Perturbation theory of commuting self-adjoint operators and related topics. Part I

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Preliminaries

- $\mathcal{H}$: separable Hilbert space, $\dim \mathcal{H} = \infty$.
- $T : \mathcal{H} \to \mathcal{H}$ linear operator.
  $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$, operator norm.
  T is bounded if and only if $\|T\| < \infty$.
  $B(\mathcal{H})$: the set of bounded operators on $\mathcal{H}$.
- For $T \in B(\mathcal{H})$, its adjoint $T^* \in B(\mathcal{H})$ is defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

$$B(\mathcal{H})_{sa} := \{T \in B(\mathcal{H}), T = T^*\}.$$

- Spectrum of T, $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - T \text{ is not invertible}\}$.
  For $A \in B(\mathcal{H})_{sa}$, $\sigma(A) \subset \mathbb{R}$.

Example

$\mathcal{H} = L^2([a,b]),\ Af(x) = xf(x),$
$A \in B(\mathcal{H})_{sa}$ *and* $\sigma(A) = [a,b]$.

Example

$\mathcal{H} = \ell_2 = \{(x_n)_{n\geq 1} : \sum_{n\geq 1} |x_n|^2 < \infty\},$
$(b_n)_{n\geq 1}$ *a bounded real sequence,*
$B(x_n) = (b_n x_n),\ B \in B(\mathcal{H})_{sa},\ \sigma(B) = \overline{(b_n)_{n\geq 1}}.$

- $T \in B(\mathcal{H})$ is diagonal $\overset{\text{def}}{\Leftrightarrow} \exists\{e_n\}_{\geq 1}$ C.O.N.S of $\mathcal{H}$ s.t. $Te_n = \lambda_n e_n$.
  B is diagonal, but A is not diagonal.

- $\mathcal{K}(\mathcal{H}) :=$ the set of compact operators of $\mathcal{H}$
  $= \{T \in B(\mathcal{H}) : \{Tx : \|x\| \leq 1\}$ is relatively compact$\}$.
- For $T \in B(\mathcal{H})$,
  $\|T\|_2 := \mathrm{Tr}(T^*T)^{1/2} \in [0, \infty]$,
  $\mathcal{C}_2 := \{T \in B(\mathcal{H}) : \|T\|_2 < \infty\} \subset \mathcal{K}(\mathcal{H})$. Hilbert-Schmidt class.

Theorem (Weyl-von Neumann, 1909, 1935)

$\forall A \in B(\mathcal{H})_{sa}, \forall \varepsilon > 0, \exists K \in \mathcal{C}_2 \cap B(\mathcal{H})_{sa}$ *s.t.*
$B = A + K$ *is diagonal and* $\|K\|_2 < \varepsilon$.

Spectral theorem and Lebesgue decomposition

- Every $A \in B(\mathcal{H})_{sa}$ is unitarily equivalent to

$$(\mathcal{H}, A) \backsimeq \oplus_i (L^2(\sigma(A), \mu_i), M_i)$$

  $M_i h(x) = x h(x)$ for $h \in L^2(\sigma(A), \mu_i)$.
- For $f : \sigma(A) \to \mathbb{C}$ bounded Borel function, define
  $f(A) = \oplus_i M_f | L^2(\sigma(A), \mu_i)$
  where M_f is the multiplication operator by f.
- Lebesgue decomposition
  $\mu_i = \mu_{i,a} + \mu_{i,s}$.
  $\mu_{i,a}$ absolutely continuous w.r.t. dx, i.e. $d\mu_{i,a} = \frac{d\mu_{i,a}}{dx} dx$.
  $\mu_{i,s}$ singular w.r.t. dx
  $L^2(\sigma(A), \mu_i) = L^2(\sigma(A), \mu_{i,a}) \oplus L^2(\sigma(A), \mu_{i,s})$
  $(\mathcal{H}, A) = (\mathcal{H}_{ac}, A_{ac}) \oplus (\mathcal{H}_s, A_s)$

