Perturbation theory of commuting self-adjoint operators and related topics. Part II

Hongyin Zhao (joint work with Alexei Ber, Fedor Sukochev, Dmitriy Zanin)

School of Mathematics and Statistics of UNSW

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Preliminaries

- $\mathcal{H}$: separable Hibert space, $\dim(\mathcal{H}) = \infty$.
- $\mathcal{A}$: a norm-closed $*$-algebra of $B(\mathcal{H})$, C^*-algabra.
- commutant of $\mathcal{A}$, $\mathcal{A}' := \{B \in B(\mathcal{H}) : AB = BA, \forall A \in B(\mathcal{H})\}$.
- $\mathcal{M} \subset B(\mathcal{H})$: a $*$-algebra of $B(\mathcal{H})$ s.t. $\mathcal{M}'' = \mathcal{M}$, von Neumann algebra.
  A von Neumann algebra is a C^*-algebra.
- $\mathcal{U}(\mathcal{M}) := \{\text{unitary operators in } \mathcal{M}\}$.
- $\mathcal{N} \subset B(\mathcal{H})$: a factor, i.e. a von Neumann algebra with trivial center, i.e. $\mathcal{Z}(\mathcal{N}) := \mathcal{N} \cap \mathcal{N}' = \mathbb{C}\mathbf{1}$.

- $\tau : \mathcal{M}_+ \to [0, \infty]$: ia called a trace on $\mathcal{M}$ if
  1. $\tau(\lambda A + B) = \lambda\tau(A) + \tau(B), \quad \lambda \in \mathbb{R}_+, A, B \in \mathcal{M}_+.$
  2. $\tau(U^* A U) = \tau(A), \quad A \in \mathcal{M}_+$ and $U \in \mathcal{U}(\mathcal{M})$.

     τ is called:
  3. faithful if $\tau(A) = 0 \Rightarrow A = 0$.
  4. normal if $\tau(\sup_{k\geq 1} A_k) = \sup_{k\geq 1} \tau(A_k)$ for every bounded increasing sequence $\{A_k\}_{k\geq 1} \subset \mathcal{M}_+$.
  5. semifinite if $\forall A \in \mathcal{M}_+, \exists 0 \neq B \in \mathcal{M}_+$ s.t. $B \leq A$ and $\tau(B) < \infty$.
- A von Neummann algebra $\mathcal{M}$ equipped with a normal semifinite faithful (n.s.f.) trace τ will be a called a semifinite von Neumann algebra. We will only consider semifinite von Neumann algebras.

Example

When $\mathcal{M} = B(\mathcal{H})$,
the matrix trace
$\tau(A) = \mathrm{Tr}(A) = \sum_{k\geq 1} \langle Ae_k, e_k \rangle, \quad A \geq 0,$ *is a trace,*
here $\{e_k\}_{k\geq 1}$ *is any C.O.N.S of* $\mathcal{H}$.

- For $x \in \mathcal{M}$:
  $l(x)$ is the projection onto $\overline{x(\mathcal{H})}$, left support of x
  $r(x)$ is the projection onto $(\ker x)^{\perp}$, right support of x
  $s(x) = l(x) \vee r(x)$, support of x
  $x \in \mathcal{M}_{sa} \Rightarrow s(x) = l(x) = r(x)$
- $\mathcal{P}(\mathcal{M}) := \{\text{projections in } \mathcal{M}\}$.
  $\mathcal{F}(\mathcal{M}) := \{T \in \mathcal{M} : \tau(l(T)) < \infty\}$, operators with τ-finite support
  $\mathcal{K}(\mathcal{M}) := \overline{\mathcal{F}(\mathcal{M})}^{\|\cdot\|}$, τ-compact operators

Non-commutative symmetric function spaces

- $S(\mathcal{M},\tau)$: the set of τ-measurable operators.
- For $a \in S(\mathcal{M},\tau)_{sa}$, let e^a be the spectral measure corresponding to a. For any Borel function $f : \mathbb{R} \to \mathbb{C}$, the normal operator $f(a)$ is defined by the spectral integral
  $f(a) = \int_{\mathbb{R}} f(\lambda) de^a(\lambda) = \int_{\sigma(a)} f(\lambda) de^a(\lambda)$.
- For $x \in S(\mathcal{M},\tau)$,
  $d_x(s) := \tau(e^{|x|}(s,\infty)), s \geq 0$, distribution function of x
  $\mu_x(t) := \inf\{s \geq 0 : d_x(s) \leq t\}, t \geq 0$, singular value function of x
  μ_x is decreasing, right-continuous, $\mu_x(0) = \|x\|_{\mathcal{M}}$ if $x \in \mathcal{M}$.

