LECTURE NOTES ON NONSTANDARD ANALYSIS

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1. Hyperreals

The study of modern mathematics is, to a large extent, the study of infinity. Nonstandard analysis provides an alternative (but extremely effective) viewpoint of infinity. The purpose of this note is to give a brief (non-logical) introduction of nonstandard analysis.

1.1. Basic Facts about the Ordered Real Field.

A field is a set along with two operations defined on the set: the addition operation $+$ and the multiplication operation $\cdot$, both of which behave similarly as addition and multiplication in real numbers. Formally speaking:

Definition 1.1. A field is a set F along with two operations $+$ (addition) and $\cdot$ (multiplication) from $F \times F$ to F such that:

- **Associativity**: $a + (b + c) = (a + b) + c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- **Commutativity**: $a + b = b + a$ and $a \cdot b = b \cdot a$;
- **Additive and Multiplicative Identity**: there are two distinct elements $0, 1 \in F$ such that $(\forall a \in F)(a + 0 = a \wedge a \cdot 1 = a)$;
- **Additive Inverse**: for every $a \in F$, there is an element $-a \in F$ (called the *additive inverse* of a), such that $a + (-a) = 0$;
- **Multiplicative Inverse**: for every $a \in F$ such that $a \neq 0$, there exists an element $a^{-1} \in F$ (called the *multiplicative inverse* of a), such that $a \cdot a^{-1} = 1$;
- **Distributive**: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

A binary relation R on a set F is a subset of $F \times F$. For example, $\leq$ and $<$ are binary relations on $\mathbb{R}$ ($(a, b) \in \leq$ if $a \leq b$).

Definition 1.2. A (strict) total order is a binary relation $<$ on some set F, which satisfies the following for all $a, b, c \in X$:

- Not $a < a$ (irreflexive);
- If $a < b$ then not $b < a$ (antisymmetry);
- If $a < b$ and $b < c$, then $a < c$ (transitive);
- If $a \neq b$, then either $a < b$ or $b < a$ (totality).

A field F with a (strict) total order $<$ on F is an *ordered field* if $<$ satisfies the following properties for all $a, b, c \in F$:

- If $a < b$, then $a + c < b + c$;
- If $0 < a$ and $0 < b$, then $0 < a \cdot b$.

We focus on our favorite ordered field $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$ (the ordered field of the reals). We recall some basic properties:

- the set $\mathbb{Q}$ is dense in $\mathbb{R}$;
- **the triangle inequality**: $|x + y| \leq |x| + |y|$;
- **the Archimedean Property**: for every $x, y \in \mathbb{R}_{>0}$, there is $n \in \mathbb{N}$ such that $nx > y$.

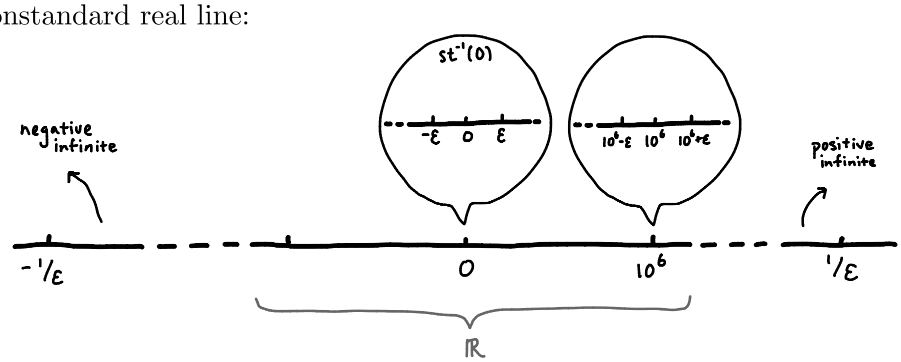
Perhaps the most important property of $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$ is:

Definition 1.3 (The Completeness Property)**.** If $A \subset \mathbb{R}$ is a subset that is bounded above, then there is a $b \in \mathbb{R}$ such that:

- for all $a \in A$, $a \leq b$ (b is an upper bound of A);
- if $a \leq c$ for all $a \in A$, then $b \leq c$ (b is the least upper bound of A).

