be an unitary operator on $L_2(X, \mathcal{F}, \mu)$, and let $P : L_2(X, \mathcal{F}, \mu)) \to F$ be a projection, where $F = \{f \in L_2(X, \mathcal{F}, \mu) : Tf = f\}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i f = Pf \ , \forall\ f \in L_2(X, \mathcal{F}, \mu).$$

- von Neumann's mean ergodic theorem, norm.
- Birkhoff's pointwise ergodic theorem, pointwise.

- Let $Tf(x) = f(\rho x)$, where $\mu \circ \rho^{-1} = \mu$. If $\mu(X) = 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i f = \mathbb{E}(f|\mathcal{F}_0),$$

  where $\mathcal{F}_0$ is the σ-algebra of all ρ-invariant sets ($\rho^{-1}(A) = A$) in $\mathcal{F}$.
- ρ is ergodic, $\mathbb{E}(f|\mathcal{F}_0) = \int_X f d\mu$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i f = \int_X f d\mu.$$

- Riesz, 1938, mean ergodic theorem, T: L_p-contraction, namely $\|T\|_{L_p \to L_p} \leq 1$, $1 \leq p < \infty$.
- Burkholder, 1962, T is not positive,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i f = \infty, \ a.e.$$

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- Akcoglu, 1975, 1977, T is positive–Dilation theorem.

- Dunford and Schwartz, 1958, [Linear Operators. I. General Theory, section VIII.5], Banach space; $\|T\|_{L_p \to L_p} \leq 1$ for all $1 \leq p \leq \infty$.
- $\ldots$

The rate of ergodic convergence

Theorem (Krengel, 1978)
Let T be an unitary operator on $L_p([0,1])$, $1 \leq p \leq \infty$. Define

$$M_n(T)f = \frac{1}{n}\sum_{i=1}^{n} T^i f.$$

For any null-sequence $(a_n)_n$ (that is $a_n \to 0$), there exists a continuous function $f \in L_p([0,1])$ such that

$$\limsup_{n\to\infty} \frac{1}{a_n}\left\| M_n(T)f - \int f \right\|_{L_p} = \infty.$$

Remark: the ergodic convergence can be arbitrarily slow.

Metastability (Kohlenbach)

for all $\epsilon > 0$*, for all function* $g : \mathbb{N} \to \mathbb{N}$*, there exists* $N \in \mathbb{N}$ *for all* $j, l \in [N, N + g(N)]$ *such that* $\|M_j(T)f - M_l(T)f\|_{L_p} \leq \epsilon$.

- Tao, 2008.
- Kohlenbach, 2008.
- Avigad et al, 2010, ϵ-fluctuations.
- $\dots$

Square function

$$S(f) = \Big(\sum_{i=1}^{\infty} |M_{n_i}(T)f - M_{n_{i+1}}(T)f|^2 \Big)^{1/2}.$$

Jones, Ostrovskii and Rosenblatt, 1996, T is an unitary operator, $\|S(f)\|_{L_2} \leq 25\|f\|_{L_2}$.

Advantage:

- quantitative estimate the rate of convergence;
- $\sup_{(n_i)} \|M_{n_i}(T)f - M_{n_{i+1}}(T)f\|_{L_2} < \infty$.

Idea: the spectral theory of unitary operators.

- $T : L_2 \to L_2$ is a contraction.
  Idea: Dilation theorem. Sz-Nagy and Foias, 1967,

$$T^i = PU^i,$$

where $\mathbb{H} \subseteq \mathbb{H}_0$, $U : \mathbb{H}_0 \to \mathbb{H}_0$ is an unitary operator and $P : \mathbb{H}_0 \to \mathbb{H}$ is a projection.

Noncommutative ergodic theory

Noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $\mathcal{S}_{\mathcal{M}_+}$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\mathrm{supp} x) < \infty$, where $\mathrm{supp} x$ denotes the support of x. Let $\mathcal{S}_{\mathcal{M}}$ be the linear span of $\mathcal{S}_{\mathcal{M}_+}$. Then $\mathcal{S}_{\mathcal{M}}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$. Given $1 \leq p < \infty$ and $x \in \mathcal{S}_{\mathcal{M}}$, if we set

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p},$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the modulus of x. Then $(\mathcal{M}, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$, and simply denoted by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm.