- A symmetric function space E is a Banach function space on the semiaxis $(0, \infty)$ with Lebesgue measure satisfying:
  If $y \in E$ and $x^*(t) \leq y^*(t)$ for all $t \in (0, \infty)$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
- Let E be a symmetric function space on $(0, \infty)$. Define
  $E(\mathcal{M}) := \{a \in S(\mathcal{M}, \tau) : \mu_a \in E\}$,
  with norm $\|a\|_{E(\mathcal{M})} := \|\mu_a\|_E, a \in E(\mathcal{M})$.
  We have $L_1 \cap L_\infty \subset E$, so $\mathcal{F}(\mathcal{M}) \subset E(\mathcal{M})$.
  $E^{(0)}(\mathcal{M}) := \overline{\mathcal{F}(\mathcal{M})}^{\|\cdot\|_{E(\mathcal{M})}}$.
  If E is separable, then $E^{(0)}(\mathcal{M}) = E(\mathcal{M})$.

Example

Let $1 \leq p \leq \infty$ and $E = L_p$ be the Lebesgue L_p space on $(0, \infty)$,
$L_p(\mathcal{M}) := \{a \in S(\mathcal{M}, \tau) : \mu_a \in L_p\}$,
with norm $\|a\|_{L_p(\mathcal{M})} := \|\mu_a\|_{L_p}, a \in L_p(\mathcal{M})$.
When $p = \infty$, $L_\infty(\mathcal{M}) = \mathcal{M}$.

Diagonality modulo non-commutative symmetric function spaces

- $D \in \mathcal{M}$ is diagonal
  $\overset{\text{def}}{\Leftrightarrow} \exists \{e_n\}_{\geq 1}$ C.O.N.S of $\mathcal{H}$ s.t. $Te_n = \lambda_n e_n$
  $\Leftrightarrow \exists \{p_n\}_{n\geq 1} \subset \mathcal{P}(\mathcal{M}), \sum_{n\geq 1} p_n = \mathbf{1}$, s.t. $D = \sum_{n\geq 1} \lambda_n p_n$.

Let $\mathcal{M}$ von Neumann algebra equiped with an n.s.f. trace τ.
$\alpha = (A_1, \ldots, A_n) \in \mathcal{M}_{sa}^n$ be a commuting self-adjoint n-tuple.
Let $E_1, \ldots, E_n$ be symmetric function spaces on $(0, \infty)$, set
$\Phi(\mathcal{M}) := E_1(\mathcal{M}) \times \cdots E_n(\mathcal{M})$.

Definition

If $\exists$ commuting diagonal n-tuple $\delta = (D_1, \ldots, D_n) \subset \mathcal{M}$ s.t. $A_i - D_i \in E_i(\mathcal{M})$, we say that α is diagonal modulo $\Phi(\mathcal{M})$.

If $E_1 = \ldots = E_n = E$, we say α is diagonal modulo $E(\mathcal{M})$.

The case when $\mathcal{M}$ is abelian (trivial)

Example

Suppose $\mathcal{M}$ is abelian and $E_1, \ldots, E_n$ are given symmetric function spaces.
$\forall$ *commuting n-tuple $\alpha \in (\mathcal{M}_{sa})^n$ and $\forall \varepsilon > 0$,*
$\exists$ *commuting diagonal n-tuple $\delta \in (\mathcal{M}_{sa})^n$ s.t.*

$$\max\{\|\alpha - \delta\|_{\mathcal{N}}, \|\alpha - \delta\|_{\Phi(\mathcal{M})}\} \leq \varepsilon,$$

where $\Phi(\mathcal{M}) = E_1(\mathcal{M}) \times \cdots \times E_n(\mathcal{M})$.

Proof.

$\exists \; \{p_k\}_{k\geq 1} \subset \mathcal{P}(\mathcal{M}), \sum_{k\geq 1} p_k = \mathbf{1}, \tau(p_k) < \infty.$
$\exists \delta_k \subset p_k \mathcal{M} p_k$ s.t. $\|\alpha p_k - \delta_k\|_{\mathcal{M}\cap\Phi(\mathcal{M})} \leq \frac{\varepsilon}{2^k}$. Set $\delta = \sum_{k\geq 1} \delta_k$. □

Perturbation of self-adjoint operators in a factor

- A semifinite factor is called properly infinite if $\tau(\mathbf{1}) = \infty$.

Let $\mathcal{N} \subset B(\mathcal{H})$ be a properly infinite factor.

Theorem 1.1 (Zsido '75, Akemann-Pedersen '77, Kaftal '78)

$\forall A \in \mathcal{N}_{sa}, \forall \varepsilon > 0$, then $\exists$ diagonal $D \in \mathcal{N}_{sa}$ s.t. $A - D \in L_2(\mathcal{N}) \cap \mathcal{N}$ and $\|A - D\|_{L_2(\mathcal{N})} < \varepsilon$.

Theorem (Li-Shen-Shi, 2020)

Let $n \geq 2$. $\forall$ commuting self-adjoint $\alpha \in (\mathcal{N}_{sa})^n$, $\forall \varepsilon > 0$, $\exists$ commuting diagonal n-tuple $\delta \in (\mathcal{N}_{sa})^n$ s.t. $\alpha - \delta \in L_n(\mathcal{N}) \cap \mathcal{N}$ and $\max\{\|\alpha - \delta\|_{\mathcal{N}}, \|\alpha - \delta\|_{L_n(\mathcal{N})}\} < \varepsilon$.

