Calderón commutators associated with the fractional differentiation

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(Joint works with Yanping Chen and Guixiang Hong)

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Outline

1. Background of the classical Calderón commutator
   - Definition and L^p boundedness of Calderón commutator
   - Application I: Cauchy integral along Lipschitz curve
   - Application II: Dirichlet and Neumann problems on bounded C^1 domain
2. Calderón commutator associated with fractional differential operator
   - Murray's result
   - Our results: generalization of Murray's result
3. Outline of proof
   - Sufficiency for L^p boundedness of $[b, T_\alpha]$
   - Sufficiency for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$
   - Sufficient for the $L^{p,\lambda}$ boundedness of $[b, T_\alpha]$
   - Necessary for $L^{p,\lambda}$ boundedness of $[b, T_\alpha]$
   - Necessary for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$
   - Implicative relationships (I)
   - $T_\alpha = D^\alpha T$ for $0 < \alpha < 1$
   - Implicative relationships (II)
   - Proof of Theorem 1

1.1 Definition

- Denote by H the **Hilbert transform**, which is defined by

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy.$$

In 1965, A. P. Calderón **(Proc. Nat. Acad. Sci.)** introduced the following commutator:

$$[\varphi, \frac{d}{dx}H](f)(x) := \varphi(x)\frac{d}{dx}Hf(x) - \frac{d}{dx}\{H(\varphi f)\}(x),$$

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where $\varphi \in \text{Lip}(\mathbb{R})$.

- By a formal computation,

$$[\varphi, \frac{d}{dx}H](f)(x) = -\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{(x-y)^2}f(y)dy =: -C_\varphi f(x),$$

where C_φ is recalled by **Calderón commutator**.

1.2 Calderón's results: L^p boundedness

Theorem A1 (Calderón, PNAS, 1965)

If $\varphi \in \mathrm{Lip}(\mathbb{R})$, then the commutator C_φ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. In particular, the commutator C_φ is bounded on $L^2(\mathbb{R})$ if and only if $\varphi \in \mathrm{Lip}(\mathbb{R})$.

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- In 1988, T. Murai (**Lecture Notes in Math. 1307**) collected 8 proofs on Calderón commutator C_φ is bounded on $L^2(\mathbb{R})$.

1.3 Cauchy integral along Lipschitz curve

- Let γ be a **Lipschitz curve** on $\mathbb{C}$, that is, γ is the graph of $\varphi \in \mathrm{Lip}(\mathbb{R})$. For $g \in L^p(\gamma)$ $(1 < p < \infty)$, the Cauchy integral of g on γ is defined by

$$F(w) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-w}dz, \quad w \notin \gamma.$$

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$$F(z_0 \pm iy) \to \pm\frac{1}{2} g(z_0) + \frac{1}{2\pi i}\,\mathrm{p.v.} \int_\gamma \frac{g(z)}{z - z_0} dz \qquad \text{as } y \to 0.$$

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- Thus, if

$$\frac{1}{2\pi i}\,\mathrm{p.v.} \int_\gamma \frac{g(z)}{z-z_0} dz < \infty \quad \text{for a.e. } z_0 \in \gamma,$$

then

$$\lim_{y\to 0}[F(z_0+iy) - F(z_0-iy)] = g(z_0) \quad \text{for a.e. } z_0 \in \gamma.$$

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- It is easy to see that the L^p-boundedness of the operator

$$\tilde{C}_\gamma(g)(w) = \frac{1}{2\pi i}\,\mathrm{p.v.}\int_\gamma \frac{g(z)}{z-w}dz \quad (w \in \gamma)$$

is equivalent to the L^p-boundedness of C_γ on $\mathbb{R}$, where C_γ is defined by

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- Calderón **conjectured** the restriction $\|\varphi'\|_\infty \le \varepsilon$ can be removed.

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Theorem A6 (Coifman-McIntosh-Meyer, 1982, Annals of Math.)

C_γ is of weak type (1.1) and bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ and any Lipschitz curve γ in $\mathbb{C}$.