Ergodic averages

Let $T : \mathcal{M} \to \mathcal{M}$ be a linear map. Set

$$M_n(T)x = \frac{1}{n} \sum_{i=1}^{n} T^i x.$$

Mean ergodic theorem (Jajte, 1981, 1985)

Assume that $\tau(1) = 1$, T is positive and satisfies $\tau \circ T \leq \tau$ for all $x \in \mathcal{M}_+$. Then there exists $\widehat{x} \in \mathcal{M}$ such that for all $x \in \mathcal{M}$

$$\lim_{n\to\infty} M_n(T)x = \widehat{x} \text{ in } L_2(\mathcal{M}).$$

T: Dunford-Schwartz operator, namely T is positive and a contraction on $\mathcal{M}$ and $\tau \circ T \leq \tau$ for all $x \in L_1(\mathcal{M}) \cap \mathcal{M}_+$ ($\Rightarrow$ T is L_p contraction for all $1 \leq p \leq \infty$.)

Pointwise ergodic theorem (Dunford-Schwartz operator)

- Yeadon, 1977, JLMS: weak type (1,1) maximal ergodic inequality.
- Junge and Xu, 2007, JAMS: maximal ergodic inequality on L_p.

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- Bekjan, 2008, JFA.
- Hu, 2008, JFA.
- Hong and Sun, 2018, JFA.
- ...

Beyond the class of Dunford-Schwartz operators

Fix $1 < p < \infty$. T: power bounded operator, namely

$$\sup_k \|T^k : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| < \infty.$$

- Hong, Liao and Wang, 2020, Duke Math. J.

  T: positive group action and power bounded.

- Hong, Ray and Wang, 2020, arXiv:1907.12967

  T: A a large subclass of positive operators (including positive Lamperti contraction).

  The first Akcoglu's maximal ergodic inequalities in the noncommutative setting.

Noncommutative Hilbert-valued L_p-spaces

For $1 \le p < \infty$, let (x_n) be a finite sequence in $L_p(\mathcal{M})$.
Define

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^r)} = \|(\sum_n |x_n^*|^2)^{\frac{1}{2}}\|_p, \quad \|(x_n)\|_{L_p(\mathcal{M};\ell_2^c)} = \|(\sum_n |x_n|^2)^{\frac{1}{2}}\|_p.$$

Row space: $L_p(\mathcal{M};\ell_2^r)$; Column space: $L_p(\mathcal{M};\ell_2^c)$.

The mixed space $L_p(\mathcal{M};\ell_2^{rc})$:

- If $2 \le p \le \infty$,

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^{rc})} = \left(\|(x_n)\|^p_{L_p(\mathcal{M};\ell_2^r)} + \|(x_n)\|^p_{L_p(\mathcal{M};\ell_2^c)}\right)^{\frac{1}{p}}.$$

- If $1 \le p < 2$,

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^{rc})} = \inf_{\substack{x_n = y_n + z_n \\ y_n, z_n \in L_p(\mathcal{M})}} \left(\|(y_n)\|^p_{L_p(\mathcal{M};\ell_2^r)} + \|(z_n)\|^p_{L_p(\mathcal{M};\ell_2^c)}\right)^{\frac{1}{p}}.$$

Main result

Theorem 1

Let $1 < p < \infty$. Suppose that T satisfies

$$\sup_{k \in \mathbb{Z}} \| T^k : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \| < \infty.$$

Set

$$M_n(T)x = \frac{1}{2n+1} \sum_{i=-n}^{n} T^i x.$$

Then for any $x \in L_p(\mathcal{M})$

$$\sup_{(n_i) \subseteq \mathbb{N}} \left\| \left(M_{n_i}(T)x - M_{n_{i+1}}(T)x \right)_i \right\|_{L_p(\mathcal{M}; \ell_2^{cr})} \lesssim \|x\|_{L_p(\mathcal{M})}.$$