Quasicentral modulus

Let $\mathcal{M}$ be a semifinite von Neumann algebra,
$E_1, \ldots, E_n$ be symmetric function spaces on $(0, \infty)$, set
$\Phi(\mathcal{M}) := E_1(\mathcal{M}) \times E_2(\mathcal{M}) \times \cdots \times E_n(\mathcal{M})$,
$\alpha = (A_1, \ldots, A_n) \in \mathcal{M}^n$, $\|\alpha\|_{\Phi(\mathcal{M})} := \max_{1 \le i \le n} \|A_i\|_{E_i(\mathcal{M})}$.

- $\mathcal{F}_1^+ := \{R \in \mathcal{M} : 0 \le R \le \mathbf{1}, \tau(s(R)) < \infty\}$.
- Quasicentral modulus:

$$k_{\Phi(\mathcal{M})}(\alpha) := \inf\{\limsup_{k\to\infty} \|[R_k, \alpha]\|_{\Phi(\mathcal{M})} : R_k \in \mathcal{F}_1^+, R_k \uparrow \mathbf{1}\}.$$

If $E_1 = \ldots = E_n = E$, $k_{\Phi(\mathcal{M})}(\alpha) = k_{E(\mathcal{M})}(\alpha)$.

Extension of Voiculescu's results to properly infinite factor

Let $\mathcal{N} \subset B(\mathcal{H})$ be a properly infinite factor.

Theorem 2.1 (Ber-Sukochev-Zanin-Zhao, 2022, under review)

Let $E_1, \ldots, E_n$ be symmetric function spaces on $(0, \infty)$ s.t.
$E_i \not\subseteq L_\infty, 1 \leq i \leq n,$
$\Phi(\mathcal{N}) := E_1^{(0)}(\mathcal{N}) \times \cdots \times E_n^{(0)}(\mathcal{N}).$
$\forall$ *commuting self-adjoint n-tuple $\alpha \in (\mathcal{M}_{sa})^n$, T.F.A.E.*

1. $k_{\Phi(\mathcal{N})}(\alpha) = 0$;
2. $\forall \varepsilon > 0$, $\exists$ *diagonal commuting n-tuple $\delta \in (\mathcal{N}_{sa})^n$ s.t.* $\alpha - \delta \in \Phi(\mathcal{N}) \cap \mathcal{N}$ *and* $\max\{\|\alpha - \delta\|_{\mathcal{N}}, \|\alpha - \delta\|_{\Phi(\mathcal{N})}\} < \varepsilon$.

Properties of Quasicentral modulus

In the remaining part we will assume $E_j \not\subseteq L_\infty, 1 \leq j \leq n$, $\Phi(\mathcal{N}) := E_1^{(0)}(\mathcal{N}) \times \cdots \times E_n^{(0)}(\mathcal{N})$.

Proposition

Let $p \in \mathcal{M}$ be a projection that commutes with α, then

1. $k_{\Phi(\mathcal{M})}(p\alpha) \leq k_{\Phi(\mathcal{M})}(\alpha)$.
2. $k_{\Phi(\mathcal{M})}(\alpha) \leq k_{\Phi_{((\mathbf{1}-p)\mathcal{M}(\mathbf{1}-p))}}((\mathbf{1}-p)\alpha) + k_{\Phi_{(p\mathcal{M}p)}}(p\alpha)$. *(subadditivity)*
3. *if $p\alpha = \alpha$, then*
$$k_{\Phi(p\mathcal{M}p)}(\alpha) = k_{\Phi(\mathcal{M})}(\alpha).$$

Techniques of proof of Theorem 2.1

The hard part of the proof of Theorem 2.1 is (1)⇒(2), i.e. the following theorem:

Theorem 2.2

Suppose $k_{\Phi(\mathcal{N})}(\alpha) = 0$. $\forall \varepsilon > 0$, $\exists$ *commuting diagonal* n*-tuple* $\delta \in (\mathcal{N}_{sa})^n$ *s.t.* $\alpha - \delta \in \Phi(\mathcal{N}) \cap \mathcal{N}$ *and* $\max\{\|\alpha - \delta\|_{\mathcal{N}}, \|\alpha - \delta\|_{\Phi(\mathcal{N})}\} < \varepsilon$.

A general way to construct a commuting diagonal n-tuple, is to construct a monomorphism $\psi : C^*(\alpha) \to \mathcal{N}$ s.t.

$$\delta := (\psi(\alpha(1)), \dots, \psi(\alpha(n)))$$

is a commuting diagonal n-tuple.

The problem is then reduced to prove that α is approximately equivalent to $\psi(\alpha)$ modulo $\Phi(\mathcal{N})$. Precise definitions will be given.

