Bilinear Riesz means on the Heisenberg group

Liu Heping
(joint work with Wang Min)

School of Mathematical Sciences, Peking University

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Abstract

We investigate the bilinear Riesz means S^α associated to the sublaplacian on the Heisenberg group. The operator S^α is bounded from $L^{p_1} \times L^{p_2}$ into L^p for $1 \le p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2$ when α is large than a suitable smoothness index $\alpha(p_1, p_2)$.

(Heping Liu, Min Wang, Bilinear Riesz means on the Heisenberg group, arXiv:1712.09238V1.)
(Heping Liu, Min Wang, Boundedness of the bilinear Bochner-Riesz means in the non-Banach triangular case, arXiv:1712.09235V1.)

Bilinear Bochner-Riesz means

The bilinear Bochner-Riesz means problem originates from the study of the summability of the product of two n-dimensional Fourier series. This leads to the study of the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the bilinear Bochner-Riesz multiplier

$$B^\alpha(f,g)(x) = \int\int_{|\xi|^2+|\eta|^2\leq 1}(1-|\xi|^2-|\eta|^2)^\alpha \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

Bernicot et al. gave a comprehensive study on the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the operator B^α.

(F. Bernicot, L. Grafakos, L. Song, L. Yan, The bilinear Bochner-Riesz problem, J. Anal. Math. 127(2015), 179–217.)

Heisenberg group

Heisenberg group $\mathbb{H}^n$:

Underlying manifold: $\mathbb{C}^n \times \mathbb{R}$.

Multiplication: $(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\mathrm{Im}(z\cdot\overline{w}))$.

Non-isotropic dilations: $\delta_r(z,t) = (rz, r^2 t)$.

Homogeneous norm: $|x| = \left(\frac{1}{16}|z|^4 + t^2\right)^{\frac{1}{4}}$.

Homogeneous dimension: $Q = 2n+2$.

Convolution: $(f*g)(x) = \int_{\mathbb{H}^n} f(xy^{-1})g(y)\,dy$.

Plancherel formula

Fourier transform of f in central variable t:

$$f^{\lambda}(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)\, dt.$$

λ-twisted convolution:

$$f *_{\lambda} g = \int_{\mathbb{C}^n} f(z-\omega) g(\omega) e^{\frac{i}{2}\lambda \mathrm{Im}(z\cdot\overline{\omega})}\, d\omega.$$

Laguerre polynomials L_k^{μ} of type μ:

$$L_k^{\mu}(t) e^{-t} t^{\mu} = \frac{1}{k!} \left(\frac{d}{dt}\right)^k (e^{-t} t^{k+\mu}).$$

Plancherel formula

Set

$$\varphi_k(z) = L_k^{n-1}\big(\frac{1}{2}\,|z|^2\,\big)\,e^{-\frac{1}{4}|z|^2},$$

$$e_k^{\lambda}(z,t) = e^{-i\lambda t}\varphi_k^{\lambda}(z) = e^{-i\lambda t}\varphi_k(\sqrt{|\lambda|}z).$$

$$\widetilde{e}_k^{\lambda}(z,t) = e_k^{\frac{\lambda}{2k+n}}(z,t),$$

Plancherel formula:

$$\|f\|_2^2 = (2\pi)^{-2n-1}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{C}^n}\Big|f^{\lambda} *_{\lambda} \varphi_k^{\lambda}(z)\Big|^2\,\lambda^{2n}\,dzd\lambda.$$

Plancherel formula

Inversion formula:

$$
\begin{aligned}
f(z,t) &= \sum_{k=0}^{\infty}\int_{-\infty}^{\infty} f * e_k^{\lambda}(z,t)\, d\mu(\lambda) \\
&= \int_{-\infty}^{\infty}\sum_{k=0}^{\infty}(2k+n)^{-n-1} f * \widetilde{e}_k^{\lambda}(z,t)\, d\mu(\lambda) \\
&= \int_{0}^{\infty} P_{\lambda} f(z,t)\, d\mu(\lambda).
\end{aligned}
$$

where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$ and

$$
P_{\lambda} f(z,t) = \sum_{k=0}^{\infty}(2k+n)^{-n-1} f * (\widetilde{e}_k^{\lambda} + \widetilde{e}_k^{-\lambda})(z,t).
$$

Sublaplacian

left invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n,$$
$$Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n.$$

Sublaplacian: $\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$.