Remark: Junge, Le Merdy and Xu, 2006, Astérisque

The infinite summations over $i \in \mathbb{N}$ can be understood as a consequence of the corresponding uniform boundedness for all finite summations

Lamperti operator (or separates supports) T: for any two τ-finite projections $e, g \in \mathcal{M}$ satisfies $eg = 0$, then

$$(Te)^* Tg = Te(Tg)^* = 0.$$

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$$(Te)^* Tg = Te(Tg)^* = 0.$$

- Kan, 1978, for $1 \leq p \neq 2 < \infty$, any isometry $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is Lamperti.
  Moreover, if T is positive isometry on $L_2(\mathcal{M})$, then T is Lamperti.
- positive invertible operators which are not Lamperti.

Theorem 2

Let $1 < p < \infty$. Suppose that T belongs to the family

$$\mathfrak{S} = \overline{\text{conv}}^{\text{sot}}\{S : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \text{ Lamperti contractions}\},$$

that is, the closed convex hull of all Lamperti contractions on $L_p(\mathcal{M})$ in the sense of strong operator topology.
Then for any $x \in L_p(\mathcal{M})$ with $2 \leq p < \infty$,

$$\sup_{(n_i)\subseteq\mathbb{N}} \left\| \left(M_{n_i}(T)x - M_{n_{i+1}}(T)x\right)_i \right\|_{L_p(\mathcal{M};\ell_2^{rc})} \lesssim \|x\|_{L_p(\mathcal{M})}.$$

If adding the positive property of the operator, the above inequalities hold for $1 < p < \infty$.

Proof

Theorem 1, Theorem 2 $\rightsquigarrow$ operator-valued case.

Operator-valued averaging operator

Let $I \subset \mathbb{Z}$ be an interval. Let $f : \mathbb{Z} \to S_{\mathcal{M}}$ be a locally integrable operator-valued function, where $S_{\mathcal{M}}$ is the subset of $\mathcal{M}$ with τ-finite support. The average operator over A is defined by

$$M_I f(v) = \frac{1}{|I|} \sum_{y \in I} f(v+y), \quad v \in \mathbb{Z}.$$

- Nested sequence: $I_{n_i} = [-n_i, n_i]$ or $I_{n_i} = [0, n_i]$.
- $\mathcal{N} = L_\infty(\mathbb{Z}) \overline{\otimes} \mathcal{M}$.

Proof of Theorem 1

The noncommutative variant of Calderón's transference principle

Theorem 3

Let T satisfy

$$\sup_{k\in\mathbb{Z}} \|T^k : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| < \infty.$$

If

$$\|(M_{I_{n_i}} f - M_{I_{n_{i+1}}} f)_i\|_{L_p(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_{L_p(\mathcal{N})}$$

then

$$\left\|\left(M_{n_i}(T)x - M_{n_{i+1}}(T)x\right)_i\right\|_{L_p(\mathcal{M};\ell_2^{rc})} \lesssim \|x\|_{L_p(\mathcal{M})}.$$

- Idea: Hong, Liao and Wang, 2020, Duke Math. J.

The strategy of proof of Theorem 2

step 1: Dilation

The operator T is said to satisfy the dilation property if there exists two linear contraction operators Q, J on $L_p(\mathcal{A}, \tau_{\mathcal{A}})$, and an isometry U on $L_p(\mathcal{M})$ such that

$$T^n = QU^nJ, \ \forall n \in \mathbb{N} \cup \{0\}.$$

Moreover, if T is positive, then Q, J and U as above can be taken to be positive.

$$\begin{array}{ccc} L_p(\mathcal{M}, \tau) & \xrightarrow{T^n} & L_p(\mathcal{M}, \tau) \\ {\scriptstyle J}\downarrow & & \uparrow{\scriptstyle Q} \\ L_p(\mathcal{A}, \tau_{\mathcal{A}}) & \xrightarrow{U^n} & L_p(\mathcal{A}, \tau_{\mathcal{A}}) \end{array}$$

- Hong, Ray and Wang, 2020, arXiv:1907.12967: $T \in \mathfrak{S}$, T satisfies N-dilation for all $N \in \mathbb{N}$.

