Analytic subalgebras of weighted Fourier algebras and complexification of Lie groups

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The Fourier algebra of a Lie group

- G: a Lie group with a fixed left Haar measure
- $\big(L^2(G), \langle \cdot\,,\,\cdot \rangle\big)$: the L^2-Hilbert space
- $\lambda : G \to \mathcal{B}\big(L^2(G)\big)$: the left regular representation

$$\big(\lambda(s)f\big)(t) = f(s^{-1}t), \quad s, t \in G$$

Definition (Fourier algebra)

The **Fourier algebra of** G is defined as

$$A(G) := \Big\{\big\langle f, \lambda(\,\cdot\,)g\big\rangle : f, g \in L^2(G)\Big\} \subseteq C_0(G).$$

- $A(G)$ is a subalgebra of $C_0(G)$ w.r.t. the pointwise operations.
- It beomces a Banach algebra with the norm

$$\|u\|_{A(G)} = \inf\Big\{\|f\|_2 \cdot \|g\|_2 : u = \big\langle f, \lambda(\,\cdot\,)g\big\rangle\Big\}, \quad u \in A(G).$$

The spectrum of $A(G)$

- The spectrum of an algebra A is defined as

$$\mathrm{Spec}A = \left\{0 \neq \chi : A \to \mathbb{C} \,\middle|\, \chi \text{ is an algebra homomorphism}\right\}.$$

- Each point $s \in G$ gives rise to an algebra homomorphism

$$\mathrm{ev}_s : A(G) \ni u \longmapsto u(s) \in \mathbb{C}.$$

Theorem (Eymard '1964)

When $\mathrm{Spec}A(G)$ is endowed with the weak-$*$ topology, the following map is a homeomorphism.

$$G \ni s \longmapsto \mathrm{ev}_s \in \mathrm{Spec}A(G)$$

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Motivation

- If A is a topological algebra and $\mathcal{A} \subseteq A$ is a dense subalgebra, $\mathcal{A}$ tends to have more "information" than A itself.
- (Example)
  M: a compact smooth manifold
  $A := C(M) \rightsquigarrow$ Topology of M
  $\mathcal{A} := C^\infty(M) \rightsquigarrow$ Topology + Smooth structure of M

Motivation

$A(G)$ has the information about the topology of G (Eymard's duality).

Q. Can we find some dense subalgebras of $A(G)$ which have more "information" about G than $A(G)$ itself?

The compact case

- G: comapct Lie group
- The space of matrix coefficients of G is

$$\mathrm{Pol}(G) := \left\{ \langle v, \pi(\,\cdot\,) w \rangle \;\middle|\; G \xrightarrow{\pi} GL(V) \text{ f.dim'l repn and } v, w \in V \right\}.$$

- By the Peter-Weyl theorem, $\mathrm{Pol}(G)$ is a dense subalgebra of $A(G)$.
- The spectrum of $\mathrm{Pol}(G)$?

Complexification of a compact connected Lie group

- G: a compact connected Lie group with Lie algebra $\mathfrak{g}$.

Theorem (Chevalley)

There exsits an embedding $G \hookrightarrow G_{\mathbb{C}}$ into a (unique) **complex Lie group** $G_{\mathbb{C}}$ such that:
for any Lie group homomorhpism $\pi : G \to H$ into a *complex* Lie group H, there exsits a unique **holomorphic homomorphism** $\tilde{\pi} : G_{\mathbb{C}} \to H$ s.t.

$$\begin{array}{ccc} G_{\mathbb{C}} & & \\ \uparrow & \overset{\tilde{\pi}}{\searrow} & \\ G & \xrightarrow[\pi]{} & H \end{array}$$

- The Lie algebra of $G_{\mathbb{C}}$ is given by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.
- The following map is a diffeomorphism **(Cartan Decomposition)**:

$$G \times \mathfrak{g} \ni (s, X) \longmapsto s \exp_{G_{\mathbb{C}}}(iX) \in G_{\mathbb{C}}$$

The spectrum of Pol(G)

- Let $u := \langle v, \pi(\,\cdot\,)w\rangle \in \mathrm{Pol}(G)$ with $\pi : G \to GL(n,\mathbb{C})$ a f.dim'l repn.
- By the universal property,

$$\begin{array}{ccc} G_{\mathbb{C}} & & \\ \uparrow & \overset{\exists!\tilde{\pi}}{\searrow} & \\ G & \xrightarrow{\ \pi\ } & GL(n,\mathbb{C}) \end{array}$$