Up to a constant multiple, $\mathcal{L}$ is the unique left invariant, rotation invariant differential operator that is homogeneous of degree two. Therefore, it is regarded as the counterpart of the Laplacian on $\mathbb{R}^n$. The sublaplacian $\mathcal{L}$ is a positive and essentially self-adjoint operator.

Bilinear Riesz means

Spectral decomposition: $\mathcal{L}f = \int_0^\infty \lambda P_\lambda f \, d\mu(\lambda)$.

Bilinear Riesz means:

$$\begin{aligned} S_R^\alpha(f,g) &= \int_0^\infty \int_0^\infty \left(1 - \frac{\lambda_1 + \lambda_2}{R}\right)_+^\alpha P_{\lambda_1} f \, P_{\lambda_2} g \, d\mu(\lambda_1) d\mu(\lambda_2) \\ &= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} f(x\omega_1^{-1}) g(x\omega_2^{-1}) S_R^\alpha(\omega_1, \omega_2) \, d\omega_1 d\omega_2, \end{aligned}$$

with the kernel

$$\begin{aligned} & S_R^\alpha\big((z_1,t_1),(z_2,t_2)\big) \\ = & \sum_{k=0}^\infty \sum_{l=0}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \left(1 - \frac{(2k+n)\,|\lambda_1| + (2l+n)\,|\lambda_2|}{R}\right)_+^\alpha \\ & \times e_k^{\lambda_1}(z_1,t_1) e_l^{\lambda_2}(z_2,t_2) \, d\mu(\lambda_1) d\mu(\lambda_2). \end{aligned}$$

Bilinear Riesz means

$$S_R^\alpha((z_1,t_1),(z_2,t_2)) = R^Q S_1^\alpha((\sqrt{R}z_1, Rt_1),(\sqrt{R}z_2, Rt_2)).$$

Because $1/p = 1/p_1 + 1/p_2$, the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of S_R^α is equivalent to the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of $S^\alpha := S_1^\alpha$.

We will concentrate on the boundedness of S^α.

Pointwise estimate for the kernel

Theorem: *If $\alpha > 4m - 1$ where m is a positive integer, then for any $\omega_1 = (z_1, t_1), \omega_2 = (z_2, t_2) \in \mathbb{H}^n$,*

$$\left|S^\alpha(\omega_1, \omega_2)\right| \leq C(1 + |\omega_1|)^{-2m}(1 + |\omega_2|)^{-2m}.$$

Corollary: *Let $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$. If $\alpha > 2Q + 3$, then S^α is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$.*

Here we have seen the first significant difference between the Euclidean space and the Heisenberg group: The pointwise estimate of the kernel on the Heisenberg group is very worse than the Euclidean space. The same thing occurs even for the linear counterparts.

Pointwise estimate for the kernel

The kernel $B^\alpha(x_1, x_2)$ of the bilinear Bochner-Riesz operator on $\mathbb{R}^n$ coincides with the kernel of the Bochner-Riesz operator on $\mathbb{R}^{2n}$. The pointwise estimate of this kernel is well known.