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1.4 Dirichlet and Neumann problems on bounded C^1 domain

- Suppose U is a bounded C^1 domain in $\mathbb{R}^{n+1}$ consider the following **Dirichlet problem** for Δ on U:

$$\begin{cases} \Delta u = 0 & \text{in} \quad U, \\ u|_{\partial U} = f & \text{on} \quad \partial U. \end{cases} \qquad (\mathcal{D})$$

Neumann problem for Δ on U, that is,

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- Using Calderón theorem on Cauchy integral on C^1 curves and method of layer potentials, Fabes, Jodeit and Riviére (**Acta Math., 1978**) gave the uniquely solvability of the Dirichlet problem (D) and Neumann problem (N) with $L^p(\partial U)\,(1 < p < \infty)$ data on C^1 domain. Their techniques rely also on the compactness of the double layer potentials in the C^1 case.

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- B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, **Annals of Math.,** 125, (1987), 437-465.

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Suppose $0 < \alpha < 1$, then the Calderón commutator of fractional order $[b, D^\alpha H]$ is bounded on $L^2(\mathbb{R})$ if and only if $D^\alpha b \in BMO(\mathbb{R})$, i.e., $b \in I_\alpha(BMO)$, where H is the Hilbert transform and I_α denotes the Riesz potential of α order.

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- Theorem B1 can be seen an extension of Theorem A1. However, it needs to point out that $[b, D^\alpha H] \neq [b, \frac{d}{dx} H]$ for $\alpha = 1$.

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- It remains an open problem whether Theorem B1 holds or not for $\alpha = 1$.

2.2 Our results

Theorem 1 (Chen-Ding-Hong, Analysis and PDE, 2016.)

Suppose $\alpha \in (0,1)$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the following five statements are equivalent:

(i) $b \in I_\alpha(BMO)$;

(ii) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded on $L^p(\mathbb{R}^n)$;

(iii) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

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- Here

  R_j: the Riesz transforms $j = 1, \cdots, n$;

  $L^{1,\infty}(\mathbb{R}^n)$: the weak L^1 space;

$$L^{p,\lambda}(\mathbb{R}^n) := \left\{ f : \|f\|_{L^{p,\lambda}} = \left(\sup_{x \in \mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{1/p} < \infty \right\}.$$

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- **Remark 1**: If $\alpha = 0$, then $I_0(BMO) = BMO$ and $[b, D^0 R_j] = [b, R_j]$. In this case, the following equivalents are well known:

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- **Remark 3**: It is not clear whether the conclusions of Theorem 1 hold or not for $\alpha = 1$.

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  (ii) $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$;
  (iii) $\Omega \in L^1(S^{n-1})$.
  Then for $0 \le \alpha \le 1$, the commutator associated with b, Ω, α is defined by

$$[b, T_\alpha]f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}} \, (b(x) - b(y))f(y)dy.$$

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  (iii) $\Omega \in L^1(S^{n-1})$.
  Then for $0 \le \alpha \le 1$, the commutator associated with b, Ω, α is defined by

$$[b, T_\alpha]f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}}\,(b(x) - b(y))f(y)dy.$$

- If $\alpha = 0$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$, by Coifman-Rocherberg-Weiss (**Annals Math., 1976**), $[b, T_0]$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$.

3.1 Sufficiency for L^p boundedness of $[b, T_\alpha]$

Theorem 2

Suppose $\alpha \in (0,1)$ and $b \in I_\alpha(BMO)$. If $\Omega \in L\log^+ L(S^{n-1})$ with mean zero on S^{n-1}, then for $1 < p < \infty$, $\|[b, T_\alpha]f\|_{L^p} \lesssim \|D^\alpha b\|_{BMO}\|f\|_{L^p}$.

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- Proof of Theorem 2: **Littlewoof-Paley decomposition + Fourier transform estimates**.

3.2 Sufficiency for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

- Note that when $b \in I_\alpha(BMO)$ for $0 < \alpha \leq 1$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ with mean zero on S^{n-1}, it is easy to check that the kernel

$$k(x, y) = \frac{\Omega(x - y)}{|x - y|^{n+\alpha}}(b(x) - b(y))$$

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- Hence, by Theorem 2 and the C-Z singular integral theory, we see that $[b, T_\alpha]$ for $0 < \alpha < 1$ is of weak type (1,1).