Approximately equivalence of $*$-homomorphisms

- Let $\alpha \in (\mathcal{N}_{sa})^n$ be a given commuting self-adjoint n-tuple, $C^*(\alpha) \subset \mathcal{N}$ be the C^*-subalgebra generated by α and $\mathbf{1}$.
  Let π, ψ be unital $*$-homomorphism from $C^*(\alpha)$ into $\mathcal{N}$.
  We say that $\pi(\alpha)$ is approximately equivalent to $\psi(\alpha)$ modulo $\Phi(\mathcal{N})$, denoted by $\pi \sim_{\Phi(\mathcal{N})} \psi$,
  if $\exists\ (U_k)_{k\geq 1} \subset \mathcal{U}(\mathcal{N})$ s.t.
  1. $\pi(A_j) - U_k\psi(A_j)U_k^* \in E_j(\mathcal{N}), \quad 1 \leq j \leq n,\ k \geq 1.$
  2. $\lim_{k\to\infty} \|\pi(A_j) - U_k\psi(A_j)U_k^*\|_{E_j(\mathcal{N})} = 0, \quad 1 \leq j \leq n.$
- If U_k in the above definition is only an isometry (or partial isometry), we write

$$\pi \sim_{isometry,\Phi(\mathcal{N})} \psi \ \text{ or } \pi \sim_{U_k,\Phi(\mathcal{N})} \psi.$$

Construction of diagonal representations

- Let $\Omega := \{\rho : \rho : C^*(\alpha) \to \mathbb{C}$ is a nonzero $*$-homomorphism$\}$.
  $C^*(\alpha) \cong C(\Omega)$. (Gelfand representation)
  Ω is weak-$*$ compact Hausdorff topological space.
  $C(\Omega)$ is separable, so by Riesz's theorem, Ω is metrizable.
  Ω metrizable and compact $\Rightarrow$ Ω is separable, so
  $\exists\{\rho_k\}_{k\geq 1} \subset \Omega, \overline{\{\rho_k\}_{k\geq 1}} = \Omega$,
  then the representation $\oplus_k \rho_k$ is faithful on $C^*(\alpha)$.
- $\mathcal{N}$ is properly infinite $\Rightarrow \exists\{q_n\}_{n\geq 1} \subset \mathcal{P}(\mathcal{N})$ s.t. $\mathbf{1}_{\mathcal{N}} = \sum_{n\geq 1} q_n$ and $\tau(q_n) = \infty$.
  Suppose $W^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$. (Technical assumption)
  Set
  $$\psi(x) = \sum_{k\geq 1} \rho_k(x) q_k, \quad x \in C^*(\alpha).$$

Clearly, $\psi : C^*(\alpha) \to \mathcal{N}$ is a unital $*$-monomorphism.
$W^*(\psi(\alpha)) \subset W^*(\{q_k\}_{k\geq 1})$ and $\tau(q_k) = \infty$ for any $k \geq 1 \Rightarrow$
$$W^*(\psi(\alpha)) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}.$$

Techniques of proof

An important step to prove Theorem 2.2 is the following theorem:

Theorem 2.3

Let $\psi : C^*(\alpha) \to \mathcal{N}$ *be a unital* $*$*-monomorphism, s.t.* $W^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = W^*(\psi(\alpha)) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$, *and* $k_{\Phi(\mathcal{N})}(\alpha) = k_{\Phi(\mathcal{N})}(\psi(\alpha)) = 0$. *Then* $\forall \varepsilon > 0$, $\exists u \in \mathcal{U}(\mathcal{N})$ *s.t.* $\alpha - u\psi(\alpha)u^{-1} \in \Phi(\mathcal{N})$ *and*

$$\|\alpha - u\psi(\alpha)u^{-1}\|_{\Phi(\mathcal{N})} < \varepsilon.$$

Step 1 for proving Theorem 2.3

The first step for proving Theorem 2.3 is to establish the existence of a smooth partition of the identity with good properties.

Theorem (Step 1)

Suppose $k_{\Phi(\mathcal{M})}(\alpha) = 0$. *For every* $\varepsilon > 0$, *there is a sequence* $\{e_m\}_{m \geq 1} \subset \mathcal{F}_1^+(\mathcal{M})$ *s.t.*

$$\sum_{m \geq 1} e_m^2 = \mathbf{1}_{\mathcal{M}}, \quad \sum_{m \geq 1} \|[\alpha, e_m]\|_{\Phi(\mathcal{M})} \leq \varepsilon,$$

where the first series converges in strong operator topology.

Step 2 for proving Theorem 2.3

Theorem (Step 2)

Let $\psi : C^*(\alpha) \to \mathcal{N}$ *be a unital* $*$*-monomorphism.*
Suppose $C^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$.
$\forall \varepsilon > 0$, $\exists$ *an isometry* $v \in \mathcal{N}$ *s.t.* $\|v\psi(\alpha) - \alpha v\|_{\mathcal{N}} < \varepsilon$.

It follows from the following extension of Voiculescu's theorems to properly infinite factors.