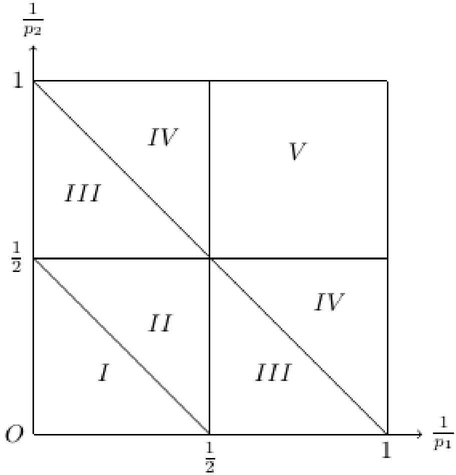
$$B^\alpha(x_1, x_2) = \frac{\Gamma(1+\alpha)}{\pi^\alpha \big|(x_1, x_2)\big|^{n+\alpha}} J_{n+\alpha}\big(2\pi\big|(x_1, x_2)\big|\big).$$

The Bessel function J_k satisfies the asymptotic estimate

$$J_k(r) = O\big(r^{-\frac{1}{2}}\big), \quad r \to \infty.$$

As a consequence, a prior result holds : B^α is bounded from $L^{p_1} \times L^{p_2}$ into L^p when $\alpha > n - \frac{1}{2}$, which is optimal in case of $(p_1, p_2) = (1, 1)$.

Boundedness of bilinear Riesz means



Boundedness of bilinear Riesz means

$$
\begin{aligned}
\alpha(1,1) &= Q, \\
\alpha(2,2) &= 0, \\
\alpha(\infty,\infty) &= Q - \frac{1}{2}, \\
\alpha(1,\frac{1}{2}) &= \frac{Q}{2}, \\
\alpha(1,\infty) &= \frac{Q}{2}, \\
\alpha(2,\infty) &= \frac{Q-1}{2}.
\end{aligned}
$$

Boundedness of bilinear Riesz means

Our full results are summarized in the following theorem.

Main Theorem: *Let* $1 \leq p_1, p_2 \leq \infty$ *and* $1/p = 1/p_1 + 1/p_2$.

(1) (region I) For $2 \leq p_1, p_2 \leq \infty$ *and* $p \geq 2$*, if* $\alpha > Q\left(1 - \frac{1}{p}\right) - \frac{1}{2}$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*.*

(2) (region II) For $2 \leq p_1, p_2 \leq \infty$ *and* $1 \leq p \leq 2$*, if* $\alpha > (Q-1)\left(1 - \frac{1}{p}\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*.*

Boundedness of bilinear Riesz means

(3) (region III) For $1 \le p_1 \le 2 \le p_2 \le \infty$ *and* $p \ge 1$*, if* $\alpha > Q\left(\frac{1}{2} - \frac{1}{p_2}\right) - \left(1 - \frac{1}{p}\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*; For* $1 \le p_2 \le 2 \le p_1 \le \infty$ *and* $p \ge 1$*, if* $\alpha > Q\left(\frac{1}{2} - \frac{1}{p_1}\right) - \left(1 - \frac{1}{p}\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*.*

(4) (region IV) For $1 \le p_1 \le 2 \le p_2 \le \infty$ *and* $p \le 1$ *, if* $\alpha > Q\left(\frac{1}{p_1} - \frac{1}{2}\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*; For* $1 \le p_2 \le 2 \le p_1 \le \infty$ *and* $p \le 1$*, if* $\alpha > Q\left(\frac{1}{p_2} - \frac{1}{2}\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*.*

(5) (region V) For $1 \le p_1, p_2 \le 2$*, if* $\alpha > Q\left(\frac{1}{p} - 1\right)$*, then* S^α *is bounded from* $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$*.*

Comparison of mathods

On Euclidean space, the Fourier transform of the product of two functions is the convolution of Fourier transforms of two functions because the dual of the Euclidean space is itself. But this convenience false on the Heisenberg group. This is the second significant difference between the Euclidean space and the Heisenberg group. We illuminate this point by comparing two different ways to obtain the estimate in case of $(p_1, p_2) = (2, \infty)$.