We will investigate these properties in the "nonstandard extension" of $\mathbb{R}$ in the upcoming sections.

1.2. **Nonstandard Extension.** Nonstandard analysis is introduced by Abraham Robinson (see Robinson [Rob66]) with the aim of formalizing infinitesimals and infinite numbers in calculus. As of today, nonstandard analysis (as a powerful machinery derived from model theory) is not only a deep field on its own but also has many fruitful applications in diverse areas in pure and applied mathematics, including measure theory, stochastic processes, mathematical physics, mathematical economics, functional analysis, combinatorics, Lie theory (Hilbert 5-th problem), and geometric group theory. The starting point of nonstandard analysis is the construction of the nonstandard real line:



The construction of the nonstandard universe requires knowledge on model theory and set theory (ultraproducts). We instead pose some postulates that the nonstandard real line should possess, assume the existence of the nonstandard real line, and start "doing" nonstandard analysis as soon as possible.

The *nonstandard real line* ${}^*\mathbb{R}$ has the following properties:

- **Extension**: the ordered field of reals is an ordered sub-field of $\langle {}^*\mathbb{R}, +, \cdot, 0, 1, < \rangle$;
- **Infinitesimal**: ${}^*\mathbb{R}$ has a positive infinitesimal. There is $\epsilon \in {}^*\mathbb{R}$ such that $\epsilon > 0$ and $\epsilon < r$ for all $r \in \mathbb{R}_{>0}$;

- **Functions and Sets Extension**: For every $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a "natural extension" (called the nonstandard extension) ${}^*f : {}^*\mathbb{R}^n \to {}^*\mathbb{R}$. The nonstandard extension of $+, \cdot$ coincide with field operations on ${}^*\mathbb{R}$. Similarly, for every $A \subset {}^*\mathbb{R}^n$, there is a "natural extension" (nonstandard extension) ${}^*A \subset {}^*\mathbb{R}^n$ such that $\mathbb{R}^n \cap {}^*A = A$;
- **Transfer**: ${}^*\mathbb{R}$, equipped with the above assignments of extensions of functions and susbsets, "behaves logically" as $\mathbb{R}$ (preserves the truth value of first-order logical statements).

$\langle {}^*\mathbb{R}, +, \cdot, 0, 1, < \rangle$ is called the *ordered field of hyperreals.* Our statement of the **Transfer** item is extremely vague. This item is formally known as the transfer principle and is at the heart of nonstandard analysis. However, we only provide rule of thumbs for using the transfer principle:

- Any statement expressible in first order logic and mentioning only *elements of* $\mathbb{R}$ is true in $\mathbb{R}$ if and only if it is true in ${}^*\mathbb{R}$;
- typical transferrable statement involve quantifiers over numbers but not sets of numbers;
- typical transferrable statement involve the relation $\in$ but not the relation $\subset$.

1.3. **Arithmetic in** ${}^*\mathbb{R}$**.** Since $\langle {}^*\mathbb{R}, +, \cdot, 0, 1, < \rangle$ is an ordered field, field operations are defined for hyperreals (we can do algebra on hyperreals!). Since there is a positive infinitesimal ϵ, we have:

- ϵ has an additive inverse $-\epsilon$, which is a *negative infinitesimal*;
- Since $\epsilon > 0$, it has a multiplicative inverse ϵ^{-1}, which is a positive infinite number. Then, $-\epsilon^{-1}$ is a negative infinite number;
- the transfer principle ensures that all algebraic operations can be applied to hyperreals. So we can consider $\epsilon^2, \sqrt{\epsilon}, 100 + \epsilon, \ldots$.

Definition 1.4. We define a few important subsets of ${}^*\mathbb{R}$.