Theorem (Ciuperca et al, 2013)

Let $\mathcal{A}$ *be a nuclear* C^**-subalgebra of* $\mathcal{N}$.
Suppose that $\psi : \mathcal{A} \to \mathcal{N}$ *is a unital* $*$*-homomorphism s.t.* $\psi|_{\mathcal{A}\cap\mathcal{K}(\mathcal{N})} = 0$.
$\forall$ *finite subset* $\mathfrak{F} \subset \mathcal{A}$ *and* $\forall \varepsilon > 0$, $\exists$ *a partial isometry* v *s.t.*

$$\|\psi(a) - v^* a v\|_{\mathcal{N}} < \varepsilon, \quad a \in \mathfrak{F}.$$

Step 3 for proving Theorem 2.3

- Let $\mathcal{N}\bar{\otimes}B(\ell_2)$ be the von Neumann algebra generated by the algebraic tensor product $\mathcal{N}\otimes B(\ell_2)$.
  Let $\{E_{i,j}\}_{i,j\geq 1}$ be a matrix unit of $B(\ell_2)$ such that $\mathrm{Tr}(E_{1,1})=1$.

Theorem (Step 3, Technical result)

Suppose $W^*(\alpha)\cap\mathcal{K}(\mathcal{N},\tau)=\{0\}$.
$\exists$ *a sequence of isometries* $\{v_j\}_{j\geq 0}\subset\mathcal{N}\bar{\otimes}B(\ell_2)$ *s.t.*

$$v_{j_1}^*v_{j_2}=\delta_{j_1,j_2}\mathbf{1}_{\mathcal{N}}\otimes\mathbf{1}_{B(\ell_2)},\quad v_jv_j^*\leq\mathbf{1}_{\mathcal{N}}\otimes E_{1,1},\quad j,j_1,j_2\geq 0,$$

$$\|v_j(b\otimes\mathbf{1}_{B(\ell_2)})-(b\otimes E_{1,1})v_j\|_{\mathcal{N}}\to 0,\quad b\in\alpha.$$

Step 4 and 5 for proving Theorem 2.3

Theorem (Step 4)

Suppose $W^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$.
If $\psi : C^*(\alpha) \to \mathcal{N}$ *is a unital* $*$*-homomorphism s.t.* $k_{\Phi(\mathcal{N})}(\psi(\alpha)) = 0$, *then* $\forall \varepsilon > 0, \exists$ *an isometry* $v \in \mathcal{N} \bar{\otimes} B(\ell_2)$ *s.t.*
$\|v(\psi(\alpha) \otimes \mathbf{1}_{B(\ell_2)}) - (\alpha \otimes \mathbf{1}_{B(\ell_2)})v\|_{\Phi(\mathcal{N} \bar{\otimes} B(\ell_2))} \leq \varepsilon$.

Theorem (Step 5)

Suppose $W^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$ *and* $k_{\Phi(\mathcal{N})}(\alpha) = 0$.
$\forall \varepsilon > 0$, $\exists$ *an isometry* $v \in \mathcal{N} \bar{\otimes} B(\ell_2)$ *s.t.*

$$v(\alpha \otimes \mathbf{1}_{B(\ell_2)}) - (\alpha \otimes E_{1,1})v \in \Phi(\mathcal{N} \bar{\otimes} B(\ell_2)),$$

$$\|v(\alpha \otimes \mathbf{1}_{B(\ell_2)}) - (\alpha \otimes E_{1,1})v\|_{\Phi(\mathcal{N} \bar{\otimes} B(\ell_2))} \leq \varepsilon, \quad vv^* \leq \mathbf{1}_{\mathcal{N}} \otimes E_{1,1}.$$

Combine all the pieces

Proof of Theorem 2.3.

$$\psi^{\oplus\infty} \sim_{isometry,\Phi(\mathcal{N})} \mathrm{id} \quad \text{(Step 4 and 5)},$$

$$\Rightarrow \mathrm{id} \sim_{isometry,\Phi(\mathcal{N})} \mathrm{id} \oplus \psi,$$

$$\text{i.e. } \alpha \sim_{isometry,\Phi(\mathcal{N})} \alpha \oplus \psi(\alpha).$$

Swap $\psi(\alpha)$ with α, repeat the above process for ψ^{-1} on $C^*(\psi(\alpha))$,

$$\psi(\alpha) \sim_{isometry,\Phi(\mathcal{N})} \psi(\alpha) \oplus \alpha.$$

Obviously $\psi(\alpha) \oplus \alpha$ is unitarily equivalent to $\alpha \oplus \psi(\alpha)$, thus

$$\alpha \oplus 0 \sim_{w,\Phi(\mathcal{N})} \psi(\alpha) \oplus 0,$$

for some partial isometry w satisfying $w^*w = \mathbf{1}_{\mathcal{N}} \oplus 0$, $ww^* = \mathbf{1}_{\mathcal{N}} \oplus 0$. Thus $w = u \oplus 0$ for some $u \in \mathcal{U}(\mathcal{N})$, i.e. $\alpha \sim_{\Phi(\mathcal{N})} \psi(\alpha)$. □

Corollary 2.4

Suppose $k_{\Phi(\mathcal{N})}(\alpha) = 0$ *and* $W^*(\alpha) \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$. $\forall\ \varepsilon > 0$, $\exists$ *a diagonal* n*-tuple* $\delta \subset \mathcal{N}$ *s.t.*

1. $\alpha - \delta \in \Phi(\mathcal{N}) \cap \mathcal{N}$;
2. $\|\alpha - \delta\|_{\Phi(\mathcal{N})} < \varepsilon$.

