Lipschitz Retract of the Space of Bounded Linear Operators

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The Ribe Program

The study of the nonlinear geometry of Banach spaces can be dated back to the begining of the functional analysis. Roughly speaking, the nonlinear geometry aims to investigate metric structures of Banach spaces ignoring their linear structures.

The Mazur-Ulam theorem

Theorem (Mazur-Ulam, 1932)

Let X and Y be Banach spaces and $T : X \to Y$ be a surjective isometry. Then T is affine.

Recall that a mapping $T : X \to Y$ is said be affine if for every $\lambda \in \mathbb{R}$ one has $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$ for all $x, y \in X$.

Uniform classification of spheres

Theorem (Mazur, 1933)

For any $1 \leq p,\ q < \infty$, suppose that (Ω, μ) and (W, ν) are two measures spaces such that $L_p(\Omega)$ and $L_q(W)$ have the same density character. Then $S_{L_p(\Omega)}$ is uniformly homeomorphic to $S_{L_q(W)}$.

Recall that $S_{L_p(\Omega)} := \{f \in L_p(\Omega) : \|f\|_{L_p} = 1\}$ is the unit sphere of $L_p(\Omega)$.

Snowfalke embeddings

Theorem (Wells-Williams, 1970)

For the case $1 \leq q \leq p \leq 2$ the metric space $(L_q, \|\cdot\|_{L_q}^{\theta})$ is emebdded into L_p isometrically for every $0 < \theta \leq q/p$.

Recall that the snowfalke metric $\|\cdot\|_{L_q}^{\alpha}$ is given by $\|f-g\|_{L_q}^{\alpha}$ for every $f, g \in L_q$ and $0 < \alpha < 1$.

The Ribe theroem

Theorem (Ribe, 1978)

Let X and Y be Banach spaces. If X and Y are uniformly homeomorphic, then X is crudely finitely representable in Y and visa versa.

Recall that a Banach space X is said to be crudely finitely representable in Y if there exsits $K > 1$ such that for each finite dimensional subspace $X_1 \subseteq X$, then there is a subspace $Y_1 \subseteq Y$ with $d_{B-M}(X_1, Y_1) < K$, where the Banach-Mazur distance $d_{B-M}(\cdot, \cdot)$ is given by

$d_{B-M}(E, F) := \inf\{\|T\|\|T^{-1}\| : T : E \to F \text{ is linear isomorphism}\}$, otherwise put $d_{B-M}(E, F) = \infty$.

The Ribe program

A detailed exposition of this program will appear in J. Lindenstrauss' forthcoming survey [...] in our "dictionary" linear operators are translated in Lipschitz maps, the operator norm by the Lipschitz constant of the map [...] The translations of "Banach-Mazur distance" and "finite representability" in linear theory are immediate...

J. Bourgain, 1986.

Assume from now on $(\varepsilon_j)_{j=1}^{\infty}$ is a sequence of independent Rademacher random variables, that is, for each $j \in \mathbb{N}$, $\mathbb{P}\{\varepsilon_j = 1\} = \mathbb{P}\{\varepsilon_j = -1\} = \frac{1}{2}$.

Definition (Rademacher type)

A Banach space B is said to have *Rademacher type* $p \in [1, 2]$, if there exists $T > 0$ such that for all $n \geq 0$ and $x_1, \cdots, x_n \in B$,

$$\mathbb{E}\left\|\sum_{j=1}^{n} \varepsilon_j x_j\right\|^p \leq T^p \sum_{j=1}^{n} \|x_j\|^p.$$

We denote by $T_p^R(B)$ the smallest possible constant T fulfills the above inequality.

Definition (Enflo type)

A Banach space B is said to have *Enflo type* $p \in [1,2]$ if there exists $T > 0$ such that for all $n \geq 1$ and $f : \{-1,1\}^n \to B$,

$$\mathbb{E}\|f - \mathbb{E}f\|^p \leq T^p \sum_{j=1}^{n} \mathbb{E}\|D_j f\|^p,$$

where the discrete derivatives on the cube is defined by, for each $j = 1, \cdots, n, \vec{\varepsilon} \in \{-1,1\}^n$,

$$D_j f(\vec{\varepsilon}) = \frac{f(\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_n) - f(\varepsilon_1, \cdots, -\varepsilon_j, \cdots, \varepsilon_n)}{2}.$$

We denote by $T_p^E(B)$ the smallest possible constant T fulfills the above inequality.

The equivalence of Rademacher type and Enflo type

One of the most important long-standing open problem in the Ribe Program is whether Rademacher type p implies Enflo type p. This problem has been opened for more than 40 years!