- The set of *finite hyperreals* is ${}^*\mathbb{R}_{\text{fin}} = \{x \in {}^*\mathbb{R} | (\exists n \in \mathbb{N})(|x| \leq n)\}$;
- The set of *infinite hyperreals* is ${}^*\mathbb{R}_{\text{inf}} = {}^*\mathbb{R} \setminus {}^*\mathbb{R}_{\text{fin}}$;
- The set of *infinitesimal hyperreals* is

$$\text{monad}(0) = \{x \in {}^*\mathbb{R} | (\forall n \in \mathbb{N})(|x| \leq \frac{1}{n})\}.$$

The following lemma is self-evident:

Lemma 1.5. *For all $x, y \in {}^*\mathbb{R}_{\text{fin}}$, $x \pm y$ and $x \cdot y$ are both elements of ${}^*\mathbb{R}_{\text{fin}}$. For all $x, y \in \text{monad}(0)$, $x \pm y$ and $x \cdot y$ are both elements of $\text{monad}(0)$. Moreover, for all $x \in {}^*\mathbb{R}_{\text{fin}}$ and all $y \in \text{monad}(0)$, we have $xy \in \text{monad}(0)$.*

Definition 1.6. For $x, y \in {}^*\mathbb{R}$, we say x and y are *infinitely close*, written, $x \approx y$, if $x - y \in \text{monad}(0)$.

It is straightforward to verify that $\approx$ is an equivalence relation on ${}^*\mathbb{R}$.

Theorem 1.7. *If $r \in {}^*\mathbb{R}_{\text{fin}}$, then there is a unique $s \in \mathbb{R}$ such that $r \approx s$. We call s the* standard part *of r.*

Proof. The set $A = \{x \in \mathbb{R} | x < r\}$ is bounded. By the completeness property, A has a least upper bound $\sup A$, which is the standard part of r. □

Remark 1.8. The proof of Theorem 1.7 makes use of the completeness property of $\mathbb{R}$. We will later shown that the existence of standard part implies the completeness property of $\mathbb{R}$. Hence, completeness of $\mathbb{R}$ is equivalent to the existence of standard part.

Definition 1.9. The *standard part map* $\mathsf{st} : {}^*\mathbb{R}_{\text{fin}} \to \mathbb{R}$ maps every finite hyperreal to its standard part. The *near-standard part* of ${}^*\mathbb{R}$, denoted by $\text{NS}({}^*\mathbb{R})$, is the set of hyperreals with standard part.

By Theorem 1.7, we have ${}^*\mathbb{R}_{\text{fin}} = \text{NS}({}^*\mathbb{R})$. It is easy to verify that $\mathsf{st}(x+y) = \mathsf{st}(x) + \mathsf{st}(y)$ and $\mathsf{st}(xy) = \mathsf{st}(x)\mathsf{st}(y)$ for all $x, y \in {}^*\mathbb{R}_{\text{fin}}$.

1.4. **The Structure of ${}^*\mathbb{N}$.** In this section, we focus on the set ${}^*\mathbb{N}$ of nonstandard natural numbers.

Lemma 1.10. *The set ${}^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$. Moreover, if $y \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $y \in {}^*\mathbb{R}_{\text{inf}}$, i.e., every $y \in {}^*\mathbb{N} \setminus \mathbb{N}$ is infinite.*

Proof. Pick $a \in {}^*\mathbb{R}_{\text{inf}}$. By the transfer principle, there exists $N \in {}^*\mathbb{N}$ such that $a \leq N$. Such N must be infinite, hence ${}^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$.

We prove that, if $y \in {}^*\mathbb{N} \cap {}^*\mathbb{R}_{\text{fin}}$, then $y \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $n \leq y \leq n+1$. As $y \in {}^*\mathbb{N}$, y must either be n or $n+1$, which implies that $y \in \mathbb{N}$. □

Definition 1.11. A set A is *hyperfinite* if and only if there is an (internal) bijection between A and $\{1, 2, \ldots, N\}$ for some $N \in {}^*\mathbb{N}$.