- The map $\tilde{u} : G_{\mathbb{C}} \ni z \longmapsto \langle v, \tilde{\pi}(z)w\rangle \in \mathbb{C}$ is an extension of u.
- Every point $z \in G_{\mathbb{C}}$ gives rise to an algebra homomorphism

$$\mathrm{ev}_z : \mathrm{Pol}(G) \ni u \longmapsto \tilde{u}(z) \in \mathbb{C}.$$

Theorem

$$\mathrm{Spec}\big(\mathrm{Pol}(G)\big) = \{\mathrm{ev}_z : z \in G_{\mathbb{C}}\} \cong G_{\mathbb{C}}$$

Remarks

- So, the dense subalgebra $\mathrm{Pol}(G) \subseteq A(G)$ indeed has more "information" than $A(G)$ itself, namely **the complexification of the group**.
- However, if G is noncompact, then $\mathrm{Pol}(G)$ isn't that useful. (For example, it is not dense in $A(G)$.)

Motivation made precise

Let G be a noncompact Lie group. We seek to find other **dense subalgebras** of the Fourier algebra whose **spectra** can reveal the structure of the **complexification** of the group.

Complexification of Lie group

- G: connected Lie group with Lie algebra $\mathfrak{g}$

Definition (Complexification)

A complex Lie group $G_{\mathbb{C}}$ is called a **complexification** of G if

1. $G \subseteq G_{\mathbb{C}}$
2. The Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$
3. G is the connected subgroup of $G_{\mathbb{C}}$ corresponding to the subalgebra $\mathfrak{g} \leq \mathfrak{g}_{\mathbb{C}}$

- (Example)

$$\mathbb{T}_{\mathbb{C}} = \mathbb{C}^{\times}, \quad SU(2)_{\mathbb{C}} = SL(2, \mathbb{C}), \quad \mathbb{R}_{\mathbb{C}} = \mathbb{C}, \quad SL(2, \mathbb{R})_{\mathbb{C}} = SL(2, \mathbb{C})$$

- Not every connected Lie group possesses a complexification. (e.g., the double cover of $SL(2, \mathbb{R})$)

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Weighted Fourier algebras (Giselsson/Turowska '22)

- W is called a **weight of the Lie group** G if
  1. W is a positive (unbounded) operator on $L^2(G)$
  2. W is invertible and $W^{-1} \in VN(G)_+$
  3. $W^{-2} \otimes W^{-2} \leq \Gamma(W^{-2})$

  where Γ is **the comultiplication** $\Gamma : VN(G) \to VN(G)\overline{\otimes}VN(G)$.

Definition (Weighted Fourier algebras)

The weighted Fourier algebra with weight W is defined as

$$A(G, W) := \left\{ \langle f, \lambda(\,\cdot\,)g\rangle : f \in L^2(G),\ g \in \mathcal{D}(W) \right\} \subseteq A(G).$$

- $A(G, W)$ is a dense subalgebra of $A(G)$.
- It is a Banach algebra with the norm

$$\left\| \langle f, \lambda(\,\cdot\,)g\rangle \right\|_{A(G,W)} := \left\| \langle f, \lambda(\,\cdot\,)Wg\rangle \right\|_{A(G)}$$

The spectrum of $A(G, W)$: the compact case

Ludwig/Spronk/Turowska '12

- G: compact connected Lie group
- For any weight W of G, the following *dense* inclusions hold.

$$\mathrm{Pol}(G) \subseteq A(G, W) \subseteq A(G)$$

- Hence, for any weight W,

$$G \cong \mathrm{Spec} A(G) \subseteq \mathrm{Spec} A(G, W) \subseteq \mathrm{Spec}\mathrm{Pol}(G) \cong G_{\mathbb{C}}.$$