3.2 Sufficiency for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

- On the other hand, if $\alpha = 1$, the commutator $[b, T_1]$ was defined by Calderón in 1965.

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Theorem C1 (**Calderón, PNAS, 1965**)

If $\Omega \in L\log^+ L(S^{n-1})$ *is odd and satisfies*

$$\int_{S^{n-1}} \Omega(x')x'_j \, d\sigma(x') = 0, \quad j = 1, 2, \cdots, n \tag{3.1}$$

and $\nabla b \in L^r(\mathbb{R}^n)$ $(1 < r \leq \infty)$*. Then for* $1 < p < \infty$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$,

$$\|[b, T_1]f\|_{L^q(\mathbb{R}^n)} \lesssim \|\nabla b\|_{L^r(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

3.2 Sufficiency for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

- Note that if $\Omega \in \mathrm{Lip}(S^{n-1})$ is odd and satisfies (3.1), then the kernel

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Corollary 3

(i) If $b \in \mathrm{Lip}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ is odd and satisfies (3.1), then $[b, T_1]$ is of weak type (1,1).

(ii) If $b \in I_\alpha(BMO)$ for $0 < \alpha < 1$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ with mean zero on S^{n-1}, then $[b, T_\alpha]$ for $0 < \alpha < 1$ is of weak type (1,1).

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- By the way, the conclusion (i) in Corollary 3 has been improved by Ding and Lai (to appear in Trans. Amer. Math. Soc.)

3.3 Sufficient condition of $L^{p,\lambda}$ boundedness

- To get the Morrey space $L^{p,\lambda}$ boundedness of $[b, T_\alpha]$, we need to use an implying relationship.

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Theorem C2 (Chen-Ding-Wang, Canad. J. Math., 2012)

Suppose $\Omega \in L^q(S^{n-1})$ for $q > n/(n-\lambda)$ and $\mathcal{S}$ is a sublinear operator satisfying

$$|\mathcal{S}f(x)| \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy.$$

Let $1 < p < \infty$. If the operator $\mathcal{S}$ is bounded on $L^p(\mathbb{R}^n)$, then $\mathcal{S}$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.

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- Thus, applying Theorem 2 and Theorem C2, we have

Corollary 4

Let $0 < \lambda < n$. Suppose $\alpha \in (0,1)$ and $b \in I_\alpha(BMO)$. If $\Omega \in L^q(S^{n-1})$ for $q > n/(n-\lambda)$, then for $1 < p < \infty$,

$$\|[b, T_\alpha]f\|_{L^{p,\lambda}} \lesssim \|D^\alpha b\|_{BMO}\|f\|_{L^{p,\lambda}}.$$

3.4 Necessary for $L^{p,\lambda}$ boundedness of $[b, T_\alpha]$

- We gave a necessary condition for $L^{p,\lambda}$ boundedness of $[b, T_\alpha]$.

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Theorem 5

Suppose $0 < \alpha \le 1$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\Omega \in {\rm Lip}(S^{n-1})$ satisfying mean zero on S^{n-1} or (3.1). If for some $1 < p < \infty$ and $0 \le \lambda < n$, $[b, T_\alpha]$ is a bounded on $L^{p,\lambda}(\mathbb{R}^n)$, then $b \in {\rm Lip}_\alpha(\mathbb{R}^n)$.

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- In particular, if $[b, T_\alpha]$ is a bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

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- In particular, if $[b, T_\alpha]$ is a bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $b \in {\rm Lip}_\alpha(\mathbb{R}^n)$.
- In the proof of Theorem 5, we used the following equivalent, which was given by N. Meyers in [**PAMS, 1964**]:

$$b \in {\rm Lip}_\alpha(\mathbb{R}^n) \Longleftrightarrow \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^{1+\frac{\alpha}{n}}} \int_Q |b(x) - b_Q| dx \le C.$$

3.5 Necessary for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

Theorem 6

Suppose $0 < \alpha \leq 1$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Omega \in \text{Lip}(S^{n-1})$ satisfying mean zero on S^{n-1} or (3.1). If $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then $b \in \text{Lip}_\alpha(\mathbb{R}^n)$.

