A New Weak Norm with Applications to Geometric Inequalities

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Fractional Integral:

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad \alpha > 0.$$

I_α : $L^p \mapsto L^q$, where $1/q = 1/p - \alpha/n$.

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i.e., for any $f \in L^p$, $g \in L^{q'}$,

$$\int_{\mathbb{R}^n} I_\alpha f(x) g(x) \lesssim \|f\|_p \|g\|_{q'}.$$

The Hardy-Littlewood-Sobolev inequality: for any $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^n)$, where $1 < p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 > 1$, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n(2-1/p_1-1/p_2)}} dxdy \leq C_{\vec{p},n} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}. \tag{1}$$

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Problem: what will happen if $f(x)g(y)$ is replaced by a general function $h(x,y)$?

Geometric inequality: for any $f \in L^{p_1}$ and $g \in L^{p_2}$.

$$\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \leq C_{\vec{p},n} \sup_{x,y\in\mathbb{R}^n} f(x)g(y)|x-y|^{n/p_1+n/p_2} \tag{2}$$

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Again, what will happen if $f(x)g(y)$ is replaced by a general function $h(x,y)$?

For $\vec{p} = (p_1, \ldots, p_r)$ and a measurable function f defined on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$, where p_i are positive numbers and n_i are positive integers, $1 \le i \le r$, we define the $L^{\vec{p}}$ norm of f by

$$\|f\|_{L^{\vec{p}}} := \Big\| \|f\|_{L^{p_1}_{x_1}} \cdots \Big\|_{L^{p_r}_{x_r}}.$$

The Lebesgure space $L^{\vec{p}}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ with mixed norms consists of all measurable functions f for which $\|f\|_{L^{\vec{p}}} < \infty$.

Define

$$L_\gamma f(x,y) = \frac{f(x,y)}{|x-y|^\gamma}, \ \gamma > 0.$$

For $\gamma = n(2 - 1/p_1 - 1/p_2)$, the Hardy-Littlewood-Sobolev inequality says that

$$\|L_\gamma f \otimes g\|_{L^{\vec{1}}} \lesssim \|f \otimes g\|_{L^{\vec{p}}}, \qquad f \in L^{p_1}(\mathbb{R}^n), g \in L^{p_2}(\mathbb{R}^n).$$

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It is natural to ask if the above inequality is still true whenever $f \otimes g$ is replaced by a general function in $L^{\vec{p}}(\mathbb{R}^n \times \mathbb{R}^n)$? More precisely, do we have

$$\|L_\gamma f\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{p}}}, \qquad \forall f \in L^{\vec{p}}(\mathbb{R}^n \times \mathbb{R}^n)$$

for appropriate $\vec{p}, \vec{q}$ and γ?

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It is natural to ask if the above inequality is still true whenever $f \otimes g$ is replaced by a general function in $L^{\vec{p}}(\mathbb{R}^n \times \mathbb{R}^n)$? More precisely, do we have

$$\|L_\gamma f\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{p}}}, \qquad \forall f \in L^{\vec{p}}(\mathbb{R}^n \times \mathbb{R}^n)$$

for appropriate $\vec{p}, \vec{q}$ and γ? The answer is false in general.

Next we consider another variant of (1). By replacing g with $g(-\cdot)$ and a change of variable, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x+y|^{n(2-1/p_1-1/p_2)}}dxdy \leq C_{\vec{p},n}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Observe that

$$\frac{1}{(|x+y|+|x-y|)^{\gamma}} \leq \frac{1}{|x+y|^{\gamma}}+\frac{1}{|x-y|^{\gamma}}.$$

This prompts us to consider the following operator

$$T_{\gamma}f(x,y)=\frac{f(x,y)}{(|x+y|+|x-y|)^{\gamma}}, \qquad \gamma>0.$$

We see from the Hardy-Littlewood-Sobolev inequality that for $\gamma = n(2 - 1/p_1 - 1/p_2)$,

$$\|T_\gamma f \otimes g\|_{L^{\vec{1}}} \leq \|f \otimes g\|_{L^{\vec{p}}}.$$

We ask if the following inequality

$$\|T_\gamma f\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{p}}}, \qquad \forall f \in L^{\vec{p}}$$

is true for some $\vec{p}$ and $\vec{q}$? The answer is again negative. Moreover, the following inequality

$$\|T_\gamma f\|_{L^{\vec{q},\infty}} \lesssim \|f\|_{L^{\vec{p}}}, \qquad \forall f$$

is also false whenever $\vec{q} \neq (\infty, \infty)$, where

$$\|f\|_{L^{\vec{q},\infty}} := \sup_{\lambda > 0} \lambda \|\chi_{\{|f| > \lambda\}}\|_{L^{\vec{q}}}$$

is the weak $L^{\vec{q}}$ norm of f.