Comparison of methods

We choose a nonnegative function $\varphi \in C_0^\infty(\frac{1}{2}, 2)$ satisfying $\sum_{-\infty}^{\infty} \varphi(2^j s) = 1$. Set $B^\alpha = \sum_{j=0}^{\infty} B_j^\alpha$ where B_j^α corresponds to the multiplier $m_j^\alpha(\xi, \eta) = (1 - |\xi|^2 - |\eta|^2)_+^\alpha \varphi\big(2^j(1 - |\xi|^2 - |\eta|^2)\big)$. Let $B_j(y, s) = \{x : |x - y| < s2^{j(1+\gamma)}\}$ with $\gamma > 0$ small enough. The key is to prove

$$\left\| B_j^\alpha(f, g) \right\|_2^2 \leq C2^{-\varepsilon j} \|f\|_2^2 \|g\|_\infty^2 ,$$

when $\text{Supp} f, \text{Supp} g \subset B_j(y, \frac{3}{4})$.

Comparison of methods

$$B_j^\alpha(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_j^\alpha(\xi-\eta,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)e^{2\pi i x\cdot\xi}\,d\eta d\xi.$$

$$\begin{aligned}
& \left\|B_j^\alpha(f,g)\right\|_2^2 \\
= & \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}(m_j^\alpha(\xi-\eta,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta\right|^2 d\xi \\
\leq & \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|(m_j^\alpha(\xi-\eta,\eta)\right|^2 d\eta\right)\left(\int_{\mathbb{R}^n}\left|\widehat{f}(\xi-\eta)\right|^2\left|\widehat{g}(\eta)\right|^2 d\eta\right) d\xi \\
\leq & C2^{-(2\alpha+1)j}\left\|f\right\|_2^2\left\|g\right\|_2^2 \\
\leq & C2^{-(2\alpha+1)j+(1+\gamma)Qj}\left\|f\right\|_2^2\left\|g\right\|_\infty^2.
\end{aligned}$$

Thus $\alpha > \frac{n-1}{2}$ is only needed.

Comparison of methods

Let us see how to get the same estimate on the Heisenberg group.

In the same way, $S^\alpha = \sum_{j=0}^{\infty} S_j^\alpha$ where S_j^α corresponds to the multiplier $\varphi_j^\alpha(\lambda_1, \lambda_2) = (1 - |\lambda_1| - |\lambda_2|)_+^\alpha \varphi\big(2^j(1 - |\lambda_1| - |\lambda_2|)\big)$.
Let $B_j(\xi, s) = \{\eta : |\xi^{-1}\eta| < s2^{j(1+\gamma)}\}$.
We want to prove

$$\left\| S_j^\alpha(f, g) \right\|_2^2 \leq C 2^{-\varepsilon j} \left\| f \right\|_2^2 \left\| g \right\|_\infty^2 ,$$

when $\text{Supp} f, \text{Supp} g \subset B_j(\xi, \frac{3}{4})$ provided $\alpha > \frac{Q-1}{2}$.

Comparison of methods

$$
\begin{aligned}
& S_j^\alpha(f,g)(z,t) \\
= \ & \int_0^\infty \int_0^\infty \varphi_j^\alpha(\lambda_1,\lambda_2) \sum_{k=0}^\infty (2k+n)^{-n-1} f * (\widetilde{e}_k^{\lambda_1} + \widetilde{e}_k^{-\lambda_1})(z,t) \\
& \times \sum_{l=0}^\infty (2l+n)^{-n-1} g * (\widetilde{e}_l^{\lambda_2} + \widetilde{e}_l^{-\lambda_2})(z,t)\, d\mu(\lambda_1) d\mu(\lambda_2) \\
= \ & C \int_{-\infty}^\infty e^{-i\lambda_1 t} \int_{-\infty}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \varphi_j^\alpha\left((2k+n)\left|\lambda_1-\lambda_2\right|, (2l+n)\left|\lambda_2\right|\right) \\
& \times \left(f^{\lambda_1-\lambda_2} *_{\lambda_1-\lambda_2} \varphi_k^{\lambda_1-\lambda_2}\right)(z) \left(g^{\lambda_2} *_{\lambda_2} \varphi_l^{\lambda_2}\right)(z) \\
& \times \left|\lambda_1-\lambda_2\right|^n \left|\lambda_2\right|^n \, d\lambda_1 d\lambda_2.
\end{aligned}
$$

