交换函数空间和非交换算子空间的球覆盖性质

刘敏曾

南开大学数学科学学院

哈尔滨工业大学 数学研究院 2022.8.16

定义 ([17])

X是赋范空间.

若存在X中的闭球列$(B(x_n, r_n))_{n=1}^{\infty}$，使得

$$S_X \subset \bigcup_{n=1}^{\infty} B(x_n, r_n)$$

且$0 \notin B(x_n, r_n)$，$n \in \mathbb{N}$，则称X有球覆盖性质（ball-covering property，BCP）.

CHENG L X. Ball-covering property of Banach spaces. Israel journal of mathematics, 2006, 156(1):111-123.

定义

X是赋范空间.

(1)[1] 若X有球覆盖性质且球心序列$(x_n)_{n=1}^{\infty}$有界（即$\sup\limits_{n\in\mathbb{N}}||x_n|| < \infty$），则称$X$有强球覆盖性质（strong ball-covering property，SBCP）.

(2)[1] 若X有强球覆盖性质且$\exists r > 0$，使得$B(x_n, r_n) \cap B(0, r) = \emptyset$，$n \in \mathbb{N}$，则称$X$有一致球覆盖性质（uniform ball-covering property，UBCP）.

LUO Z H, ZHENG B T. Stability of ball covering property. Studia mathematica, 2020, 250(1):19-34.

- $\ell^p(\Gamma)(1 \leqslant p \leqslant \infty)$没有球覆盖性质，其中$\Gamma$是不可数指标集.
- ℓ^∞有一致球覆盖性质.
  $S_{\ell^\infty} \subset \bigcup_{n=1}^{\infty} B(\pm p e_n, 1)$，$1 < p < 2$.
- $L^\infty[0,1]$没有球覆盖性质.

问题:

- $B(H)$有没有球覆盖性质? 有! 用谱分解.
- $B(l_p)$有没有球覆盖性质$(1 \leq p \leq \infty)$? 有! 依赖于序列范数.
- $B(L^p[0,1])$有没有球覆盖性质? 目前仅知道$p = 1$时没有，其余$1 < p < \infty$仍未知.
- $H^\infty(\mathbb{D})$有没有球覆盖性质?
- $BMO(\mathbb{R}^d)$有没有球覆盖性质?

由Hahn-Banach凸集分隔性定理，

定理

X是赋范空间. 若X有球覆盖性质，则X^*弱*可分，但反之不然.

$(L^\infty[0,1])^* = (L^1[0,1])^{**}$弱*可分，但$L^\infty[0,1]$没有球覆盖性质.

定理

X是赋范空间. 若X^*有球覆盖性质，则X可分，但反之不然.

$L^1[0,1]$可分，但$(L^1[0,1])^* = L^\infty[0,1]$没有球覆盖性质.

$(C[a,b])^*$没有球覆盖性质.

综上：X^*可分$\Longrightarrow$ X^*有球覆盖性质$\Longrightarrow$ X可分$\Longrightarrow$ X有球覆盖性质$\Longrightarrow$ X^*弱*可分，且其中每个逆命题均不成立.

定理 ([17])

S_X存在一个可数的半径一致小于1的球覆盖，则X是可分，反之亦然.

定理 ([17])

当X Gateaux具有可微性或局部一致凸性时，X有球覆盖性质，当且仅当X是w^*可分的.

CHENG L X. Ball-covering property of Banach spaces[J]. Israel journal of mathematics, 2006, 156(1):111-123.

定理

$(X, ||\cdot||)$是Banach空间，则X^*弱*可分，当且仅当$\forall \epsilon > 0$，存在$||\cdot||$的$(1+\epsilon)$等价范数$|||\cdot|||$，使得$X' = (X, |||\cdot|||)$有(强)球覆盖性质.

定理 ([21])

$(X, ||\cdot||)$是Banach空间，则X^*弱*可分，当且仅当S_X可被可数个中心对称的有界凸体的平移所覆盖，且这些集合都不包含零点.

Cheng, LiXin; Shi, HuiHua; Zhang, Wen Every Banach space with a w^*-separable dual has a $1+\epsilon$-equivalent norm with the ball covering property. Sci. China Ser. A 52 (2009), no. 9, 1869 – 1874.