When the weak norm is replaced by the iterated weak norm defined by

$$\|f\|_{L^{(p_r,\infty)}(\cdots(L^{(p_1,\infty)}))} := \left\| \|f\|_{L^{p_1,\infty}_{x_1}} \cdots \right\|_{L^{p_r,\infty}_{x_r}},$$

we get a positive conclusion. Specifically, we have the following.

Theorem

Let f be a nonnegative measurable function defined on $\mathbb{R}^{2n}$.

1. *For all $0 < q_1 \le p_1 \le \infty$ and $0 < q_2 \le p_2 \le \infty$ satisfying the homogeneity condition $1/q_1 + 1/q_2 = 1/p_1 + 1/p_2 + \gamma/n$, we have*

$$\|T_\gamma f\|_{L^{q_2,\infty}(L^{q_1,\infty})} \le C_{\vec{p},\vec{q}n}\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})}. \tag{3}$$

However, neither

$$\|T_\gamma f\|_{L^{\vec{q},\infty}} \le C_{\vec{p},\vec{q},n,\gamma}\|f\|_{L^{\vec{p},\infty}} \tag{4}$$

nor

$$\|T_\gamma f\|_{L^{\vec{q},\infty}} \le C_{\vec{p},\vec{q},n,\gamma}\|f\|_{L^{\vec{p}}} \tag{5}$$

is true in general.

Theorem (Continued)

2. *For all* $0 < p_1 \leq q_1 \leq \infty$ *and* $0 < p_2 \leq q_2 \leq \infty$ *satisfying the homogeneity condition*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{\gamma}{n},$$

we have

$$\| T_\gamma^{-1} f \|_{L^{q_2,\infty}(L^{q_1,\infty})} \geq C_{\vec{p},\vec{q},n} \| f \|_{L^{p_2,\infty}(L^{p_1,\infty})}. \tag{6}$$

Weak Norms

For simplicity, we consider only the case of $r = 2$.

In this case, the iterated weak norm on $\mathbb{R}^n \times \mathbb{R}^m$ is

$$\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})} = \sup_{\gamma>0} \gamma \left| \left\{ y : \sup_{\lambda>0} \lambda |\{x : |f(x,y)| > \lambda\}|^{1/p_1} > \gamma \right\} \right|^{1/p_2}$$
$$= \left\| \sup_{\lambda>0} \lambda |E_{y,\lambda}|^{1/p_1} \right\|_{L^{p_2,\infty}},$$

where

$$E_{y,\lambda} := \{x : |f(x,y)| > \lambda\}. \tag{7}$$

And the mixed weak norm is

$$\|f\|_{L^{\vec{p},\infty}} = \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{L^{\vec{p}}} = \sup_{\lambda>0} \left\| \lambda E_{y,\lambda} \right|^{1/p_1} \Big\|_{L^{p_2}}.$$

Theorem

Suppose that $0 < p_1, p_2 < \infty$ and m and n are positive integers. We have

1. $L^{p_2,\infty}(L^{p_1,\infty})(\mathbb{R}^n \times \mathbb{R}^m) \not\subset L^{\vec{p},\infty}(\mathbb{R}^n \times \mathbb{R}^m)$ *and* $L^{\vec{p},\infty}(\mathbb{R}^n \times \mathbb{R}^m) \not\subset L^{p_2,\infty}(L^{p_1,\infty})(\mathbb{R}^n \times \mathbb{R}^m)$;
2. $L_x^{p_1,\infty}(L_y^{p_2,\infty}) \not\subset L_y^{p_2,\infty}(L_x^{p_1,\infty})$ *and* $L_y^{p_2,\infty}(L_x^{p_1,\infty}) \not\subset L_x^{p_1,\infty}(L_y^{p_2,\infty})$;
3. $L^{\vec{p}} \subsetneq L^{\vec{p},\infty} \bigcap L^{p_2,\infty}(L^{p_1,\infty})$;