- $G_{\mathbb{C}}$ is covered by the spectra of weighted Fourier algebras. I.e.,

$$G_{\mathbb{C}} = \bigcup_{W} \mathrm{Spec} A(G, W).$$

The spectrum of $A(G, W)$: a few noncompact cases

Ghandehari/Lee/Ludwig/Spronk/Turowska, '22

- $G = \mathbb{H}^3, \mathbb{H}^3_r, E(2)$, or $\tilde{E}(2)$
- $^\exists \mathcal{A} \subseteq A(G)$ a dense subalgebra s.t.
  1. Every $u \in \mathcal{A}$ admits a holomorphic extension to $G_{\mathbb{C}}$.
  2. The following correspondence is a bijection.
  $$G_{\mathbb{C}} \ni z \longmapsto \mathrm{ev}_z \in \mathrm{Spec}\mathcal{A}$$
  3. For some weights W, the *dense* inclusions $\mathcal{A} \subseteq A(G, W) \subseteq A(G)$ hold.
- Thus, $G \cong \mathrm{Spec}A(G) \subseteq \mathrm{Spec}A(G, W) \subseteq \mathrm{Spec}\mathcal{A} \cong G_{\mathbb{C}}$.
- $G_{\mathbb{C}}$ is covered by the spectra of weighted Fourier algebras. I.e.,
  $$G_{\mathbb{C}} = \bigcup_W \mathrm{Spec}A(G, W).$$

A limitation

- The definition of $\mathcal{A}$ in this work was **highly dependent on the representation theory of each group**.
- As a result, it could not be generalized to more general class of Lie groups.

The spectrum of $A(G, W)$: general case

L./Lee, '24

- G: *any* connected Lie group which has a complexification $G_{\mathbb{C}}$
- We constructed a family of subalgebras $\mathcal{A}_r \subseteq A(G)$ $(0 < r \leq \infty)$, called **analytic subalgebras** with the following properties:

  There exists $0 < R \leq \infty$ such that for all $0 < r \leq R$,

  1. Every $u \in \mathcal{A}_r$ admits a holomorphic extension to a neighborhood G_r in $G_{\mathbb{C}}$ containing G.
  2. The following correspondence is an injection.
  $$G_r \ni z \longmapsto \mathrm{ev}_z \in \mathrm{Spec}\mathcal{A}_r$$
  3. For *a class of* weights W depending on r, the following *dense* inclusions hold.
  $$\mathcal{A}_r \subseteq A(G, W) \subseteq A(G)$$

The spectrum of $A(G, W)$: general case

L./Lee, '24

- For this class of weights W,

$$\mathrm{Spec} A(G, W) \subseteq G_r \subseteq \mathrm{Spec} \mathcal{A}_r.$$

- $G_r = \bigcup^!_W \mathrm{Spec} A(G, W)$.
- If G is simply-connected nilpotent, we can choose $R = \infty$ and thus

$$G_{\mathbb{C}} = \bigcup_{r>0} G_r = \bigcup^!_W \mathrm{Spec} A(G, W).$$

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The construction of analytic subalgebras

Motivation

- When G is compact, each $u = \langle v, \pi(\,\cdot\,)w\rangle \in \mathrm{Pol}(G)$ admits a holomorphic extension

$$G_{\mathbb{C}} \ni s\exp_G(iX) \longmapsto \langle v, \tilde{\pi}(s\exp_G(iX))w\rangle = \langle v, \pi(s)e^{i\pi_* X}w\rangle \in \mathbb{C}.$$

- Is there a subalgebra $\mathcal{A} \subseteq A(G)$ consisting of $u = \langle f, \lambda(\,\cdot\,)g\rangle \in A(G)$ s.t. the following expression makes sense?

$$G \times \mathfrak{g} \ni (s, X) \longmapsto \langle f, \lambda(s)e^{i\partial\lambda(X)}g\rangle \in \mathbb{C}$$

Here, $\partial\lambda(X)$ is **the infinitesimal generator** of the one-parameter group of unitaries $\mathbb{R} \ni t \longmapsto \lambda\big(\exp_G(tX)\big) \in U\big(L^2(G)\big)$. E.g.,

$$\Big(\partial\lambda(X)g\Big)(s) = \frac{d}{dt}\Big|_{t=0} g\big(\exp_G(-tX)s\big), \quad g \in C_c^\infty(G) \subseteq L^2(G).$$

The construction of analytic subalgebras

- G: connected Lie group which has a complexification $G_{\mathbb{C}}$
- For $0 < r \leq \infty$, define

$$\mathcal{H}_r^a := \left\{ g \in L^2(G) : E_s(g) < \infty,\ 0 <^{\forall} s < r \right\}$$

where $E_s : L^2(G) \to [0, \infty]$ is defined as, for $g \in L^2(G)$,

$$E_s(g) := \sum_{n=0}^{\infty} \frac{s^n}{n!} \left(\sum_{1 \leq j_1, \cdots, j_n \leq d} \|\partial\lambda(X_{j_1}) \cdots \partial\lambda(X_{j_n}) g\|_2^2 \right)^{\frac{1}{2}}.$$

- There exists $0 < R \leq \infty$ such that $\mathcal{H}_r^a$ is dense in $L^2(G)$ for all $0 < r \leq R$ (Nelson, 1959).