Hyperfinite sets are infinite sets but "behaves logically" as finite sets (so we can take addition, multiplication, find max/min elements of a hyperfinite set). Hyperfinite sets naturally link discrete math with their continuous analogues.

1.5. **More Practice with the Tranfer.** As we have mentioned, the transfer principle is at the heart of the nonstandard analysis. We illustrate a few applications of the transfer principle:

Example 1.12. We illustrate three basic applications of the nonstandard analysis in this example:

- For every $a, b \in {}^*\mathbb{R}$, $a + b = b + a$ and $a \cdot b = b \cdot a$;
- Given a bounded above set $A \subset \mathbb{R}$, then ${}^*A$ has an upper bound;
- In fact, ${}^*A$ in the above item has a least upper bound.

On the other hand, one has to be very careful applying the transfer principle:

Lemma 1.13. *There exists a bounded subset of* $^*\mathbb{R}$ *with no least upper bound. In fact, if* $A \subset \mathbb{R}$ *is not bounded above, then* A *does not have a least upper bound in* $^*\mathbb{R}$.

We conclude this section with the following result:

Theorem 1.14. *The statement "every* $y \in {}^*\mathbb{R}_{\text{fin}}$ *has a standard part" implies the completeness property of of* $\mathbb{R}$.

Proof. Suppose $A \subset \mathbb{R}$ is non-empty and bounded above. We need to show that $\sup A$ exists. Let $f : \mathbb{N} \to \mathbb{N}$ be the function such that

$$f(n) = \inf\{k \in \mathbb{N} : \frac{k}{n} \text{ is an upper bound of } A\}.$$

Such f is well-defined because of the Archimedean property and the well-ordering property. The nonstandard extension $^*f$ is a function from $^*\mathbb{N}$ to $^*\mathbb{N}$. By the transfer, for $n \in {}^*\mathbb{N}$:

$$^*f(n) = \inf\{K \in {}^*\mathbb{N} : \frac{K}{n} \text{ is an upper bound of } {}^*A\}.$$

Pick $N \in {}^*\mathbb{N} \setminus \mathbb{N}$.

Claim 1.15. $\frac{^*f(N)}{N} \in {}^*\mathbb{R}_{\text{fin}}$.

Proof. Suppose $\frac{^*f(N)}{N} \in {}^*\mathbb{R}_{\text{inf}}$. Then $^*f(N) \in {}^*\mathbb{R}_{\text{inf}}$. There exists $a_0 \in {}^*A$ such that $\frac{^*f(N)-1}{N} \leq a_0$. Note that $\frac{^*f(N)-1}{N} \in {}^*\mathbb{R}_{\text{inf}}$, so a_0 must also be infinite, contradicts with A being bounded. □

Let $r = \mathsf{st}(\frac{^*f(N)}{N})$. We shall show that r is the least upper bound of A. For every $a \in A$, $a \leq \frac{^*f(N)}{N} \approx r$. So r is an upper bound. Pick any $\delta > 0$

and consider $r - \delta$. As $r - \delta < \frac{^*f(N)-1}{N}$, we have:

$$(\exists b \in {}^*A)(r - \delta < b).$$

By the transfer, we have $(\exists b \in A)(r - \delta < b)$, which is what we need. □

2. Limit and Continuity

In this section, we use nonstandard analysis to study the concept of limit and continuity. As we shall see, nonstandard analysis eliminates quantifiers in many arguments, hence drastically simplify a lot of arguments.

2.1. **Sequence Limit.** A *real sequence* is a function $s : \mathbb{N} \to \mathbb{R}$. We often use $(s_n)_{n\in\mathbb{N}}$ to denote the sequence, where $s_n = s(n)$. The nonstandard extension $^*s$ is a map from $^*\mathbb{N}$ to $^*\mathbb{R}$, which we often write $(^*s_n)_{n\in{}^*\mathbb{N}}$.