Lemma 1(noncommutative Khintchine inequalities)

(i) If $1 < p < \infty$, then

$$\Big\| \sum_n \varepsilon_n x_n \Big\|_{L_p(L_\infty(\Omega)\overline{\otimes}\mathcal{M}))} \approx \|(x_n)\|_{L_p(\mathcal{M};\ell_2^{rc})}.$$

(ii) If $p = 1$, then

$$\| \sum_n \varepsilon_n x_n \|_{L_{1,\infty}(L_\infty(\Omega)\overline{\otimes}\mathcal{M})} \approx \|(x_n)\|_{L_{1,\infty}(\mathcal{M};\ell_2^{rc})},$$

where (ε_n) is a Rademarcher sequence on probability space (Ω, P)

- Lust-Piquard, 1986; Lust-Piquard, Pisier, 1991.

Lemma 2

Let $1 \leq p < \infty$, $T : L_p(\mathcal{A}, \tau_{\mathcal{A}}) \to L_p(\mathcal{M}, \tau)$ is a bounded liner operator. Then T extends to a bounded operator from $L_p(\mathcal{A}; \ell_2^{rc})$ to $L_p(\mathcal{M}; \ell_2^{rc})$.

Lemma 2
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By Lemma 2 and $M_{n_i}(T)x = QM_{n_i}(U)Jx$, then

$$\begin{aligned} & \left\| \left(M_{n_i}(T)x - M_{n_{i+1}}(T)x \right)_i \right\|_{L_p(\mathcal{M}; \ell_2^{rc})} \\ & \lesssim \left\| \left(M_{n_i}(U)Jx - M_{n_{i+1}}(U)Jx \right)_i \right\|_{L_p(\mathcal{A}; \ell_2^{rc})}. \end{aligned}$$

Step 2: Erogodic square function associated with isometry

Theorem 4

Let $x \in L_p(\mathcal{M})$ with $2 \leq p < \infty$. Let U be an isometry on $L_p(\mathcal{M})$. Then

$$\sup_{(n_i)\subseteq\mathbb{N}} \left\| \big(M_{n_i}(U)x - M_{n_{i+1}}(U)x\big)_i \right\|_{L_p(\mathcal{M};\ell_2^{rc})} \lesssim \|x\|_{L_p(\mathcal{M})}.$$

Moreover, if additionally U is positive, then the above inequalities hold for every $1 < p < \infty$.

The proof of Theorem 4

Key lemma

Let $1 \leq p < \infty$ and $U : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be an isometry.

For $2 \leq p < \infty$, U can be extended an isometry on $L_p(\mathcal{M}; \ell_2^{cr})$; for $1 \leq p < 2$, U can be extended a contraction on $L_p(\mathcal{M}; \ell_2^{cr})$.

If U is positive, U can be extended an isometry on $L_p(\mathcal{M}; \ell_2^{cr})$ for $1 \leq p < \infty$.

The structural description of isometry operators

Lemma (Yeadon, 1981;Junge, Ruan and Sherman, 2005).

Let $1 \leq p \neq 2 < \infty$. $T : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau)$ is an isometry if and only if there exist uniquely
a partial isometry $w \in \mathcal{M}$,
a normal Jordan $*$-monomorphism $J : \mathcal{M} \to \mathcal{M}$,
and a positive self-adjoint operator b affiliated with $\mathcal{M}$,
such that

(i) $w^*w = suppb = J(1)$;

(ii) For all $x \in \mathcal{M}$, $J(x)$ commutes with every spectral projection of b;

(iii) $T(x) = wbJ(x)$ for all $x \in \mathcal{S}_{\mathcal{M}}$;

(iv) $\tau(b^pJ(x)) = \tau(x)$ for all $x \in \mathcal{M}_+$.