Theorem (Ivanisvili-van Handel-Volberg, 2020)

For a Banach space B we have

$$T_p^R(B) \leq T_p^E(B) \leq \frac{\pi}{2} T_p^R(B). \tag{1}$$

Lipschitz retractions

Linear projections

Let Y be a closed subspace of Banach space X and a mapping P defined from X onto Y is said to be a projection if $P|_Y = \mathrm{id}_Y$, that is, $P(y) = y$ for every $y \in Y$. In additional, if P is linear, then P is called a linear projection.

A closed subspace Y is said to be complemented in X if there exists a bounded linear projection from X onto Y.

In the infinite dimensional setting, we shall impose some additional topological assumptions on projections.

Linear projections

Theorem (Phillips-Sobczyk, 1940)

c_0 is uncomplemented in ℓ_∞.

Lipschitz retractions

Nonlinear projections

Given a closed subspace Y of X, if a projection $P : X \to Y$ is continuous, unformly continuous, and Lipshitz, then P is called a continuous, uniformly continuous, and Lipschitz retraction respectively.

If such retract $P : X \to Y$, the subspace Y will be called a continuous, uniformly continuous, Lipschitz retract of X.

In the study of Lipschitz retraction of Banach spaces, a breakthrough was made by Lindenstrauss, who showed that c_0 is a Lipschitz retract of ℓ_∞. Specifically, he obtained the following theorem.

Theorem (Lindenstrauss, 1964)

There exists a Lipschitz retraction $F : \ell_\infty \to c_0$ such that $\|F\|_{\mathrm{Lip}} \leq 2$. Suppose that K is a compact topological space, then there exists a Lipschitz retraction $F : C(K)^{**} \to C(K)$ such that $\|F\|_{\mathrm{Lip}} \leq 2$.

Recall that $\|F\|_{\mathrm{Lip}} := \sup_{x,\, y\in X,\, x\neq y} \frac{\|F(x)-F(y)\|}{\|x-y\|}$ is the Lipschitz norm of F.

Lindenstrauss' mapping

The Lindenstrauss mapping $F : \ell_\infty \to c_0$ is given as follows

$$F(x)_n = \begin{cases} 0, & \text{if } |x_n| < d(x) \\ (|x_n| - d(x))sign(x_n), & \text{if } |x_n| \geq d(x), \end{cases} \tag{2}$$

where $d(x) := \mathrm{dist}(x, c_0) = \limsup_{n\to\infty} |x_n|$, for $x \in \ell_\infty$.

A conjecture of Lindenstrauss

The following problem, posed by Lindenstrauss, has been opened for more than 40 years!

Problem (Lindenstrauss, 1964)

Whether every Banach is a Lipschitz retract of its bidual.

Until 2011, Kalton answered this problem negatively by construct a nonseparable conterexample which is not a Lipschitz retract in its bidual. Later, in 2017, Surárz de la Fuente further modified Kalton's construction by showing a nonseparable Banach space with Schur property, which is not a Lipschitz retract of its bidual.

However, Lindenstrauss' problem remains open for the case of separable Banach spaces.

Kalton's investigations in Lipschitz retracts in Banach spaces

Another remarkable result of Kalton asserts that a Banach space X is a Lipschitz retract of its bidual if X satisfies some specific conditions.

Theorem (Kalton, 2011)

A Banach space X is a Lipschitz retract of X^{**} if X satisfies one of the following conditions

i. *X is a Banach space with an unconditional finite-dimensional decomposition;*
ii. *X is a separable order continuous lattice.*

Tanaka's result

Theorem (Lindenstrauss, 1964)

c_0 is a 2-Lipschitz retract of ℓ_∞.

Theorem (Tanaka, 2019)

Denote that $K(\ell_2)$ and $B(\ell_2)$ are the space of all compact and bounded linear operators on ℓ_2 respectively. Then $K(\ell_2)$ is a $64\sqrt{2}+4$-Lipschitz retract of $B(\ell_2)$.

Interesting questions

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- *How to sharpen the Lpischitz consntant in Tanaka's result?*

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- *Interesting question: What about the case $K(\ell_p)$ and $B(\ell_p)$ for $p \neq 2$?*
  Caution: $B(\ell_p)$ does not form a $*$-algebra!
- *More interesting question: What about $K(\ell_p, \ell_q)$ and $B(\ell_p, \ell_q)$ for $p \neq q$?*
  Caution: $B(\ell_p, \ell_q)$ does not form an algebra for $p \neq q$!

Main results

Theorem (Cheng-He-Luo, 2022)

For $1 \leq p,\ q < \infty$, $K(\ell_p, \ell_q)$ is a 7-Lipschitz retract of $B(\ell_p, \ell_q)$.

Our approach

The following spaces, introduced by Xue, Li and Bu, play an essential role in our approach. Suppose that $(U, \|\cdot\|_U)$ is a Banach space having a K-unconditional basis $(e_j)_{j=1}^{\infty}$, and X is a Banach space.