Definition 2.1. $(s_n)_{n\in\mathbb{N}}$ converges to L, written $\lim_{n\to\infty} s_n = L$, if: for all $\epsilon \in \mathbb{R}_{>0}$, there is $m \in \mathbb{N}$ such that for all $n \geq m$, $|s_n - L| < \epsilon$.

Theorem 2.2. $\lim_{n\to\infty} s_n = L$ *if and only if* $^*s(N) \approx L$ *for all* $N \in {}^*\mathbb{N} \setminus \mathbb{N}$.

The transfer is at the heart of Theorem 2.2. The "if" part uses the "downward transfer" while the "only if" part uses the "upward" transfer.

2.2. **Continuity.** Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ a function. Then f is continuous at $c \in A$ if: for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. We provide a nonstandard characterization of continuity:

Theorem 2.3. *Suppose* $f : A \to \mathbb{R}$ *and* $c \in A$*. Then the following statements are equivalent:*

- f *is continuous at* c*;*
- $^*f(x) \approx {}^*f(c) = f(c)$ *for all* $x \approx c$*;*

- *there is* $\delta \in {}^*\mathbb{R}_{>0} \cap \mathrm{monad}(0)$ *such that, for all* $x \in {}^*A$, *if* $|x - c| < \delta$, *then* ${}^*f(x) \approx {}^*f(c)$.

As in Theorem 2.2, the proof of Theorem 2.3 uses "upward" and "downward" transfer.

Corollary 2.4. *Suppose* f *is continuous at* $c \in \mathbb{R}$ *and* g *is continuous at* $f(c)$. *Then* $g \circ f$ *is continuous at* c.

Example 2.5. As $\sin x$ is continuous at 0, so $\sin \epsilon \approx 0$ for all $\epsilon \approx 0$. Similarly, $\cos \epsilon \approx 1$ for all $\epsilon \approx 0$. We now show that sin is continuous in $\mathbb{R}$: if $c \in \mathbb{R}$ and $x \approx c$, then $x = c + \epsilon$ for some $\epsilon \approx 0$. Thus, we have:

$$\sin x = \sin c + \epsilon = \sin c \cos \epsilon + \cos c \sin \epsilon \approx \sin c.$$

We now present the intermediate value theorem, the proof of which makes use of hyperfinite partition.

Theorem 2.6 (The Intermediate Value Theorem)**.** *Suppose* $f : [a, b] \to \mathbb{R}$ *is continuous. Let* $d \in \mathbb{R}$ *be a value strictly in between of* $f(a)$ *and* $f(b)$. *Then, there exists* $c \in [a, b]$ *such that* $f(c) = d$.

Proof. Without loss of generality, assume that $f(a) < f(b)$. Then we have $f(a) < d < f(b)$. Pick $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $\{p_1, p_2, \dots, p_N\}$ be a hyperfinite partition of ${}^*[a, b]$ such that into N pieces of equal width $\frac{b-a}{N}$, so $p_1 = a$ and $p_N = b$. Let $s_N = \max\{p_k | {}^*f(p_k) < d\}$. Note that $a < s_N < b$.

Note that $c = \mathsf{st}(s_N) \in [a, b]$. As $s_N < b$, we conclude that $s_N + \frac{b-a}{N} \leq b$. By definition, we have ${}^*f(s_N) < d \leq {}^*f(s_N + \frac{b-a}{N})$. As $s_N \approx s_N + \frac{b-a}{N} \approx c$ and f is continuous at c, we have

$$f(c) \approx {}^*f(s_N) < d \leq {}^*f(s_N + \frac{b-a}{N}) \approx f(c).$$

Hence, we must have $f(c) = d$. □

Similarly, we prove the extreme value theorem.

Theorem 2.7 (The Extreme Value Theorem)**.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous. Then there exist* $c, d \in [a, b]$ *such that* $f(c) \leq f(x) \leq f(d)$ *for all* $x \in [a, b]$.