Transference principle for isometry operator

By key lemma, we prove that
let U be an isometry on $L_p(\mathcal{M})$, if

$$\|(M_{I_{n_i}} f - M_{I_{n_{i+1}}} f)_i\|_{L_p(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_{L_p(\mathcal{N})}$$

then

$$\left\|\left(M_{n_i}(U)x - M_{n_{i+1}}(U)x\right)_i\right\|_{L_p(\mathcal{M};\ell_2^{rc})} \lesssim \|x\|_{L_p(\mathcal{M})}.$$

Operator-valued square function

Theorem 5

Let $(n_i)_{i\in\mathbb{N}}$ be the increasing sequence of positive integers. Set $T_i = M_{I_{n_i}} - M_{I_{n_{i+1}}}$. Let $1 \leq p \leq \infty$. Then

(i) for $p = 1$,

$$\|(T_i f)_{i\in\mathbb{N}}\|_{L_{1,\infty}(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_1;$$

(ii) for $p = \infty$,

$$\Big\|\sum_{i:i\in\mathbb{N}} T_i f \otimes e_{1i}\Big\|_{\mathrm{BMO}_d(\mathcal{R})} + \Big\|\sum_{i:i\in\mathbb{N}} T_i f \otimes e_{i1}\Big\|_{\mathrm{BMO}_d(\mathcal{R})} \lesssim \|f\|_\infty;$$

(iii) for $1 < p < \infty$,

$$\|(T_i f)_{i\in\mathbb{N}}\|_{L_p(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_p,$$

where $\mathcal{N} = L_\infty(\mathbb{Z})\overline{\otimes}\mathcal{M}$ equipped with the tensor trace $\varphi = \int_{\mathbb{Z}} \otimes \tau$ and $\mathcal{R} = \mathcal{N}\overline{\otimes}\mathcal{B}(\ell_2)$.

Sketch for the proof of Theorem 5

Fix an increasing sequence $(n_i)_{i\in\mathbb{N}}$ and let $(I_{n_i} = [0, n_i])_i$ be the associated interval sequence. Observe that there are two cases for the interval $J_i = [n_i, n_{i+1})$:

- Case 1: $J_i \subset [2^k, 2^{k+1})$ for some $k \in \mathbb{N}$;
- Case 2: J_i contains some dyadic points 2^k for $k \in \mathbb{N}$.

In Case 2, there exist t_i such that $2^{t_i} = \min\{2^k : 2^k \in J_i\}$ and s_i such that $2^{s_i} = \max\{2^k : 2^k \in J_i\}$. Then, in this case, one can see that

$$J_i = [n_i, 2^{t_i}) \cup [2^{t_i}, 2^{s_i}) \cup [2^{s_i}, n_{i+1}). \tag{1.1}$$

Note that $[n_i, 2^{t_i})$ and $[2^{s_i}, n_{i+1})$ belong to Case 1 (if $t_i = s_i$, then the middle interval of (1.1) is an empty set).

For any interval J_i, the worst case is when we divide J_i into three parts; in this case we have the decomposition

$$T_i = M_{I_{n_i}} - M_{I_{n_{i+1}}} = \left(M_{I_{n_i}} - M_{I_{2^{t_i}}}\right) + \left(M_{I_{2^{t_i}}} - M_{I_{2^{s_i}}}\right) + \left(M_{I_{2^{s_i}}} - M_{I_{n_{i+1}}}\right).$$

We introduce two collections of the indices i with respect to J_i:

- $\mathcal{S}$ consists of all i such that the corresponding intervals J_i belongs to Case 1, or $[n_i, 2^{t_i})$, $[2^{s_i}, n_{i+1})$ in (1.1).
- $\mathcal{L}$ consists of all i intervals such that the corresponding intervals $J_i = [2^{t_i}, 2^{s_i})$ in Case 2.