The construction of analytic subalgebras

- Let $\mathfrak{g}$ be the Lie algebra of G with a basis $\{X_1, \cdots, X_d\}$. Define a norm $|\cdot| : \mathfrak{g} \to [0, \infty)$ by

$$\left|a_1 X_1 + \cdots + a_d X_d\right| = \left(\sum_{j=1}^{d} a_j^2\right)^{\frac{1}{2}}, \quad a_j \in \mathbb{R}$$

and denote $\mathfrak{g}_r := \{X \in \mathfrak{g} : |X| < r\}$.

Proposition

For each $f \in \mathcal{H}_r^a$, the map

$$\mathfrak{g}_r \ni X \longmapsto e^{i\partial\lambda(X)} f \in L^2(G)$$

is well-defined.

The construction of analytic subalgebras

Definition (Analytic subalgebras)

Fix $0 < r \leq \infty$. Let

$$\mathcal{A}'_r := \left\{ \langle f, \lambda(\,\cdot\,)g \rangle : f \in L^2(G),\ g \in \mathcal{H}^a_r \right\} \subseteq A(G).$$

Its *completion*, denoted as $\mathcal{A}_r$, w.r.t. *a certain locally convex topology* becomes a subalgebra of $A(G)$, called the **analytic subalgebra of** $A(G)$ **with radius** r.

- Each element $u = \langle f, \lambda(\,\cdot\,)g \rangle \in \mathcal{A}'_r$ with $g \in \mathcal{H}^a_r$ admits an extension

$$G \times \mathfrak{g}_r \ni (s, X) \longmapsto \langle f, \lambda(s) e^{i\partial\lambda(X)} g \rangle \in \mathbb{C}.$$

* Why completion? For all $f, f' \in L^2(G)$ and $g, g' \in \mathcal{H}^a_r$,

$$\langle f, \lambda(\,\cdot\,)g \rangle \langle f', \lambda(\,\cdot\,)g' \rangle = \int_G \langle F_t, \lambda(\,\cdot\,)G_t \rangle dt$$

where $F_t(\cdot) = f(\cdot\, t)f'(\cdot) \in L^2(G)$, $G_t(\cdot) = g(\cdot\, t)g'(\cdot) \in \mathcal{H}^a_r$.

The spectrum of $\mathcal{A}_r$: holomorphic evaluations

L./Lee '24

There exists $0 < R \leq \infty$ such that for all $0 < r \leq R$,

- The following subset is a neighborhood of G in $G_{\mathbb{C}}$.

$$G_r := \left\{ s \exp_{G_{\mathbb{C}}}(iX) \in G_{\mathbb{C}} : s \in G,\ X \in \mathfrak{g}_r \right\}$$

- $u = \langle f, \lambda(\,\cdot\,) g \rangle \in \mathcal{A}'_r$ with $g \in \mathcal{H}^a_r$ admits a (unique) holomorhpic extension to G_r given by

$$\tilde{u} : G_r \ni s \exp_{G_{\mathbb{C}}}(iX) \longmapsto \langle f, \lambda(s) e^{i\partial\lambda(X)} g \rangle \in \mathbb{C}.$$

- Each element of $\mathcal{A}_r$ admits a (unique) holomorphic extension to G_r.
- Thus, each element $z \in G_r$ gives rise to a homomorphism $\mathrm{ev}_z : \mathcal{A}_r \ni u \mapsto \tilde{u}(z) \in \mathbb{C}$ and we get an embedding

$$G_r \ni z \longmapsto \mathrm{ev}_z \in \mathrm{Spec}\mathcal{A}_r.$$

The spectra of some weighted Fourier algebras

L./Lee '24

- For each $X \in \mathfrak{g}$, the operator $e^{|\partial\lambda(X)|}$ is a weight of G.
- For all $0 < r \leq R$,
$$\mathcal{A}_r \subseteq A\big(G, e^{|\partial\lambda(X)|}\big)$$
densely for all $X \in \mathfrak{g}_r$.
- Hence, for all $X \in \mathfrak{g}_r$, we get an embedding provided by the restriction map
$$\mathrm{Spec} A\big(G, e^{|\partial\lambda(X)|}\big) \hookrightarrow \mathrm{Spec}\mathcal{A}_r.$$

The spectra of some weighted Forier algebras

L./Lee '24

- In this identification,

$$\mathrm{Spec}A\big(G, e^{|\partial\lambda(X)|}\big) \cong \big\{s\exp_{G_{\mathbb{C}}}(itX) : s \in G,\ -1 \le t \le 1\big\} \subseteq G_r \subseteq \mathrm{Spec}\mathcal{A}_r$$

  for all $X \in \mathfrak{g}_r$.