Proof. We only prove the existence of the maximum of f. Pick $N \in {}^*\mathbb{N}$, and let $\{p_1, p_2 \ldots, p_N\}$ be a hyperfinite partition of ${}^*[a, b]$ of equal width $\frac{b-a}{N}$. By the transfer principle, there is some $1 \leq k_0 \leq N$ such that ${}^*f(p_i) \leq {}^*f(p_{k_0})$ for all $1 \leq i \leq N$. Then, f achieves its maximum at $d = \mathsf{st}(p_{k_0})$. □

2.3. **Uniform Continuity.** The difference between continuity and uniform continuity is subtle. For continuity, the δ depends on both ϵ and the point at which the function is continuous. On the other hand, for uniform continuity, the δ depends only on ϵ. Nonstandard analysis provides an alternative (and clearer!) characterization of uniform continuity.

Definition 2.8. $f : A \to \mathbb{R}$ is *uniformly continuous* if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in A$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Uniform continuity is perhaps one of the best examples of elucidating a standard concept by nonstandard means.

Definition 2.9. A function $F : {}^*A \to {}^*\mathbb{R}$ is *S-continuous* if $F(x) \approx F(y)$ for all $x, y \in {}^*A$ such that $x \approx y$.

Theorem 2.10. *A* $f : A \to \mathbb{R}$ *is uniformly continuous if and only if* ${}^*f : {}^*A \to {}^*\mathbb{R}$ *is S-continuous.*

The proof of Theorem 2.10 also uses the "downward" and the "upward" transfer.

Theorem 2.11. *A function* $f : A \to \mathbb{R}$ *is continuous if and only if* ${}^*f(x) \approx {}^*f(y)$ *for all* $x, y \in \mathrm{NS}({}^*A)$ *such that* $x \approx y$.

Proof. The proof follows from Theorem 2.3. □

Corollary 2.12. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Example 2.13. The function $f(x) = x$ on $\mathbb{R}$ is uniformly continuous. In fact, for any $a \approx b \in {}^*\mathbb{R}$, we have ${}^*f(a) \approx {}^*f(b)$.

The function $g(x) = x^2$ on $\mathbb{R}$ is continuous but not uniformly continuous. To see this, if $a, b \in {}^*\mathbb{R}_{\text{fin}} = \text{NS}({}^*\mathbb{R})$ such that $a \approx b$, then $\mathsf{st}(a) = \mathsf{st}(b) \in \mathbb{R}$ exists. Let $c = \mathsf{st}(a) = \mathsf{st}(b)$ and ϵ_1, ϵ_2 be two infinitesimals such that $a = c + \epsilon_1$ and $b = c + \epsilon_2$. It is easy to see ${}^*g(a) \approx {}^*g(b)$. Hence g is continuous. On the other hand, let y be a positive infinite element in ${}^*\mathbb{R}$. Then $\frac{1}{y}$ is a positive infinitesimal. We have:

$$(y + \frac{1}{y})^2 = y^2 + 2 + \frac{1}{y^2} \not\approx y^2$$

2.4. **Sequence of Functions.** In this section, we consider sequences of functions and different type of convergence of these functions. For $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function and let $f : A \to \mathbb{R}$ be another function.

Definition 2.14. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions converges pointwisely to f if, for every $x \in A$, the sequence $\big(f_n(x)\big)_{n\in\mathbb{N}}$ converges to $f(x)$.

To get a nonstandard characterization of this concept, we first need to consider the nonstandard extension of a sequence of functions: Define the function $F : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by letting $F(n, x) = f_n(x)$. We consider the nonstandard extension ${}^*F : {}^*\mathbb{N} \times {}^*\mathbb{R} \to {}^*\mathbb{R}$. We now define, for $n \in {}^*\mathbb{N}$, ${}^*f_n : {}^*A \to {}^*\mathbb{R}$. By the transfer, one can verify that, for $n \in \mathbb{N}$, this agrees with taking nonstandard extension of f_n. Hence, we have the nonstandard extension $({}^*f_n)_{n\in{}^*\mathbb{N}}$ of the sequence $(f_n)_{n\in\mathbb{N}}$.