Definition

A sequence $\{e_j\}_{j=1}^{\infty}$ is said to be a K-unconditional basis in U with $K \geq 1$, if the following conditions hold

- for every x there exists a unique sequence of scalars $(a_j)_{j=1}^{\infty}$ such that $x = \sum_{j=1}^{\infty} a_j x_j$;
- for every $n \in \mathbb{N}$ and any sequences of scalars $(a_j)_{j=1}^{n}$ and $(b_j)_{j=1}^{n}$ with $|a_j| \leq |b_j|$ for each $j = 1, \ldots, n$, we have

$$\left\| \sum_{j=1}^{n} a_j e_j \right\|_U \leq K \left\| \sum_{j=1}^{n} b_j e_j \right\|_U . \tag{3}$$

Xue-Li-Bu's spaces

The space $U_w(X)$ is defined by

$$U_w(X) := \left\{ \bar{x} = (x_j)_{j=1}^{\infty} \in X^{\mathbb{N}} : \sum_{j=1}^{\infty} x^*(x_j)e_j \text{ converges in } U, \ \forall x^* \in X^* \right\}.$$

$U_w(X)$ forms a Banach space with the norm $\|\cdot\|_{U_w(X)}$ given by

$$\|\bar{x}\|_{U_w(X)} := \sup \left\{ \left\| \sum_{j=1}^{\infty} x^*(x_j)e_j \right\|_U : x^* \in B_{X^*} \right\}$$

Xue-Li-Bu's spaces

For each $\bar{x} \in U_w(X)$ and $n \in \mathbb{N}$, we denote

$$\bar{x}(\geq n) := (0, \dots, 0, x_n, x_{n+1}, \dots).$$

Let $U_{w,0}(X)$ be a subspace of $U_w(X)$ consisting of elements $\bar{x}$ with $\lim_{n\to\infty} \|\bar{x}(\geq n)\|_{U_w(X)} = 0$, i.e.,

$$U_{w,0}(X) := \left\{\bar{x} \in U_w(X) : \lim_{n\to\infty} \|\bar{x}(\geq n)\|_{U_w(X)} = 0\right\}.$$

It is clear that $U_{w,0}(X)$ is a closed subspace of $U_w(X)$.

Xue-Li-Bu's representations

Theorem (Xue-Li-Bu, 2007)

Assume that U is a Banach space with boundedly complete 1-unconditional basis $\{e_j\}_{j=1}^{\infty}$, and $\{e_j^*\}_{j=1}^{\infty}$ is the orthogonal linear functionals associated with $\{e_j\}_{j=1}^{\infty}$. Let $V = \overline{\mathrm{span}}\{e_j^* : j \in \mathbb{N}\}$. Then there exists a surjective linear isometry $T : U_w(X) \to B(V, X)$ such that the restriction $T|_{U_{w,0}(X)}$ is a surjective linear isometry from $U_{w,0}(X)$ onto $K(V, X)$.

Main results

Theorem (Cheng-He-Luo, 2022)

Let U be a Banach space with 1-unconditional basis and X be a Banach space. Then $U_{w,0}(X)$ is a 7-Lipschitz retract of $U_w(X)$.

Main results

Combing with the Xue-Li-Bu representation theorem, we obtain the following.

Corollary (Cheng-He-Luo, 2022)

For $1 < p < \infty$, $1 \leq q \leq \infty$, $K(\ell_p, \ell_q)$ is a 7-Lipschitz retract of $B(\ell_p, \ell_q)$.

Theorem (Cheng-He-Luo, 2022)

For $1 \leq q < \infty$, $K(\ell_1, \ell_q)$ is a 7-Lipschitz retract of $B(\ell_1, \ell_q)$.

Given $A \in B(\ell_1, \ell_q)$, A could be represented as an infinite dimensional matrix as $A = (a_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ such that the following holds

$$\|A\| = \sup\left\{\left(\sum_{i=1}^{\infty}|a_{i,j}|^q\right)^{1/q} : j \in \mathbb{N}\right\}$$

Lipschitz retractions

Some further questions

- Our method could not tackle the following two cases: $p = 1$, $q = \infty$ and $p = \infty$, $1 \leq q \leq \infty$. The main obstructions are the nonseparability of ℓ_∞ and the complexity of the dual space of ℓ_∞ (i.e., $\ell_\infty^* = \ell_1 \oplus c_0^\perp$).
- We have already known if there exists a Lipschitz retraction $F : B(\ell_p) \to K(\ell_p)$, then $\|F\|_{\mathrm{Lip}} \geq 2$. But we do not know whether there is a Lipschitz retraction $F : B(\ell_p) \to K(\ell_p)$ such that $\|F\|_{\mathrm{Lip}} = 2$.
- Whether every separable Banach is a Lipschitz retract of its bidual.

Thank you for your attention!