Proof.

Let ψ be the diagonal representation constructed above, so $\psi(\alpha)$ is diagonal in $\mathcal{N}$, this implies $k_{\Phi(\mathcal{N})}(\psi(\alpha)) = 0$. By Theorem 2.3, $\mathrm{id} \sim_{\Phi(\mathcal{N})} \psi$. □

Proof of Theorem 2.2.

Let $\mathcal{W} = W^*(\alpha)$ be the von Neumann subalgebra in $\mathcal{N}$ generated by α and $\mathbf{1}$, $\mathcal{W}$ is abelian.
Set

$$p_{\mathcal{W}} = \bigvee\{s(x) : x \in \mathcal{W} \cap \mathcal{K}(\mathcal{N}, \tau)\}.$$

$p_{\mathcal{W}}\mathcal{W}p_{\mathcal{W}}$ is semifinite and $p_{\mathcal{W}}$ commutes with α.
Note that

$$k_{\Phi(p_{\mathcal{W}}\mathcal{N}p_{\mathcal{W}})}(p_{\mathcal{W}}\alpha) = k_{\Phi((\mathbf{1}_{\mathcal{N}}-p_{\mathcal{W}})\mathcal{N}(\mathbf{1}-p_{\mathcal{W}}))}((\mathbf{1}-p_{\mathcal{W}})\alpha) = 0,$$

it suffices to consider the case $p_{\mathcal{W}} = \mathbf{1}$ and $p_{\mathcal{W}} = 0$ respectively.
Case 1. $p_{\mathcal{W}} = \mathbf{1}$, this is just the commutative semifinite case.
Case 2. $p_{\mathcal{W}} = 0$, then $x = xp_{\mathcal{W}} = 0$ for any $x \in \mathcal{W} \cap \mathcal{K}(\mathcal{N}, \tau)$, so $\mathcal{W} \cap \mathcal{K}(\mathcal{N}, \tau) = \{0\}$. Thus Corollary 2.4 can be applied. □

Noncommutative Lorentz $(p, 1)$-ideals

- $L_{p,1} := \{f \in L_1 + L_\infty : \int_0^\infty f^*(t)t^{\frac{1}{p}-1}dt < \infty\}, \quad 1 \le p \le \infty.$
  Here f^* denotes the decreasing rearrangement of $f \in L_1 + L_\infty$.
- $L_{p,1}(\mathcal{M}) := \left\{a \in (L_1 + L_\infty)(\mathcal{M}) : \int_0^\infty \mu_a(t)t^{\frac{1}{p}-1}dt < \infty\right\},$
  with norm $\|a\|_{L_{p,1}(\mathcal{M})} := \frac{1}{p}\int_0^\infty \mu_a(t)t^{\frac{1}{p}-1}dt$ for any $a \in L_{p,1}(\mathcal{M})$.

Recall that:

Theorem (Voiculescu, 1979 & 2018)

Let $\mathcal{M} = B(\mathcal{H})$. Let $\Phi = \mathcal{C}_{p_1,1} \times \cdots \times \mathcal{C}_{p_n,1}$, where $\sum_{i=1}^n \frac{1}{p_i} = 1, 1 \le p_i < \infty, 1 \le i \le n.$
$k_\Phi(\alpha) = 0 \Leftrightarrow$ the spectral measure of α is singular.

Spectral measure of α is singular $\Leftrightarrow$
α is diagonal modulo $\mathcal{C}_{p_1,1} \times \cdots \times \mathcal{C}_{p_n,1}$.

Singularity implies vanishing of quasicentral modulus

Theorem 3.1 (Ber-Sukochev-Zanin-Zhao, 2022, under review)

Let $n \geq 1$*. Let* $\Phi(\mathcal{M}) = L_{p_1,1}(\mathcal{M}) \times \cdots \times L_{p_n,1}(\mathcal{M})$*,*
where $1 \leq p_i \leq \infty, 1 \leq i \leq n$ *and* $\frac{1}{p_1} + \cdots + \frac{1}{p_n} \leq 1$*.*
Let $\alpha \in (\mathcal{M}_{sa})^n$ *be a commuting self-adjoint* n*-tuple.*
The spectral measure of α *is singular* $\Rightarrow k_{\Phi(\mathcal{M})}(\alpha) = 0$*.*

Corollary

The spectral measure of α *is singular* $\Rightarrow k_{L_{n,1}(\mathcal{M})}(\alpha) = 0$*.*

The converse is not true, i.e.
$k_{L_{n,1}(\mathcal{M})}(\alpha) = 0 \not\Rightarrow$ the spectral measure of α is singular.