Lemma 2.15. *The sequence $(f_n)_{n\in\mathbb{N}}$ converges to f pointwisely if and only if, for all $x \in A$ and all $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, ${}^*f_N(x) \approx f(x)$.*

Definition 2.16. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions converges uniformly to f if, for every $\epsilon > 0$, there exists $m \in \mathbb{N}$, such that, for all $n \geq m$ and all $x \in A$, $|f_n(x) - f(x)| < \epsilon$.

Using "upward" and "downward" transfer, we have:

Theorem 2.17. *The sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly if and only if, for all $x \in {}^*A$ and all $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, ${}^*f_N(x) \approx f(x)$.*

Example 2.18. Consider $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = x^n$. Then $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions, converging pointwisely to $f : [0, 1] \to \mathbb{R}$ where $f(x) = 0$ if $x \neq 1$ and $f(1) = 1$. Such function f is clearly not continuous.

Theorem 2.19. *Suppose $f_n : A \to \mathbb{R}$ is continuous for each $n \in \mathbb{N}$ and that f_n converges to f uniformly. Then f is continuous.*

Proof. Fix $c \in A$ and we shall show that f is continuous at c. Pick some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and a positive $\epsilon \in \text{monad}(0)$. Then there exists some positive $\delta \in \text{monad}(0)$ such that, for all $x \in {}^*A$ with $|x - c| < \delta$, ${}^*f_N(x) \approx {}^*f_N(c)$.

For all $x \in {}^*A$ with $|x - c| < \delta$, we have:

$$|{}^*f(x) - {}^*f(c)| \leq |{}^*f(x) - {}^*f_N(x)| + |{}^*f_N(x) - {}^*f_N(c)| + |{}^*f_N(c) - {}^*f(c)| \approx 0.$$

By Theorem 2.3, f is continuous. □

3. Application of Nonstandard analysis to Economics

Non-trivial applications of nonstandard analysis to other branches of mathematics often involve deeper concepts such as Loeb measure theory. In this section, we consider a rather simple application of nonstandard analysis to economic theory.

3.1. **The Saturation Principle.** We first introduce another important logic property of the nonstandard model:

Definition 3.1. Suppose κ is an uncountable cardinal. A nonstandard extension is κ-saturated if whenever $(A_i)_{i\in I}$ is a family of internal sets with $|I| < \kappa$ satisfying the *finite intersection property*, that is, the intersection of any finite number of A_i's is non-empty, then $\bigcap_{i\in I} A_i \neq \emptyset$.

Example 3.2. Suppose the nonstandard extension ${}^*\mathbb{R}$ is $\aleph_1$-saturated. For each $n \in \mathbb{N}$, let $A_n = \{x \in {}^*\mathbb{R} | x < \frac{1}{n}\}$. The family $(A_n)_{n\in\mathbb{N}}$ has the finite intersection property. Hence, $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ so there is a positive infinitesimal.

The proof of the following (very important) theorem is beyond the scope of this note.

Theorem 3.3. *For any uncountable cardinal κ, there is a κ-saturated non-standard extension.*

Hence, we always work in nonstandard extensions that are *as saturated as needed*. A direct consequence of the saturation principle and Theorem 3.3 is:

- Given any standard infinite set A, then there is a hyperfinite set $B \subset {}^*A$ such that $A \subset B$.

Nonstandard analysis allows us to embed an infinite set into a larger set which possess the same first-order logic properties as finite sets (Magic!).

3.2. **Matching Theory.** In this section, we consider the simple one-to-one "marriage" matching market. Such a market is defined as a quadruplet $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ such that:

- M and W are (possibly infinite) sets of men and women, respectively;

- $\mathcal{P}_M = \{P_m : m \in M\}$ is a set of preferences for the men over the women. For each $m \in M$, its preference P_m is a strict total ordered set of women (each P_m is a subset of W) that is either finite, or is countably infinite (so it specifies man m's n-th choice of woman for every $n \in \mathbb{N}$). Any woman on m's list is preferred by m over being unmatched, while any woman not on m's list is considered unacceptable to m.
- The set $\mathcal{P}_W = \{Q_w : w \in W\}$ of preferences for the women over men is defined similarly.