Then

$$\|(T_i f)_{i\in\mathbb{N}}\|_{L_p(\mathcal{N};\ell_2^{rc})} \lesssim \|(T_i f)_{i\in\mathcal{L}}\|_{L_p(\mathcal{N};\ell_2^{rc})} + \|(T_i f)_{i\in\mathcal{S}}\|_{L_p(\mathcal{N};\ell_2^{rc})}.$$

- $p = 1$, the quasi-inequality of $L_{1,\infty}(\mathcal{N};\ell_2^{rc})$ norm.

The long part

Let $f : \mathbb{Z} \to \mathcal{M}$ be a locally integrable operator-valued function. The operator-valued dyadic martingale $(\mathsf{E}_n f)$ is defined by

$$\mathsf{E}_{2^n}(f) := \sum_{I \in \mathcal{F}_{2^n}} \frac{1}{|I|} \sum_{y \in I} f(y) \chi_I,$$

where $\mathcal{F}_{2^n}$ is the σ-algebra generated by the dyadic interval I with $|I| = 2^n$.

$\forall\ i \in \mathcal{L}$, one can decompose

$$M_{I_{2^{n_i}}} f - M_{I_{2^{n_{i+1}}}} f$$
$$= \left(M_{I_{2^{n_i}}} f - \mathsf{E}_{2^{n_i}} f\right) + \left(\mathsf{E}_{2^{n_i}} f - \mathsf{E}_{2^{n_{i+1}}} f\right) + \left(\mathsf{E}_{2^{n_{i+1}}} f - M_{I_{2^{n_{i+1}}}} f\right).$$

- Hong and Xu, 2020, JFA.: for part $M_{I_{2^{n_i}}} f - \mathsf{E}_{2^{n_i}} f$.
- Mei and Parcet, 2009, IMRN: for part $\mathsf{E}_{2^{n_i}} f - \mathsf{E}_{2^{n_{i+1}}} f$.

The short part

Let $1 \leq p \leq \infty$. Set $T_i = M_{I_{n_i}} - M_{I_{n_{i+1}}}$. Then

(i) for $p = 1$,

$$\|(T_i f)_{i \in \mathcal{S}}\|_{L_{1,\infty}(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_1;$$

(ii) for $p = \infty$,

$$\Big\| \sum_{i \in \mathcal{S}} T_i f \otimes e_{1i} \Big\|_{\mathrm{BMO}_d(\mathcal{R})} + \Big\| \sum_{i \in \mathcal{S}} T_i f \otimes e_{i1} \Big\|_{\mathrm{BMO}_d(\mathcal{R})} \lesssim \|f\|_\infty;$$

(iii) for $1 < p < \infty$,

$$\|(T_i f)_{i \in \mathcal{S}}\|_{L_p(\mathcal{N};\ell_2^{rc})} \lesssim \|f\|_p.$$

Some technical lemmas

$p = 2$: Almost orthogonality principle

Lemma (Hong and Ma, 2017, Math. Z)

Let $(S_{k,i})_{k,i\in\mathbb{Z}}$ be a sequence of bounded linear operators on $L_2(\mathcal{N})$. Let $h \in L_2(\mathcal{N})$. If $(u_n)_{n\in\mathbb{Z}}$ and $(v_n)_{n\in\mathbb{Z}}$ are two sequences of operator-valued functions in $L_2(\mathcal{N})$ such that $h = \sum_{n\in\mathbb{Z}} u_{n\in\mathbb{Z}}$ and $\sum_{n\in\mathbb{Z}} \|v_n\|_2^2 < \infty$, then

$$\sum_{k\in\mathbb{Z}} \|(S_{k,i}h)_i\|^2_{L_2(\mathcal{N};\ell_2^{rc})} \leq w^2 \sum_{n\in\mathbb{Z}} \|v_n\|_2^2$$

provided that there exists a sequence $(\sigma(j))_{j\in\mathbb{Z}}$ of positive numbers with $w = \sum_{j\in\mathbb{Z}} \sigma(j) < \infty$ such that

$$\|(S_{k,i}u_n)_i\|_{L_2(\mathcal{N};\ell_2^{rc})} \leq \sigma(n-k)\|v_n\|_2$$

for every n, k.