FONF V P,ZANCO C. Covering spheres of Banach spaces by balls. Mathematische Annalen, 2009, 344(4):939-945.

定义函数$p: \ell^\infty \to \mathbb{R}$，若$u = (u(n))_{n=1}^\infty \in \ell^\infty$，$p(u) = \limsup_{n\to\infty} |u(n)|$. $\forall \lambda \in [0,1]$，

$$||\cdot||_\lambda \triangleq \lambda||\cdot|| + (1-\lambda)p(\cdot),$$

其中$||\cdot||$为ℓ^∞的上确界范数$||u|| = \sup_{n\in\mathbb{N}} |u(n)|$. 记$X_\lambda = (\ell^\infty, ||\cdot||_\lambda)$.

定理 ([6])

$X_\lambda = (\ell^\infty, ||\cdot||_\lambda)$有球覆盖性质，当且仅当$\lambda \in \left(\frac{1}{2}, 1\right]$.

问题: 重赋范的构造关于$B(H)$?

CHENG L X, CHENG Q J, LIU X Y. Ball-covering property of Banach spaces that is not preserved under linear isomorphisms[J]. Science in China Series A: Mathematics, 2008, 51(1):143-147.

定理 ([6])

子空间不一定保持球覆盖性质.

$\ell^1[0,1] \subset \ell^\infty$.

定理 ([1])

稠子空间保持球覆盖性质.

问题:

如果存在一个稠子空间满足球覆盖性质，全空间是否满足球覆盖性质

LUO Z H, ZHENG B T. Stability of ball covering property. Studia mathematica, 2020, 250(1):19-34.

ℓ^∞可被赋予一个等价范数使其失去球覆盖性质.

定理 ([6])

同构映射不一定保持球覆盖性质.

空间$X_\lambda = (\ell^\infty, ||\cdot||_\lambda)$, 易见当$\lambda = 0$时, $||\cdot||_0$是商空间ℓ^∞/c_0中的范数, 且定理4.1（必要性）的证明过程对$\lambda = 0$仍成立, 从而ℓ^∞/c_0没有球覆盖性质.

定理 ([6])

商空间不一定保持球覆盖性质.

问题:

1. 如果既是子空间又是商空间(即1-可补子空间)时, 是否传递球覆盖性质? 否!

2. Calkin代数$B(H)/K(H)$是否满足球覆盖性质?

重要问题:

如果一个Banach空间X的任意重赋范后都满足球覆盖性质，是否X可分???

定理 ([1])

$\{X_k : k \in \mathbb{N}\}$是一列赋范空间，$1 \leqslant p < \infty$，则直和空间$X = (\sum \oplus X_k)_{\ell^p}$有球覆盖性质，当且仅当每个$X_k$ 有球覆盖性质.

定理 ([1])

$\{X_k : k \in \mathbb{N}\}$是一列赋范空间，$E = c_0$或$\ell^\infty$，则直和空间$X = (\sum \oplus X_k)_E$有球覆盖性质，当且仅当每个$X_k$有球覆盖性质.

LUO Z H, ZHENG B T. Stability of ball covering property. Studia mathematica, 2020, 250(1):19-34.

定理 ([1])

X是赋范空间，$1 \leqslant p < \infty$，则$L^p([0,1],X)$有球覆盖性质，当且仅当X有球覆盖性质.

若(Ω,Σ,μ)是可分的测度空间，$1 \leqslant p < \infty$，则存在可数的指标集I，s.t. $L^p(\mu,X) = L^p([0,1],X) \oplus_p \ell^p(I,X)$. 从而由推论5.3和定理5.6，

定理 ([1])

(Ω,Σ,μ)是可分的测度空间，$1 \leqslant p < \infty$，则$L^p(\mu,X)$有球覆盖性质，当且仅当X有球覆盖性质.

定理 (Liu, Liu, Lu, Zheng, 2022)

K是局部紧Haussdorff空间，则下列等价

(1)$C_0(K)$有球覆盖性质.

(2)K有可列π-基(这是比拓扑基还弱的开集族!).

(3) $C_0(K)$有一致球覆盖性质.

问题:

C^*-代数满足球覆盖性质当且仅当满足一致球覆盖性质?