Example

1. $F(x,y) := 1/(|x|^{n/p_1}|y|^{m/p_2}) \in L^{p_2,\infty}(L^{p_1,\infty}) \setminus L^{\vec{p},\infty}$

2. $G(x,y) = a^{|y|^m}\chi_{[0,a^{-p_1|y|^m/n}]}(|x|) \in L^{\vec{p},\infty} \setminus L^{p_2,\infty}(L^{p_1,\infty})$, where $a > 1$.

It is easy to see that $f \otimes g(x, y) := f(x)g(y) \in L^{q,\infty}(L^{p,\infty}) \setminus \{0\}$ if and only if $f \in L^{p,\infty}$ and $g \in L^{q,\infty}$. Next we consider the conditions for $f \otimes g \in L^{\vec{p},\infty}$.

Theorem

Suppose that $0 < p, q < \infty$ and m and n are positive integers. We have

1. *If $f \in L^{p_1,\infty}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^m)$, then $f \otimes g \in L^{\vec{p},\infty}(\mathbb{R}^n \times \mathbb{R}^m)$.*
2. *If $f \in L^{p_1}$, $g \in L^{p_2,\infty}$ and $p_1 \le p_2$, then $f \otimes g \in L^{\vec{p},\infty}$.*
3. *If $f \otimes g \in L^{\vec{p},\infty}$ and $f, g \neq 0$, then $f \in L^{p_1,\infty}$ and $g \in L^{p_2,\infty}$. But g need not to be in $g \in L^{p_2}$.*

Given $\vec{p} = (p_1, p_2)$, we compare the three mixed norms $L^{p_2,\infty}(L^{p_1})$, $L^{p_2}(L^{p_1,\infty})$, and $L^{\vec{p},\infty}$.

Theorem

Suppose that $\vec{p} = (p_1, p_2)$. We have

1. *For any measurable function F defined on $\mathbb{R}^n \times \mathbb{R}^m$, we have*

$$\|F\|_{L^{\vec{p},\infty}} \le \|F\|_{L^{p_2}(L^{p_1,\infty})}.$$

2. $L^{p_2,\infty}(L^{p_1}) \not\subset L^{\vec{p},\infty}$ *and* $L^{\vec{p},\infty} \not\subset L^{p_2,\infty}(L^{p_1})$.

It is well known that Hölder's inequality holds for both L^p and $L^{p,\infty}$. For $1/r = 1/p + 1/q$, $0 < p, q \leq \infty$, we have

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

Weak type:

$$\|fg\|_{L^{r,\infty}} \leq \left(\frac{q}{r}\right)^{1/q} \left(\frac{p}{r}\right)^{1/p} \|f\|_{L^{p,\infty}} \|g\|_{L^{q,\infty}}.$$

For mixed norms, it was shown by Benedek (1961) that if $1 \leq p_i \leq \infty$, $i = 1, 2$, then we have

$$\|fg\|_{L^{\vec{1}}} \leq \|f\|_{L^{\vec{p}}} \|g\|_{\vec{p}'},$$

where $\vec{p}' = (p_1', p_2')$.

Using the weak type Hölder's inequality, we get Hölder's inequality for iterated weak norms.

Theorem

Suppose that $0 < p_i, q_i, r_i < \infty$ *and that* $1/r_i = 1/p_i + 1/q_i$, $i = 1, 2$. *Then we have*

$$\|fg\|_{L^{r_2,\infty}(L^{r_1,\infty})} \leq C_{\vec{p},\vec{q}}\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})}\|g\|_{L^{q_2,\infty}(L^{q_1,\infty})}.$$

However, for mixed weak norms, Hölder's inequality is true only for very special cases. The following is a complete characterization of indices for which Hölder's inequality is true on mixed weak spaces.