3.5 Necessary for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

Theorem 6

Suppose $0 < \alpha \leq 1$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Omega \in \text{Lip}(S^{n-1})$ satisfying mean zero on S^{n-1} or (3.1). If $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then $b \in \text{Lip}_\alpha(\mathbb{R}^n)$.

- As far as we know, this is the first time to give a necessary condition for the $L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ boundedness of an operator.

3.5 Necessary for $L^{1,\infty}$ boundedness of $[b, T_\alpha]$

- Applying Theorem C1, Theorem C2, Corollary 3, Theorems 5 and 6 for $\alpha = 1$, we give the characterizations for the Calderón commutator $[b, T_1]$.

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Corollary 7

Let $1 < p < \infty$, $0 < \lambda < n$. Suppose that $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\Omega \in {\rm Lip}(S^{n-1})$ is odd and satisfying (3.1), then the following four statements are equivalent:

(i) $b \in {\rm Lip}(\mathbb{R}^n)$;

(ii) $[b, T_1]$ is bounded on $L^p(\mathbb{R}^n)$;

(iii) $[b, T_1]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, T_1]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.

3.6 Implicative relationships (I)

- For $0 < \alpha < 1$, there are the following implicative relationships between boundedness of $[b, T_\alpha]$.

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Theorem 8

Suppose $0 < \alpha < 1$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Omega \in \text{Lip}(S^{n-1})$ satisfying mean value zero property. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the implicative relationships (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold for the following four statements:

(i) $[b, T_\alpha]$ is bounded on $L^p(\mathbb{R}^n)$;

(ii) $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$; (by Theorem 6, $b \in \text{Lip}_\alpha(\mathbb{R}^n)$)

(iii) $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$; (by Theorem 5, $b \in \text{Lip}_\alpha(\mathbb{R}^n)$)

(iv) $[b, T_\alpha]$ is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

3.7 $T_\alpha = D^\alpha T$ for $0 < \alpha < 1$

- We now show that $T_\alpha = D^\alpha T$ for $0 < \alpha < 1$, where

$$T_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}} f(y)dy, \quad 0 < \alpha < 1, \tag{3.2}$$

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\widetilde{\Omega}(x-y)}{|x-y|^{n}} f(y)dy, \tag{3.3}$$

Here both Ω and $\widetilde{\Omega}$ are homogeneous of degree zero and with mean value zero.

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Here both Ω and $\widetilde{\Omega}$ are homogeneous of degree zero and with mean value zero.

Proposition 9

(i) For $0 < \alpha < 1$ and $\Omega \in L^2(S^{n-1})$, there exists a singular integral operator T defined by (3.3) with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ such that $T_\alpha = D^\alpha T$.

(ii) Conversely, for any singular integral operator T with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$, there exists an operator T_α defined by (3.2) with $\Omega \in L^2(S^{n-1})$ such that $T_\alpha = D^\alpha T$.

3.7 $T_\alpha = D^\alpha T$ for $0 < \alpha < 1$

- Denote by $\mathcal{H}_m$ the spaces of spherical harmonics of degree m and $\{Y_{m,j}\}_{j=1}^{d_m}$ denotes the normalized orthonormal basis of $\mathcal{H}_m$. Then using the spherical harmonic decomposition,

$$L^2(S^{n-1}) = \Big\{\Omega :\ \Omega(x') = \sum_{m\geq 1}\sum_{j=1}^{d_m} a_{m,j}Y_{m,j}(x'),\ \sum_{m\geq 1}\sum_{j=1}^{d_m} a_{m,j}^2 < \infty.\Big\}$$

and for $0 < \alpha < 1$,

$$L^2_\alpha(S^{n-1}) = \Big\{\Omega :\ \Omega(x') = \textstyle\sum_{m\geq 1}\sum_{j=1}^{d_m} b_{m,j}Y_{m,j}(x'), \\ \textstyle\sum_{m\geq 1}\sum_{j=1}^{d_m}(m^\alpha b_{m,j})^2 < \infty.\Big\}.$$

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- Proof of Proposition 9: **Fourier transform estimate of spherical harmonic functions and Riesz potential.**

3.8 Implicative relationships (II)

- The following implicative relationships between boundedness of $[b, D^\alpha T]$ is an immediate consequence of Theorem 8 and Proposition 9.