Comparison of methods

$$\int_{\mathbb{C}^n}\int_{\mathbb{R}}\left|S_j^\alpha(f,g)(z,t)\right|^2 dtdz$$

$$= C\int_{\mathbb{R}}\int_{\mathbb{C}^n}\Bigg|\int_{\mathbb{R}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\varphi_j^\alpha\left((2k+n)\left|\lambda_1-\lambda_2\right|,(2l+n)\left|\lambda_2\right|\right)$$

$$\times\left(f^{\lambda_1-\lambda_2}*_{\lambda_1-\lambda_2}\varphi_k^{\lambda_1-\lambda_2}\right)(z)\left(g^{\lambda_2}*_{\lambda_2}\varphi_l^{\lambda_2}\right)(z)$$

$$\times\left|\lambda_1-\lambda_2\right|^n\left|\lambda_2\right|^n d\lambda_2\Bigg|^2 dzd\lambda_1$$

Comparison of methods

$$\leq \quad C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sum_{l=0}^{\infty} \left\| \sum_{k=0}^{\infty} \varphi_j^{\alpha}\left((2k+n)\left|\lambda_1-\lambda_2\right|, (2l+n)\left|\lambda_2\right| \right) \right.\right.$$
$$\left.\times \left(f^{\lambda_1-\lambda_2} *_{\lambda_1-\lambda_2} \varphi_k^{\lambda_1-\lambda_2} \right) \right\|_2 \left\| g^{\lambda_2} *_{\lambda_2} \varphi_l^{\lambda_2} \right\|_{\infty}$$
$$\left. \times \left|\lambda_1-\lambda_2\right|^n \left|\lambda_2\right|^n \, d\lambda_2 \right)^2 d\lambda_1$$

Comparison of methods

$$
\begin{aligned}
\leq \quad & C\left(\int_{|\lambda_2|\leq 1}\sum_{l\leq\frac{1}{|\lambda_2|}}\left\|g^{\lambda_2}\right\|_2^2\left\|\varphi_l^{\lambda_2}\right\|_2^2|\lambda_2|^{2n-\delta}\,d\lambda_2\right)\\
& \times\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left|\varphi_j^{\alpha}\left((2k+n)\left|\lambda_1-\lambda_2\right|,(2l+n)\left|\lambda_2\right|\right)\right|^2\right.\\
& \left.\times\left\|f^{\lambda_1-\lambda_2}*_{\lambda_1-\lambda_2}\varphi_k^{\lambda_1-\lambda_2}\right\|_2^2|\lambda_1-\lambda_2|^{2n}|\lambda_2|^{\delta}\,d\lambda_2 d\lambda_1\right)
\end{aligned}
$$

Comparison of methods

$$\begin{aligned}
\leq \quad & C\left(\int_{|\lambda_2|\leq 1}\left\|g^{\lambda_2}\right\|_2^2 |\lambda_2|^{n-\delta}\left(\sum_{l\leq \frac{1}{|\lambda_2|}} l^{n-1}\right) d\lambda_2\right) \\
& \times\left(\sum_{k=0}^{\infty}\int_{\mathbb{R}}\left\|f^{\lambda_1} *_{\lambda_1} \varphi_k^{\lambda_1}\right\|_2^2 |\lambda_1|^{2n}\left(\sum_{l=0}^{\infty}(2l+n)^{-1-\delta}\right)\right. \\
& \left.\times\int_{\mathbb{R}}\left|\varphi_j^{\alpha}\left((2k+n)\,|\lambda_1|\,,|\lambda_2|\,\right)\right|^2 |\lambda_2|^{\delta}\, d\lambda_2 d\lambda_1\right) \\
\leq \quad & C2^{-j(2\alpha+1)}\,\|f\|_2^2\int_{|\lambda_2|\leq 1}\left\|g^{\lambda_2}\right\|_2^2 |\lambda_2|^{-\delta}\, d\lambda_2.
\end{aligned}$$