Theorem

Suppose that $1/r_i = 1/p_i + 1/q_i$, $i = 1, 2$, *where* $0 < p_1, p_2, q_1, q_2 \leq \infty$. *Then there exists some constant* $C_{\vec{p},\vec{q}} < \infty$ *such that*

$$\|fg\|_{L^{\vec{r},\infty}} \leq C_{\vec{p},\vec{q}} \|f\|_{L^{\vec{p},\infty}} \|f\|_{L^{\vec{q},\infty}}, \qquad \forall f, g,$$

if and only if

$$p_1 q_2 = p_2 q_1.$$

Theorem

Suppose that $1/r_i = 1/p_i + 1/q_i$, $i = 1, 2$, *where* $0 < p_1, p_2, q_1, q_2 \leq \infty$. *Then there exists some constant* $C_{\vec{p},\vec{q}} < \infty$ *such that*

$$\|fg\|_{L^{\vec{r},\infty}} \leq C_{\vec{p},\vec{q}} \|f\|_{L^{\vec{p},\infty}} \|f\|_{L^{\vec{q},\infty}}, \qquad \forall f, g,$$

if and only if

$$p_1 q_2 = p_2 q_1.$$

When the condition is true, we have

$$C_{\vec{p},\vec{q}} = \begin{cases} \max\{1, 2^{1/r_1 - 1/r_2}\} \dfrac{p_2^{1/p_2} q_2^{1/q_2}}{r_2^{1/r_2}}, & 0 < p_1, p_2, q_1, q_2 < \infty, \\ \max\{1, 2^{1/r_1 - 1}\} \dfrac{p_1^{r_1/p_1} q_1^{r_1/q_1}}{r_1}, & p_2 = q_2 = \infty, 0 < p_1, p_2 < \infty, \\ \dfrac{p_2^{1/p_2} q_2^{1/q_2}}{r_2^{1/r_2}}, & p_1 = q_1 = \infty, 0 < p_2, q_2 < \infty, \\ 1, & \vec{p} = (\infty, \infty) \text{ or } \vec{q} = (\infty, \infty). \end{cases}$$

Counter Examples. Suppose that

$$\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}, \quad i = 1, 2.$$

Example

For $q_1 = \infty$, $0 < p_2 \leq \infty$ and $0 < p_1, q_2 < \infty$, set $\gamma = 1/q_2$ and $\alpha = p_1/p_2 + p_1/q_2$. Let $f(x,y) = (|x|^n + |y|^m)^\gamma \chi_E(x,y)$ and $g(x,y) = (|x|^n + |y|^m)^{-\gamma}$, where $E = \{(x,y) : 0 < |x|^n < |y|^{-m\alpha}, 1 \leq |y| \leq N\}$. Then we have

$$\lim_{N\to\infty} \frac{\|fg\|_{L^{\vec{r},\infty}}}{\|f\|_{L^{\vec{p},\infty}} \|g\|_{L^{\vec{q},\infty}}} = \infty.$$

Example

For $q_2 = \infty$, $0 < p_1 \leq \infty$ and $0 < p_2, q_1 < \infty$, set $\gamma = n/q_1$. Let $f(x,y) = |x|^{\gamma}\chi_E(x,y)$ and $g(x,y) = |x|^{-\gamma}$, where $E = \{(x,y) : |x|^n \leq |y|^{-mr_1/r_2}\}$. Then we have

$$\|fg\|_{L^{\vec{r},\infty}} \not\lesssim \|f\|_{L^{\vec{p},\infty}} \|g\|_{L^{\vec{q},\infty}}.$$

Example

For $0 < p_1, p_2, q_1, q_2 < \infty$ with $p_2/q_2 > p_1/q_1$, set

$$\frac{\alpha}{m} = \frac{1}{q_2} - \frac{\beta}{q_1}, \qquad \beta = \frac{1/p_2 + 1/q_2}{1/p_1 + 1/q_1},$$

$f(x,y) = |y|^{\alpha}\chi_E(x,y)$ and $g(x,y) = |y|^{-\alpha}\chi_E(x,y)$, where $E = \{(x,y) : |x|^n \le |y|^{-m\beta}\}$. Then we have

$$\|fg\|_{L^{\vec{r},\infty}} \not\lesssim \|f\|_{L^{\vec{p},\infty}} \|g\|_{L^{\vec{q},\infty}}.$$

Example

For $0 < p_1, p_2, q_1, q_2 < \infty$ with $p_2/q_2 < p_1/q_1$, set

$$\frac{\alpha}{m} = \frac{1}{p_2} - \frac{\beta}{p_1}, \qquad \beta = \frac{1/p_2 + 1/q_2}{1/p_1 + 1/q_1},$$