3.8 Implicative relationships (II)

- The following implicative relationships between boundedness of $[b, D^\alpha T]$ is an immediate consequence of Theorem 8 and Proposition 9.

Corollary 10

Suppose $0 < \alpha < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\widetilde{\Omega} \in C^2(S^{n-1})$ satisfying mean value zero property. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the implicative relationships $\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)}$ hold for the following four statements:

(i) $[b, D^\alpha T]$ is bounded on $L^p(\mathbb{R}^n)$;

(ii) $[b, D^\alpha T]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $[b, D^\alpha T]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha T]$ is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

3.9 Proof of Theorem 1

- Finally, applying Corollary 10 to Riesz transforms, we get the conclusion of Theorem 1.

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Suppose $0 < \alpha < 1$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the following five statements are equivalent:

(i) $b \in I_\alpha(BMO)$;

(ii) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded on $L^p(\mathbb{R}^n)$;

(iii) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iv) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded on $L^{p,\lambda}(\mathbb{R}^n)$;

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(v) For $j = 1, \cdots, n$, $[b, D^\alpha R_j]$ are bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

- In fact, let $\widetilde{\Omega}_j(x) = \frac{x_j}{|x|}$ for $j = 1, \cdots, n$, then we see that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) hold by Corollary 10. So, it remains to show that (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

3.9 Proof of Theorem 1

- Note that for $j = 1, 2, \cdots, n$, $\widehat{D^\alpha R_j f}(\xi) = -i\xi_j|\xi|^{\alpha-1}\widehat{f}(\xi)$ and

$$\eta(\alpha)\left(\text{p.v.}\,\frac{x_j}{|x|^{n+1+\alpha}}\right)^{\wedge}(\xi) = i\xi_j|\xi|^{\alpha-1},$$

where $\eta(\alpha) = \dfrac{1-n-\alpha}{2\pi}\,\dfrac{\Gamma(\frac{n+\alpha-1}{2})}{\pi^{\frac{n}{2}+\alpha-1}\Gamma(\frac{1-\alpha}{2})}$. Hence we get

$$[b, D^\alpha R_j]f(x) = \text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega_j(x-y)}{|x-y|^{n+\alpha}}(b(x)-b(y))f(y)\,dy,$$

where $\Omega_j(x) = \eta(\alpha)\frac{x_j}{|x|}$.

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$$[b, D^\alpha R_j]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega_j(x-y)}{|x-y|^{n+\alpha}}(b(x)-b(y))f(y)\,dy,$$

where $\Omega_j(x) = \eta(\alpha)\frac{x_j}{|x|}$.

- If $b \in I_\alpha(BMO)$, then by Theorem 2,

$$\|[b, D^\alpha R_j]\|_{L^p} \le C\|D^\alpha b\|_{BMO}\|f\|_{L^p}$$

for $j = 1, 2, \cdots, n$ and $1 < p < \infty$. Thus we show that $(\mathrm{i}) \Rightarrow (\mathrm{ii})$.

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- Using the relationship between BMO function and Carleson measure, Fefferman-Stein (**Acta Math.**, 1972) showed that

$$\sum_{j=1}^{n} R_j^2 f \in BMO \Longrightarrow f \in BMO. \qquad (*)$$

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$$\sum_{j=1}^{n} R_j^2 f \in BMO \Longrightarrow f \in BMO. \qquad (*)$$

- By (v), $[b, D^\alpha R_j] : L^\infty \to BMO$, the vanishing moment of Ω_j gives

$$[b, D^\alpha R_j](1)(x) = -D^\alpha R_j b(x) = -R_j D^\alpha(b)(x) \in BMO, \text{for } j = 1, 2, \cdots, n.$$

Hence, $-\sum_{j=1}^{n} R_j^2 D^\alpha(b) \in BMO$. By $(*)$, $D^\alpha(b) \in BMO$, so (i) holds.

Many thanks for your attention!