Comparison of methods

$$
\begin{aligned}
& \int_{|\lambda_2|\le 1} \left\| g^{\lambda_2} \right\|_2^2 |\lambda_2|^{-\delta} \, d\lambda_2 \\
= & \int_{2^{-2j(1+\gamma)} \le |\lambda_2| \le 1} \left\| g^{\lambda_1} \right\|_2^2 |\lambda_2|^{-\delta} \, d\lambda_2 \\
& + \int_{|\lambda_2| \le 2^{-2j(1+\gamma)}} \left\| g^{\lambda_2} \right\|_2^2 |\lambda_1|^{-\delta} \, d\lambda_2 \\
\le & \; 2^{2\delta j(1+\gamma)} \|g\|_2^2 + C 2^{j(1+\gamma)(Q+2)} \|g\|_\infty^2 \int_{|\lambda_2| \le 2^{-2j(1+\gamma)}} |\lambda_2|^{-\delta} \, d\lambda_2 \\
\le & \; C 2^{j(1+\gamma)(Q+2\delta)} \|g\|_\infty^2 .
\end{aligned}
$$

Comparison of methods

Thus

$$\left\| S_j^{\alpha}(f,g) \right\|_2^2 \leq C 2^{-j(2\alpha+1)} 2^{j(1+\gamma)(Q+2\delta)} \|f\|_2^2 \|g\|_\infty^2 ,$$

and $\alpha > \frac{Q-1}{2}$ is enough.

Non-Banach triangle

Our results in the Banach triangle case corresponds to the results of Bernicot et al. Our treatment in the non-Banach triangle case applies to the Euclidean space and improves the results of Bernicot et al in two aspects: Our partition of the non-Banach triangle is simpler and we obtain lower smoothness indices $\alpha(p_1, p_2)$ for various cases apart from $1 \leq p_1 = p_2 < 2$.

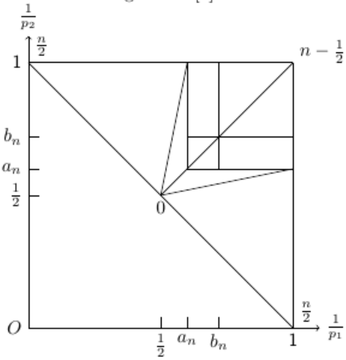
Our results on in the non-Banach triangle case the Euclidean space are summarized in the following theorem. The pictures also display the comparison of two partitions.

Non-Banach triangle

Theorem: *Assume that $n \geq 2$. Let $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$.*
(1)(region I) For $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$ and $p < 1$, if $\alpha > n\big(\frac{1}{p_1} - \frac{1}{2}\big)$, then B^α is bounded from $L^{p_1} \times L^{p_2}$ to L^p; For $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$ and $p < 1$, if $\alpha > n\big(\frac{1}{p_2} - \frac{1}{2}\big)$, then B^α is bounded from $L^{p_1} \times L^{p_2}$ to L^p.
(2)(region II) For $1 \leq p_1 \leq p_2 \leq 2$, if $\alpha > n\big(\frac{1}{p} - 1\big) - \big(\frac{1}{p_2} - \frac{1}{2}\big)$, then B^α is bounded from $L^{p_1} \times L^{p_2}$ to L^p; For $1 \leq p_2 \leq p_1 \leq 2$, if $\alpha > n\big(\frac{1}{p} - 1\big) - \big(\frac{1}{p_1} - \frac{1}{2}\big)$, then B^α is bounded from $L^{p_1} \times L^{p_2}$ to L^p.

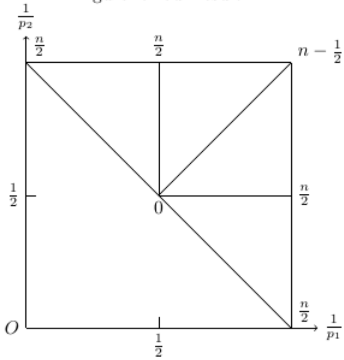
Non-Banach triangle

Figure for [1]



$a_n = \frac{n+1}{2n}$, $b_n = \frac{n+1}{2n} + \frac{n-1}{n^2+n}$

Figure for our result



Thank You !