$f(x, y) = |y|^{-\alpha}\chi_E(x, y)$ and $g(x, y) = |y|^{\alpha}\chi_E(x, y)$, where $E = \{(x, y) : |x|^n \leq |y|^{-m\beta}\}$. Then we have

$$\|fg\|_{L^{\vec{r},\infty}} \not\lesssim \|f\|_{L^{\vec{p},\infty}} \|g\|_{L^{\vec{q},\infty}}.$$

It is well known that for $p < r < q$, we have $L^p \cap L^q \subset L^r$. The same is true for weak Lebesgue spaces. Moreover, we have the following interpolation formula.

Proposition

Let $p < q \leq \infty$ and $f \in L^{p,\infty} \cap L^{q,\infty}$. Then f is in L^r for all r satisfies that $1/r = \theta/p + (1-\theta)/q$, where $0 < \theta < 1$,

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{1/r} \|f\|_{L^{p,\infty}}^{\theta} \|f\|_{L^{q,\infty}}^{1-\theta}$$

with the suitable interpretation when $q = \infty$.

However, the above proposition is not true in general if p, q, r are replaced with vector indices.

Theorem

Suppose that $\vec{p} = (p_1, p_2)$, $\vec{q} = (q_1, q_2)$ *and* $\vec{r} = (r_1, r_2)$ *satisfy that*

$$\frac{1}{r_1} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}, \qquad \frac{1}{r_2} = \frac{\theta}{p_2} + \frac{1-\theta}{q_2}, \tag{8}$$

where $0 < \theta < 1$ *is a constant. Then we have*

$$\|f\|_{L^{\vec{r},\infty}} \le \|f\|_{L^{\vec{p},\infty}}^{\theta} \|f\|_{L^{\vec{q},\infty}}^{1-\theta}.$$

However, $L^{\vec{p},\infty} \cap L^{\vec{q},\infty} \not\subset L^{\vec{r}}$ *if* $\vec{p} \neq \vec{q}$ *and* $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$*.*

When the iterated weak norms are invoked, we get again an interpolation theorem. However, four iterated weak norms are invoked.

Theorem

Suppose that

$$\frac{1}{r_1} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1},$$

$$\frac{1}{r_2} = \frac{\theta\xi}{p_{21}} + \frac{(1-\theta)\xi}{p_{22}} + \frac{\theta(1-\xi)}{q_{21}} + \frac{(1-\theta)(1-\xi)}{q_{22}},$$

where $0 < \theta, \xi < 1$ *are constants. Then we have*

$$\|f\|_{L^{\vec{r}}} \le C \|f\|_{L^{p_{21},\infty}(L^{p_1,\infty})}^{\theta\xi} \|f\|_{L^{p_{22},\infty}(L^{q_1,\infty})}^{(1-\theta)\xi} \|f\|_{L^{q_{21},\infty}(L^{p_1,\infty})}^{\theta(1-\xi)} \|f\|_{L^{q_{22},\infty}(L^{q_1,\infty})}^{(1-\theta)(1-\xi)}.$$

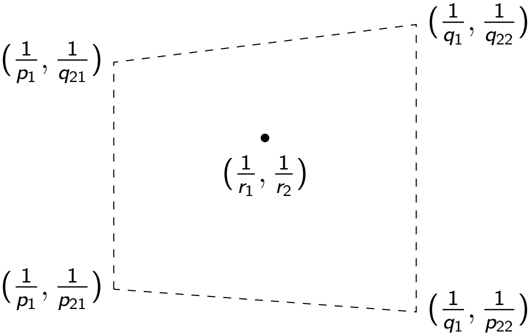

Figure: Interpolation area

Theorem

Let W be either $L^{\vec{p},\infty}$ or $L^{p_2,\infty}(L^{p_1,\infty})$, where $\vec{p}=(p_1,p_2)$ with $0<p_1,p_2\le\infty$. Suppose that $\{f_k : k\ge 1\}$ is a sequence of non-negative measurable functions such that $f_k(x,y)\le f_{k+1}(x,y)$, a.e., $k\ge 1$. Then we have

$$\left\|\lim_{k\to\infty} f_k\right\|_W = \lim_{k\to\infty}\|f_k\|_W .$$

$$\left\|\liminf_{k\to\infty} f_k\right\|_W \le \liminf_{k\to\infty}\|f_k\|_W .$$