$p = 1$: Noncommutative Calderón-Zygmund decomposition

Lemma (Cadilhac, Conde-Alonso and Parcet, arXiv:2105.05036)

Fix $f \in \mathcal{N}_{c,+}$ and $\lambda > 0$. Let $(q_n)_n$ and $(p_n)_n$ be the two sequences of projections appeared in the above Cuculescu's construction. Then there exist a projection $\zeta \in \mathcal{N}$ defined by

$$\zeta = \Big(\bigvee_{I \in \mathcal{F}} p_I \chi_{5I} \Big)^{\perp},$$

where $5I$ denotes the interval with the same center as I with length $|5I| = 5|I|$, and a decomposition of f,

$$f = g + b.$$

g and b satisfy the following properties:

(i) $\lambda\varphi(\mathbf{1}_{\mathcal{N}} - \zeta) \leq 5\|f\|_1$.

(ii) $g = qfq + \sum_n p_n f_n p_n$ satisfies
$\|g\|_1 \leq \|f\|_1 \quad$ and $\quad \|g\|_\infty \leq 2\lambda$.

(iii) $b = \sum_n b_n$, where

$$b_n = p_n(f - f_n)q_n + q_{n+1}(f - f_n)p_n.$$

Each b_n satisfies two cancellation conditions: $\mathsf{E}_n b_n = 0$; and for all $x, y \in \mathbb{Z}$ with $y \in 5I_{x,n}$, $\zeta(x)b_n(y)\zeta(x) = 0$, where $I_{x,n}$ is the interval in $\mathcal{F}_n$ containing x.

- By Noncommutative Khintchine inequalities,

  $(T_i f)_{i\in\mathcal{S}} \rightsquigarrow \sum_{i\in\mathcal{S}} \varepsilon_i T_i f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathcal{S}_k} \varepsilon_i T_i f,$

  where $\mathcal{S}_k$ consists of i in $\mathcal{S}$ such that $[n_i, n_{i+1})$ contains in $[2^k, 2^{k+1})$.

BMO estimate

By Operator valued Hardy spaces (Mei, 2007), the dyadic BMO space $\mathrm{BMO}_d(\mathcal{R})$ is defined as a subspace of $L_\infty(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell_2); L_2^{rc}(\mathbb{Z}; dx/(1+|x|)^2))$ with

$$\|f\|_{\mathrm{BMO}_d(\mathcal{R})} = \max\left\{\|f\|_{\mathrm{BMO}_d^r(\mathcal{R})}, \|f\|_{\mathrm{BMO}_d^c(\mathcal{R})}\right\} < \infty,$$

where the row and column dyadic BMO_d norms are given by

$$\|f\|_{\mathrm{BMO}_d^r(\mathcal{R})} = \sup_{I\in\mathcal{F}} \left\| \left(\frac{1}{|I|} \sum_{x\in I} \left| \Big(f(x) - \frac{1}{|I|} \sum_{y\in I} f(y) \Big)^* \right|^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{B}(\ell_2)},$$

$$\|f\|_{\mathrm{BMO}_d^c(\mathcal{R})} = \sup_{I\in\mathcal{F}} \left\| \left(\frac{1}{|I|} \sum_{x\in I} \left| f(x) - \frac{1}{|Q|} \sum_{y\in I} f(y) \right|^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}\overline{\otimes}\mathcal{B}(\ell_2)}.$$

By the definition of BMO spaces, we are reduced to showing

$$\Big\| \sum_{i \in \mathcal{S}} T_i f \otimes e_{i1} \Big\|_{\mathrm{BMO}_d(\mathcal{R})} \lesssim \|f\|_\infty,$$

and

$$\Big\| \sum_{i \in \mathcal{S}} T_i f \otimes e_{1i} \Big\|_{\mathrm{BMO}_d(\mathcal{R})} \lesssim \|f\|_\infty.$$

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- $1 < p < 2$, real interpolation (Pisier and Xu, 2003, Book).
- $2 < p < \infty$, complex interpolation (Musat, 2003, JFA).

Thank you for your attention !