Proposition

Le $p_1, \ldots, p_L$ *be orthogonal projections in* $\mathcal{M}$ *s.t.* $p_1 + \cdots + p_L = \mathbf{1}$ *and* $p_l \alpha_l = \alpha_l p_l = \alpha_l$ *for any* $1 \leq l \leq L$. *Let* $\theta_l \in \mathbb{R}^n$, $1 \leq l \leq L$. *We have* $k_{\Phi(\mathcal{M})}(\sum_{l=1}^{L} \alpha_l) = k_{\Phi(\mathcal{M})}(\sum_{l=1}^{L} \alpha_l - \theta_l p_l))$.

Let $\alpha \in (\mathcal{M}_{sa})^n$ be a commuting self-adjoint n-tuple with singular spectral measure.

Proposition (Technical result)

Let $p, q \in \mathcal{P}(\mathcal{M})$ *s.t.* $\alpha p = \alpha, p \leq q$. *Suppose there exists a* τ*-finite projection* e *in* $\mathcal{M}$ *s.t.* $[W^*(\alpha)e(\mathcal{H})] = q(\mathcal{H})$. *We have*

$$k_{\Phi(\mathcal{M})}(\alpha) \leq c_\Phi \max_{1 \leq j \leq n} \tau(e)^{\frac{1}{p_j}} \cdot \|\alpha\|_{\mathcal{M}},$$

where c_Φ *is a constant depends only on* $p_1, \ldots, p_n$.

Theorem (Strong continuity)

Let $\{p_j\}_{j\geq 1}$ *be a sequence of projections in* $\mathcal{M}$ *s.t.* $p_j\alpha = \alpha p_j$ *and* $p_j \to \mathbf{1}$ *in strong operator topology. Then* $k_{\Phi(\mathcal{M})}(\alpha) = \lim_{j\to\infty} k_{\Phi(\mathcal{M})}(\alpha p_j)$.

Proposition

Suppose $\exists$ *a* τ*-finite projection* $e \in \mathcal{M}$ *s.t.* $\overline{\mathrm{span}\{W^*(\alpha)e(\mathcal{H})\}} = \mathcal{H}$. *Then* $k_{\Phi(\mathcal{M})}(\alpha) = 0$.

- $\mathscr{B}(\mathbb{R}^n) := \{\text{Borel sets in } \mathbb{R}^n\}$.
  $e^\alpha : \mathscr{B}(\mathbb{R}^n) :\to \mathcal{P}(\mathcal{M})$ be the spectral measure of α.
  Set $\mu_\xi(B) := \langle e^\alpha(B)\xi, \xi\rangle, B \in \mathscr{B}(\mathbb{R}^n)$.
  Separability of $\mathcal{H} \Rightarrow \exists$ a vector $\xi \in \mathcal{H}$ such that
  $\mu_\eta \prec \mu_\xi, \quad \forall \eta \in \mathcal{H}$.

Proof.

W.L.O.G, assume that $0 \leq \alpha \leq \mathbf{1}$.
Let $\xi \in \mathcal{H}$ s.t. $\mu_\eta \prec \mu_\xi, \forall \eta \in \mathcal{H}$.
μ_ξ is singular $\Rightarrow \exists B \in \sigma(\alpha)$, s.t. $\lambda(B) = \mu_\xi(\mathbb{R}^n \backslash B) = 0$.
For every $j \in \mathbb{N}$, $\exists$ disjoint cubes $\{A_{k,j} : 1 \leq k \leq n_j\}$ in $\mathbb{R}^n$ with same side length s.t.
$\mu_\xi([-2,2]^n \backslash \cup_{k=1}^{n_j} A_{k,j}) \to 0$ as $j \to \infty$ and
$\lambda(\cup_{k=1}^{n_j} A_{k,j}) \to 0$ as $j \to \infty$.
Then $\mu_\eta([-2,2]^n \backslash \cup_{k=1}^{n_j} A_{k,j}) \to 0$ as $j \to \infty, \forall \eta \in \mathcal{H}$.
i.e. $e^\alpha(\cup_{k=1}^{n_j} A_{k,j}) \overset{s.o.t.}{\to} \mathbf{1}$.
$\alpha_j := \alpha e^\alpha(\cup_{k=1}^{n_j} A_{k,j}) = \sum_{k=1}^{n_j} \alpha e^\alpha(A_{k,j})$
choose proper $c_{k,j}$, $A'_{k,j} := A_{k,j} - c_{k,j}$ so that $\{A'_{k,j}\}_{j=1}^{n_j}$ are disjoint and $\mathrm{diam}(\cup_{k=1}^{n_j} A'_{k,j}) \to 0$ as $j \to \infty$.
$\alpha'_j := \sum_{k=1}^{n_j} (\alpha - c_{k,j}\mathbf{1}) e^\alpha(A_{k,j})$.
$k_{\Phi(\mathcal{M})}(\alpha_j) = k_{\Phi(\mathcal{M})}(\alpha'_j) \leq c_\Phi \max_{1 \leq i \leq n} (\tau(e))^{\frac{1}{p_i}} \mathrm{diam}(\cup_{k=1}^{n_j} A'_{k,j}) \to 0$.
Strong continuity $\Rightarrow k_{\Phi(\mathcal{M})}(\alpha) = \lim_{j \to \infty} k_{\Phi(\mathcal{M})}(\alpha_j) = 0$. □

Proof of Theorem 3.1.