However, the dominated convergence theorem fails in weak norm spaces. For example, set $f_0(x) = 1/|x|^{n/p_1}$ and $f_k(x) = f_0(x)\chi_{[k,\infty]}(|x|)$. Take some $g \in L^{p_2} \setminus \{0\}$. We have $\lim_{k\to\infty} f_k \otimes g(x, y) = 0$. Moreover, we see from Theorem 5 that $f_k \otimes g \leq f_0 \otimes g \in L^{\vec{p},\infty} \cap L^{p_2,\infty}(L^{p_1,\infty})$. But

$$\|f_k \otimes g\|_{L^{\vec{p},\infty}} = \|f_k \otimes g\|_{L^{p_2,\infty}(L^{p_1,\infty})} = v_n^{1/p_1} \|g\|_{L^{p_2}}, \ \ k \geq 1.$$

It is known that if $\{f_k : k \geq 1\}$ is convergent in L^p or $L^{p,\infty}$, then it is convergent in measure. However, it is not true for mixed norm. Specifically, neither the strong convergence nor the weak convergence in mixed norm spaces implies the convergence in measure.

Nevertheless, it was shown in by Benedek that if $\{f_k : k \geq 1\}$ is convergent to f in $L^{\vec{p}}$, then it contains a subsequence convergent almost everywhere to f. We show that the same is true for weak norms.

Theorem

Let W be either $L^{\vec{p},\infty}$ or $L^{p_2,\infty}(L^{p_1,\infty})$, where $\vec{p} = (p_1, p_2)$ with $0 < p_1, p_2 \leq \infty$. Let $\{f_k : k \geq 1\}$ be a Cauchy sequence in W, that is,

$$\lim_{k,l\to\infty} \|f_k - f_l\|_W = 0.$$

Then there is some $f \in W$ such that $\lim_{k\to\infty} \|f - f_k\|_W = 0$.

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Let W be either $L^{\vec{p},\infty}$ or $L^{p_2,\infty}(L^{p_1,\infty})$, where $\vec{p} = (p_1, p_2)$ with $0 < p_1, p_2 \leq \infty$. Let $\{f_k : k \geq 1\}$ be a Cauchy sequence in W, that is,

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In [Benedek1961], the Riesz theorem for mixed norm Lebesgue spaces was proved. It says that if

$$\lim_{k\to\infty} \|f_k\|_{L^{\vec{p}}} = \|f\|_{L^{\vec{p}}} \quad \text{and} \quad \lim_{k\to\infty} f_k(x,y) = f(x,y), \ \textit{a.e.}$$

where $\vec{p} = (p_1, p_2)$ with $1 \le p_1, p_2 < \infty$, then we have

$$\lim_{k\to\infty} \|f_k - f\|_{L^{\vec{p}}} = 0.$$

Whenever weak norms are considered, the above conclusion fails. For example, set $f_0(x) = 1/|x|^{n/p_1}$ and $f_k(x) = f_0(x)\chi_{[0,k]}(|x|)$. Take some $g \in L^{p_2} \setminus \{0\}$. We have

$$\lim_{k\to\infty} f_k \otimes g(x,y) = f_0(x)g(y)$$

and

$$\lim_{k\to\infty} \|f_k \otimes g\|_{L^{\vec{p},\infty}} = \lim_{k\to\infty} \|f_k \otimes g\|_{L^{p_2,\infty}(L^{p_1,\infty})} = v_n^{1/p_1}\|g\|_{L^{p_2}}.$$

However, for any $k \geq 1$,

$$\|f_k \otimes g - f_0 \otimes g\|_{L^{\vec{p},\infty}} = \|f_k \otimes g - f_0 \otimes g\|_{L^{p_2,\infty}(L^{p_1,\infty})} = v_n^{1/p_1}\|g\|_{L^{p_2}}.$$

Hence $\{f_k \otimes g : k \geq 1\}$ is not convergent to $f_0 \otimes g$ in $L^{\vec{p},\infty}$ or $L^{p_2,\infty}(L^{p_1,\infty})$.