Minzeng Liu; Rui Liu; Jimeng Lu,;Bentuo Zheng Ball covering property from commutative function spaces to non-commutative spaces of operators. Journal of Functional Analysis,283(2022) 109502

定理 (Liu, Liu, Lu, Zheng, 2022)

X是Banach空间, K是局部紧Haussdorff空间, 则下列等价

(1)$C_0(K,X)$有(一致或强)球覆盖性质.

(2)K有可列π-基并且X有(一致或强)球覆盖性质.

(3) $C_0(K)$有(一致或强)覆盖性质并且X有(一致或强)球覆盖性质.

推论 (Liu, Liu, Lu, Zheng, 2022)

$\{\Omega_k\}$ 至多可数个紧*Hausdorff*空间. $\times\Omega_k$为积空间. 则下列等价:

(1) $\times\Omega_k$有可列π-基;

(2) $C(\times\Omega_k)$有一致球覆盖性质;

(3) $C(\Omega_k)$有一致球覆盖性质对每个k;

(4) Ω_k有可列π-基对每个k.

我们给出个例子通过这个拓扑刻画. 设L是如此实函数全体$f:\mathbb{R}\to\{0,1\}$, 拓扑为逐点拓扑, 即为积空间

$$L=\{0,1\}^{\mathbb{R}}.$$

由Tychonoff定理可知是紧空间.

则L没有可数π-基. 反设若存在可数π-基$\{U_n\}$. 设$W_r=\{f\in L: f(r)=1\}$ 对$r\in\mathbb{R}$则是L中的开集. 由π-基的定义, 对每个$r\in\mathbb{R}$存在$n\in\mathbb{N}$ 使得$U_n\subseteq W_r$. 则存在$n_0\in\mathbb{N}$ 和不可数子集$A\subseteq\mathbb{R}$使得

$$U_{n_0}\subseteq\bigcap_{r\in A}W_r.$$

导出矛盾, 因为$\bigcap_{r\in A}W_r$内部为空对任意无限子集$A\subseteq\mathbb{R}$.

并且L是可分的. 实际上，有理数端点的区间的有限并集上的特征函数即为可列稠密子集.

则设$\{h_n\}_{n\in\mathbb{N}}$为L的可数稠子集, 定义

$$K = L \times \{0\} \bigcup \left\{ \left(h_n, \frac{1}{n} \right) : n \in \mathbb{N} \right\} \subseteq L \times \mathbb{R}.$$

这是$L \times [0,1]$的闭子集, 因此是紧空间. 这些点$(h_n, \frac{1}{n})$是K中的孤立点并且是K的可列稠子集. 特别的, K有可列π-基, 即为单点集$\{(h_n, \frac{1}{n})\}$.

由拓扑刻画定理，$C(K)$有一致球覆盖性质, 但$C(L)$ 没有球覆盖性质. 而且，$C(L)$是$C(K)$的1可补子空间.

设$\alpha : K \to L$为投影$\alpha(f,x) = f$，并且$\beta : L \to K$ 定义为$\beta(f) = (f,0)$ for $f \in L$.然后用复合算子按定义验证即可.

定理 (Liu, Liu, Lu, Zheng, 2022)

X是Banach空间并且X^*可分，$1 < p < \infty$，则$\mathcal{B}(X, \ell_p)$中任意包含$\mathcal{F}(X, \ell_p)$的子空间有一致球覆盖性质.

定理 (Liu, Lu, Liu, Zheng, 2022)

$\mathcal{B}(\ell_1)$和$\mathcal{B}(c_0)$有球覆盖性质.

定理 (Liu, Liu, Zheng, Lu, 2022)

设X和Y是Banach空间. 如果$\mathcal{B}(X,Y)$有球覆盖性质, 当且仅当X^*和Y有球覆盖性质.

推论 (Liu, Liu, Lu, Zheng, 2022)

$\mathcal{B}(L^1[0,1])$没有球覆盖性质.

问题

设X和Y是Banach空间. 如果$\mathcal{B}(X,Y)$有球覆盖性质,当且仅当X^*和Y有球覆盖性质.

问题

设X和Y是Banach空间. 如果$X\bigotimes_\epsilon Y$具有球覆盖性质,当且仅当X和Y有球覆盖性质.

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感谢各位老师和同学!