- Hence,

$$G_r = \bigcup_{X \in \mathfrak{g}_r} \mathrm{Spec}A\big(G, e^{|\partial\lambda(X)|}\big).$$

- If G is simply-connected nilpotent, we can choose $R = \infty$ and thus

$$G_{\mathbb{C}} = \bigcup_{X \in \mathfrak{g}} \mathrm{Spec}A\big(G, e^{|\partial\lambda(X)|}\big).$$

Restrictions on R

- Here, I collect some issues that impose restrictions on the choice of $0 < R \leq \infty$ such that the above statements hold.
  1. $\mathcal{H}_R^a$ must be dense in $L^2(G)$.
  2. The following map is a diffeomorphism.
  $$G \times \mathfrak{g}_R \ni (s, X) \longmapsto s \exp_{G_{\mathbb{C}}}(iY) \in G_R$$
  3. There exists a neighborhood $0 \in U \subseteq \mathfrak{g}$ such that for all $X \in U$ and $Y \in \mathfrak{g}_R$,
  $$\exp_{G_{\mathbb{C}}}(X) \exp_{G_{\mathbb{C}}}(iY) = \exp_{G_{\mathbb{C}}} \left(\Phi(X, iY)\right)$$
  holds where $\Phi : U \times i\mathfrak{g}_R \to \mathfrak{g}_{\mathbb{C}}$ is given by the Baker-Campbell-Hausdorff formula.
- If G is simply-connected nilpotent, all these conditions are satisfied for $R = \infty$.

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Generalization to compact quantum groups

- $C(\mathbb{G})$: compact quantum group.
- The Fourier algebra of $\mathbb{G}$ is defined as

$$A(\mathbb{G}) := \left\{ VN(\mathbb{G}) \ni T \mapsto \langle \xi, T\eta \rangle \;\middle|\; \xi, \eta \in L^2(\mathbb{G}) \right\} = VN(\mathbb{G})_*.$$

- The definition of weight carries over to the quantum case. So,

$$A(\mathbb{G}, W) := \left\{ \langle \xi, (\cdot)\eta \rangle \middle| \xi \in L^2(\mathbb{G}),\ \eta \in \mathcal{D}(W) \right\}.$$

- The following is the matrix coefficients algebra:

$$\mathrm{Pol}(\mathbb{G}) := \left\{ \langle v, \pi(\cdot) w \rangle \middle| G \xrightarrow{\pi} GL(V) \text{ f. dim'l repn and } v, w \in V \right\}$$

- For any weight W of $\mathbb{G}$, the following *dense* inclusions hold.

$$\mathrm{Pol}(\mathbb{G}) \subseteq A(\mathbb{G}, W) \subseteq A(\mathbb{G})$$

Generalization to compact quantum groups

- Hence, for any weight W of $\mathbb{G}$,

$$\mathbb{G} \cong \mathrm{Spec}A(\mathbb{G}) \subseteq \mathrm{Spec}A(\mathbb{G}, W) \subseteq \mathrm{Spec}\mathrm{Pol}(\mathbb{G}) \cong \mathbb{G}_{\mathbb{C}}.$$

- And

$$\mathbb{G}_{\mathbb{C}} = \bigcup_{W} \mathrm{Spec}A(\mathbb{G}, W).$$

Problems

- $A(\mathbb{G})$ being noncommutative, $\mathrm{Spec}A(\mathbb{G})$ doesn't give us useful information about the quantum group $\mathbb{G}$.
- What would be **"the complexification of $\mathbb{G}$"?**

The case $SU_q(2)$

- $C(SU_q(2))$ is the universal C^*-algebra generated by the generators α, γ and the relations

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2\big(C(SU_q(2))\big) \text{ is a unitary matrix.}$$

- The "quantum double" of $SU_q(2)$, which, as a C^*-algebra, is

$$C_0(SL_q(2,\mathbb{C})) := C(SU_q(2)) \otimes c_0(\widehat{SU_q(2)}).$$

  It was introduced in [Podleś & Woronowicz '1990] and is widely considered as **"the complexification of $SU_q(2)$"**.
- Can we recover the following set from $A(SU_q(2), W)$ for some classes of weights W?