It is well known that the Hardy-Littlewood maximal operator is of strong type (p, p) for $p > 1$ and weak type $(1, 1)$. Moreover, the strong maximal operator is not of weak type $(1, 1)$.

When the iterated weak norm is considered, we do not know if the maximal operator is of weak type $(1, 1)$.

Let M_s be the strong maximal operator defined by

$$M_s f(x,y) = \sup_{\substack{Q_1 \subset \mathbb{R}^n, Q_2 \subset \mathbb{R}^m \\ (x,y) \in Q_1 \times Q_2}} \frac{1}{|Q_1| \cdot |Q_2|} \int_{Q_1 \times Q_2} |f|.$$

Theorem

Let $f \in L^1(\mathbb{R}^2)$. Suppose that there is some $a \in \mathbb{R}$ such that for any $y \in \mathbb{R}$ (or any $x \in \mathbb{R}$), $|f(x,y)|$ is increasing on $(-\infty, a)$ and decreasing on (a, ∞) with respect to x (or y). Then we have

$$\|M_s f(x,y)\|_{L^{1,\infty}(L^{1,\infty})} \leq 12 \|f\|_1.$$

Theorem

$$\sup_{\alpha,\beta>0} \beta \left| \left\{ y : \alpha \left| \{ x : |Mf(x,y)| > \alpha \} \right|^{1/p_1} > \beta \right\} \right|^{1/p_2} \lesssim \|f\|_1$$

Theorem

$$\sup_{\alpha,\beta>0} \beta \left| \left\{ y : \alpha \left|\{x : |Mf(x,y)| > \alpha\}\right|^{1/p_1} > \beta \right\} \right|^{1/p_2} \lesssim \|f\|_1$$

Problem

$$\|Mf\|_{L^{1,\infty}(L^{1,\infty})} \lesssim \|f\|_1?$$

The strong maximal function is bounded on $L^{\vec{p}}$ if $p_i > 1$.

Weighted bounded for $w(x, y) = u(x)v(y)$.

Linear and multilinear CZ Operators:

we study the boundedness of T_γ and L_γ from $L^{\vec{p}}$ to $L^{\vec{q}}$. First, we consider T_γ with $\vec{p} = (\infty, \infty)$. In this case, it is more convenient to rewrite the inequality in the following form,

$$\|F\|_X \lesssim \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2},$$

where X stands for some norm defined on $\mathbb{R}^{2n}$. Recall that $L^{\vec{p}} = L^\infty$ whenever $\vec{p} = (\infty, \infty)$.

Theorem

Let F be a nonnegative measurable function defined on $\mathbb{R}^{2n}$. Then for all $0 < q_1, q_2 \leq \infty$, we have

$$\|F\|_{L^{\vec{q},\infty}} \leq C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2}, \tag{9}$$

$$\|F\|_{L^{q_2,\infty}(L^{q_1,\infty})} \leq C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2}. \tag{10}$$

However, for $\vec{q} \neq (\infty,\infty)$, we have

$$\|F\|_{L^{\vec{q}}} \leq C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2} \tag{11}$$

is not true for all $F \in L^{\vec{q}}(\mathbb{R}^{2n})$.

Next we consider the boundedness of T_γ from $L^\infty(\mathbb{R}^{2n})$ to $X(\mathbb{R}^n)$, where X stands for the mixed norm $L^{q_2,\infty}(L^{q_1})$ or $L^{q_2}(L^{q_1,\infty})$.

Theorem

Let F be nonnegative measurable functions defined on $\mathbb{R}^{2n}$. Then for all $0 < q_1, q_2 < \infty$ we have

$$\|F\|_{L^{q_2,\infty}(L^{q_1})} \le C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2}. \quad (12)$$

However,

$$\|F\|_{L^{q_2}(L^{q_1,\infty})} \le C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1+n/q_2} \quad (13)$$

does not hold.