A *one-to-one matching* between M and W is a collection of pairwise disjoint set of man-woman pairs. A *blocking pair* with respect to a matching μ is a pair (m, w) such that

- If m is matched, then m prefers w to his partner in μ. If m is unmatched, then m prefers w to being unmatched;
- If w is matched, then w prefers m to her partner in μ. If w is unmatched, then w prefers m to being unmatched.

A matching μ is *stable* if

- no participant is matched to someone he or she finds unacceptable;
- there is no blocking pair.

A classical result of Gale and Shapley [GS62] shows that stable matching always exist if M and W are both finite.

Theorem 3.4. *Suppose M and W are finite. Then every one-to-one market $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ has a stable matching.*

We extend Theorem 3.4 to allow for infinite sets of men and women[1] by nonstandard analysis.

[1] In recent years, there has been increasing interest in infinite matching theory. This is to address the so-called *substitutable* assumption in finite matching theory.

Theorem 3.5. *Every one-to-one market has a stable matching.*

Proof. Let $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ be a one-to-one matching market. Let ${}^*\mathcal{P}_M = \{\mathbb{P}_m : m \in {}^*M\}$ and ${}^*\mathcal{P}_W = \{\mathbb{Q}_w : w \in {}^*W\}$. Note that, by the transfer principle, we have $\mathbb{P}_m = {}^*P_m$ and $\mathbb{Q}_w = {}^*Q_w$ for all $m \in M$ and $w \in W$. By saturation, pick hyperfinite sets $\mathcal{M} \subset {}^*M$ and $\mathcal{W} \subset {}^*W$ such that $M \subset \mathcal{M}$ and $W \subset \mathcal{W}$. Let $\mathscr{P}_{\mathcal{M}} = \{\mathbb{P}_m \cap \mathcal{W} : m \in \mathcal{M}\}$ and $\mathscr{P}_{\mathcal{W}} = \{\mathbb{Q}_w \cap \mathcal{M} : w \in \mathcal{W}\}$. By the transfer of Theorem 3.4, the hyperfinite one-to-one market $(\mathcal{M}, \mathcal{W}, \mathscr{P}_{\mathcal{M}}, \mathscr{P}_{\mathcal{W}})$ has a stable matching π. Let μ be a matching in $(M, W, \mathcal{P}_M, \mathcal{P}_W)$ such that:

- For every $m \in M$, if m is matched to some $w \in W$ in π, then m is matched to the same $w \in W$ in μ. If m is matched to some $w \in \mathcal{W} \setminus W$ or is unmatched, then m is unmatched in μ;
- For every $w \in W$, if w is matched to some $m \in M$ in π, then w is matched to the same $m \in M$ in μ. If w is matched to some $m \in \mathcal{M} \setminus M$ or is unmatched, then w is unmatched in μ.

We now show that μ is a stable matching for $(M, W, \mathcal{P}_M, \mathcal{P}_W)$. Clearly, under μ, there is no participant being matched to a partner he or she finds unacceptable. Because otherwise the same participant is matched to someone unacceptable in π, contradict with the fact that π is a stable matching. Suppose there is a blocking pair $(m_0, w_0) \in M \times W$. Then

- Suppose both m_0 and w_0 are matched in μ.Then m_0 and w_0 are matched with the same people in π, which implies that (m_0, w_0) is a blocking pair in π, a contradiction;
- Suppose at least one of m_0 and w_0 is unmatched in μ. Without loss of generality, assume that m_0 is unmatched. Then m_0 is either unmatched or matched with some woman in $\mathcal{W} \setminus W$. By the transfer

principle, m_0 prefers w_0. Hence, (m_0, w_0) is again a blocking pair in π, a contradiction.

Hence, we conclude that μ is a stable matching. □

References

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