$$\begin{aligned} &\mathrm{sp}\, C_0(SL_q(2,\mathbb{C})) \\ &\quad := \{[\pi] : C_0(SL_q(2,\mathbb{C})) \xrightarrow{\pi} \mathcal{B}(\mathcal{H}) \text{ is an irreducible } *\text{-repn}\} \end{aligned}$$

The case $SU_q(2)$

- The set

$$C(SL_q(2,\mathbb{C})) \\ := \{\text{The ``unbounded elements affiliated to } C_0(SL_q(2,\mathbb{C}))\}$$

  is a $*$-algebra that contains $C_0(SL_q(2,\mathbb{C}))$ as a $*$-subalgebra.
- There is an **algebra embedding**

$$i : \mathrm{Pol}(SU_q(2)) \hookrightarrow C(SL_q(2,\mathbb{C})).$$

- Every $\pi \in \mathrm{sp}\, C_0(SL_q(2,\mathbb{C}))$ extends to a $*$-representation $\tilde{\pi} : C(SL_q(2,\mathbb{C})) \to \mathcal{B}(\mathcal{H}_\pi)$, inducing an **algebra representation**

$$\varphi_\pi : \mathrm{Pol}(SU_q(2)) \xrightarrow{i} C(SL_q(2,\mathbb{C})) \xrightarrow{\tilde{\pi}} \mathcal{B}(\mathcal{H}_\pi).$$

The case $SU_q(2)$

- Fix a weight W on $SU_q(2)$. We say that $\pi \in \operatorname{sp} C_0(SL_q(2,\mathbb{C}))$ is W-**extendible** if

$$\begin{array}{ccc} A(SU_q(2), W) & & \\ \uparrow & \searrow^{\exists \tilde{\varphi}_\pi} & \\ \mathrm{Pol}(SU_q(2)) & \xrightarrow[\varphi_\pi]{} & \mathcal{B}(\mathcal{H}_\pi). \end{array}$$

  such that $\tilde{\varphi}_\pi$ is completely-bounded.

Franz/Lee '21

For any weight W on $SU_q(2)$,

$$\operatorname{sp} C(SU_q(2)) \subseteq \{\pi \in \operatorname{sp} C_0(SL_q(2,\mathbb{C})) : \pi \ W\text{-extendible}\} \subseteq \operatorname{sp} C_0(SL_q(2,\mathbb{C}))$$

and

$$\operatorname{sp} C_0(SL_q(2,\mathbb{C})) = \bigcup_W \{\pi \in \operatorname{sp} C_0(SL_q(2,\mathbb{C})) : \pi \ W\text{-extendible}\}.$$

The general case

- Let K be a compact semisimple Lie group and G its complexification.
- One can analogously define the compact quantum group $C(K_q)$ and its "complexification" $C_0(G_q)$.
- There is an algebra embedding $i : \mathrm{Pol}(K_q) \to C(G_q)$ and every $\pi \in \mathrm{sp}\, C_0(G_q)$ extends to $C(G_q)$, inducing an **algebra representation**

$$\varphi_\pi : \mathrm{Pol}(K_q) \xrightarrow{i} C(G_q) \xrightarrow{\tilde{\pi}} \mathcal{B}(\mathcal{H}_\pi).$$

The general case

- Fix a weight W on K_q. We say that $\pi \in \mathrm{sp}\, C_0(G_q)$ is W-**extendible** if

$$\begin{array}{ccc} A(K_q, W) & & \\ \uparrow & \searrow^{\exists \tilde{\varphi}_\pi} & \\ \mathrm{Pol}(K_q) & \xrightarrow[\varphi_\pi]{} & \mathcal{B}(\mathcal{H}_\pi). \end{array}$$

  such that $\tilde{\varphi}_\pi$ is completely-bounded.

L./Voigt '24

For any weight W on K_q,

$$\mathrm{sp}\, C(K_q) \subseteq \{\pi \in \mathrm{sp}\, C_0(G_q) : \pi \ W\text{-extendible}\} \subseteq \mathrm{sp}\, C_0(G_q)$$

and

$$\mathrm{sp}\, C_0(G_q) = \bigcup_W \{\pi \in \mathrm{sp}\, C_0(G_q) : \pi \ W\text{-extendible}\}.$$

Thank you for your attention