Theorem (Continued)

Meanwhile, we present all the endpoint cases. For any $C_{\vec{q},n} > 0$,

$$\|F\|_{L^\infty(L^{q_1})} \not\lesssim C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1}, \tag{14}$$

$$\|F\|_{L^{q_1}(L^\infty)} \not\lesssim C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1}. \tag{15}$$

For the remaining endpoint cases, we have

$$\|F\|_{L^{q_1,\infty}(L^\infty)} \le C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1}, \tag{16}$$

$$\|F\|_{L^\infty(L^{q_1,\infty})} \le C_{\vec{q},n} \sup_{x,y\in\mathbb{R}^n} F(x,y)(|x+y|+|x-y|)^{n/q_1}. \tag{17}$$

Corollary

For all $0 < p_1, p_2 \leq \infty$,

$$\|f\|_{L^{p_1,\infty}} \|g\|_{L^{p_2,\infty}} \leq C_{\vec{p},n} \sup_{x,y\in\mathbb{R}^n} f(x)g(y)|x-y|^{n/p_1+n/p_2} \tag{18}$$

holds for any $f \in L^{p_1,\infty}$, $g \in L^{p_2,\infty}$.

Furthermore, by interpolation

$$\|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \leq C_{\vec{p},n} \sup_{x,y\in\mathbb{R}^n} f(x)g(y)|x-y|^{n/p_1+n/p_2} \tag{19}$$

holds for any $f \in L^{p_1}$, $g \in L^{p_2}$.

Theorem

Let f be nonnegative measurable functions defined on $\mathbb{R}^{2n}$. Then for all $0 < r < p_1 \leq \infty$ and $0 < p_2 \leq \infty$ satisfying the homogeneity condition $1/r = 1/p_1 + \gamma/n$,

$$\|L_\gamma f\|_{L^{p_2}(L^{r,\infty})} \leq C_{\vec{p},r,n}\|f\|_{L^{p_2}(L^{p_1,\infty})}, \tag{20}$$

$$\|L_\gamma f\|_{L^{p_2,\infty}(L^{r,\infty})} \leq C_{\vec{p},r,n}\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})}. \tag{21}$$

And for all $0 < p_1 < r \leq \infty$ and $0 < p_2 \leq \infty$ satisfying the homogeneity condition $1/p_1 = 1/r + \gamma/n$,

$$\|L_\gamma^{-1} f\|_{L^{p_2}(L^{r,\infty})} \geq C_{\vec{p},r,n}\|f\|_{L^{p_2}(L^{p_1,\infty})}, \tag{22}$$

$$\|L_\gamma^{-1} f\|_{L^{p_2,\infty}(L^{r,\infty})} \geq C_{\vec{p},r,n}\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})}. \tag{23}$$

Theorem (Continued)

However, for any multiple indices $\vec{p}$ and $\vec{q}$,

$$\|L_\gamma f\|_{L^{q_2,\infty}(L^{q_1})} \not\lesssim C_{\vec{p},\vec{q},n}\|f\|_{L^{p_2,\infty}(L^{p_1})}; \tag{24}$$

$$\|L_\gamma f\|_{L^{q_2}(L^{q_1,\infty})} \not\lesssim C_{\vec{p},\vec{q},n}\|f\|_{L^{p_2}(L^{p_1,\infty})} \text{ unless } p_2 = q_2; \tag{25}$$

$$\|L_\gamma f\|_{L^{q_2,\infty}(L^{q_1,\infty})} \not\lesssim C_{\vec{p},\vec{q},n}\|f\|_{L^{p_2,\infty}(L^{p_1,\infty})} \text{ unless } p_2 = q_2; \tag{26}$$

$$\|L_\gamma f\|_{L^{\vec{q}}} \not\lesssim C_{\vec{p},\vec{q},n}\|f\|_{L^{\vec{p}}}, \tag{27}$$

Finally, let us show that both Theorem 24 and Theorem 1 imply the classical Hardy-Littlewood-Sobolev inequality and its reverse version as follows.

Corollary

For $1 < p_1, p_2 < \infty$ *with* $1/p_1 + 1/p_2 > 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)|x-y|^{-n(2-1/p_1-1/p_2)}dxdy \leq C_{\vec{p},n}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \quad (28)$$

holds for all nonnegative functions $f \in L^{p_1}$, $g \in L^{p_2}$.
For $0 < p_1, p_2 < 1$ *and all nonnegative functions* $f \in L^{p_1}$, $g \in L^{p_2}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)|x-y|^{n(1/p_1+1/p_2-2)}dxdy \geq C_{\vec{p},n}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}. \quad (29)$$

THANKS!