Let $W^*(\alpha)$ be the von Neumann subalgebra in $\mathcal{M}$ generated by $\mathbf{1}$ and α. By Zorn's Lemma, $\exists$ $\{e_k\}_{k\geq 1}$ of τ-finite projections s.t. $\sum_{k\geq 1} q_k = \mathbf{1}$ where

$$q_k = \bigvee_{a \in W^*(\alpha)} (l(ae_k)) = \bigvee_{B \in \mathscr{B}(\mathbb{R}^n)} l(\chi_B(\alpha)e_k).$$

e_k is a τ-finite cyclic projection of αq_k on $q_k(\mathcal{H}) \Rightarrow k_{\Phi(q_k \mathcal{M} q_k)}(\alpha q_k) = 0$.
Subadditivity of $k_{\Phi(\mathcal{M})} \Rightarrow k_{\Phi((\sum_{j=1}^k q_j)\mathcal{M}(\sum_{j=1}^k q_k))}(\alpha \sum_{j=1}^k q_k) = 0$.
Strong continuity of $k_{\Phi(\mathcal{M})} \Rightarrow k_{\Phi(\mathcal{M})}(\alpha) = 0$. □

Ongoing project – Extension of Kato-Rosenblum theorem to von Neumann algebras

Let $\alpha \in (\mathcal{M}_{sa})^n$ be a commuting self-adjoint n-tuple.

- A projection $P \in \mathcal{M}$ is called norm absolutely continuous w.r.t. α if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\sum_{j=1}^{k} \|PE_\alpha(Q_j)P\|_{\mathcal{M}} < \varepsilon$ whenever $\{Q_j\}_{j=1}^{k} \subset \mathfrak{B}(\mathbb{R}^n)$ are pairwise disjoint s.t. $\sum_{j=1}^{k} \lambda(Q_j) \leq \delta$.
- $\mathscr{P}_{ac}^{\infty}(\alpha) := \{P : P \text{ is norm absolutely continuous w.r.t. } \alpha\}$. $P_{ac}^{\infty}(\alpha) = \bigvee\{P : P \in \mathscr{P}_{ac}^{\infty}(\alpha)\}, P_{ac}^{\infty}(\alpha) \leq P_{ac}(\alpha)$.
- In some cases, $P_{ac}^{\infty}(\alpha)$ is totally different to $P_{ac}(\alpha)$, there is an example that $P_{ac}(T) = \mathbf{1}, P_{ac}^{\infty}(T) = 0$.

The following theorem extends [Li-Shen-Shi-Wang, 2018] to the case when $n \geq 1$.

Theorem (Ber-Sukochev-Zanin-Zhao, ongoing)

$\alpha, \beta \in (B(\mathcal{H})_{sa})^n, \beta - \alpha \in (L_1(\mathcal{M}))^n \Rightarrow \forall t \in \mathbb{S}^{n-1}, \exists$ *a limit*

$$W_t = \textit{s.o.t.-}\lim_{r\to\infty} e^{irt\beta} e^{-irt\alpha} P_{ac}^{\infty}(\alpha).$$

For almost every $t \in \mathbb{S}^{n-1}$,

1. $W_t^* W_t = P_{ac}^{\infty}(\alpha)$, $W_t W_t^* = P_{ac}^{\infty}(\beta)$.
2. $e^{\beta}(\Delta) W_t = W_t e^{\alpha}(\Delta), \Delta \in \mathfrak{B}(\mathbb{R}^n)$.

How far we've got?

- Can we define quasicentral modulus $k_{\Phi(\mathcal{M})}(\alpha)$ for $\alpha \in \mathcal{M}^n$? Yes
- Is is true that $k_{\Phi(\mathcal{M})}(\alpha) = 0 \Leftrightarrow \alpha$ is diagonal modulo Φ? Yes
- What can we say about α_s and α_{ac}?
  $k_{\Phi(\mathcal{M})}(\alpha_s) = 0$ for $\Phi(\mathcal{M}) = L_{p_1,1}(\mathcal{M}) \times \cdots \times L_{p_n,1}(\mathcal{M}), \sum_j \frac{1}{p_j} \leq 1$.
  $P_{ac}^{\infty}(\alpha)$ is preserved (up to equivalence) under trace class perturbations.
- Spectral multiplicity function does not work well for α_{ac} in von Neumann algebras.
  Let U be the corresponding unitary operator such that $U\alpha U^*$ is the tuple of multiplication operators of coordinate functions, the obstacle is that $U \notin \mathcal{M}$, so it is meaningless to calculate $k_{\Phi(\mathcal{M})}(U\alpha U^*)$.
  What is the proper analogue of spectral multiplicity theory in von Neumann algebras?
  We do not know yet.

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Thanks for your attention!