- $\mathcal{C}_1 := \{T \in B(\mathcal{H}) : \mathrm{Tr}(|T|) < \infty\}$ trace class, here $|T| = \sqrt{T^*T}$.
  $\mathcal{C}_1 \subsetneqq \mathcal{C}_2 \subsetneqq \mathcal{K}(\mathcal{H})$

Theorem (Kato-Rosenblum, 1957)

$A, B \in B(\mathcal{H})_{sa}, A - B \in \mathcal{C}_1 \Rightarrow A_{ac}$ *and* B_{ac} *are unitarily equivalent.*

Normed ideals

- $T \in K(\mathcal{H})$.
  $\{\mu_k(T)\}_{k=1}^{\infty}$:= the singular values of T, i.e.
  the eigenvalues of $|T|$ in decreasing order.
- $\widehat{c} := \{(a_k)_{k=1}^{\infty} : a_k = 0$ for large $k\}$.
  A norm $\Phi : \widehat{c} \to [0, \infty)$ is symmetric $\overset{\text{def}}{\Leftrightarrow} \Phi(a) = \Phi(a^*)$,
  $(a_k^*)_{k=1}^{\infty}$ is the decreasing rearrangement of $(|a_k|)_{k=1}^{\infty}$.
- For $T \in K(\mathcal{H})$, define
  $\|T\|_\Phi := \lim_{k\to\infty} \Phi(\mu_1(T), \mu_2(t) \ldots, \mu_k(t), 0, 0, \ldots) \in [0, \infty]$.
  $S_\Phi := \{T \in K(\mathcal{H}) : \|T\|_\Phi < \infty\}$.
  $(S_\Phi, \|\cdot\|_\Phi)$ is a Banach space, called the (symmetric) normed ideal.
  $S_\Phi^{(0)} := \overline{F(\mathcal{H})}^{\|\cdot\|_\Phi}$, where $F(\mathcal{H})$ is the set of finite rank operators.

Normed ideals

Example

Schatten-von Neumann classes.
$1 \leq p < \infty$, $\Phi_p(a) = (\sum_{k\geq 1} |a_n|^p)^{\frac{1}{p}}$.
$\|T\|_p := \|T\|_{\Phi_p} = (\mathrm{Tr}(|T|^p))^{\frac{1}{p}}$.
$\mathcal{C}_p := S_{\Phi_p} = \{T \in K(\mathcal{H}) : \|T\|_p < \infty\} \subset K(\mathcal{H})$.
$p_1 < p_2 \Rightarrow \mathcal{C}_{p_1} \subsetneqq \mathcal{C}_{p_2}$.

Example

$(p, 1)$-*Lorentz ideals.*
$1 \leq p \leq \infty$, $\Phi_{p,1}(a) = \sum_{k=1}^{\infty} \frac{a_k^*}{k^{1-\frac{1}{p}}}$.
$\|T\|_{p,1} := \|T\|_{\Phi_{p,1}}$.
$\mathcal{C}_{p,1} := \{T \in K(\mathcal{H}) : \|T\|_{p,1} < \infty\}$
$\cup_{r<p}\mathcal{C}_r \subsetneqq \mathcal{C}_{p,1} \subsetneqq \mathcal{C}_p, \mathcal{C}_{1,1} = \mathcal{C}_1$.

The trace class $\mathcal{C}_1$ is the smallest normed ideal.

Theorem (Kuroda '58)

If a symmetric Φ is not equivalent to $\|\cdot\|_1$, the Weyl-von Neumann theorem holds w.r.t. $S_\Phi^{(0)}$.

What happens in the case of normal operators?

T is normal $\overset{\text{def}}{\Leftrightarrow} T^*T = TT^*$

$\Leftrightarrow T = A + iB, A, B \in B(\mathcal{H})_{sa}, AB = BA.$

Theorem

The Weyl-von Neumann theorem for normal operators is true w.r.t.

1. $K(\mathcal{H})$ *(Berg '71)*
2. $\mathcal{C}_2$ *(Voiculescu '79)*

Approximately equivalence of representations

- Diagonality modulo compact operators is related to the approximately equivalence of representations.
- Let $\mathcal{H}, \mathcal{L}$ be two Hilbert spaces. Suppose $\mathcal{A} \subset B(\mathcal{H})$ is a unital separable C^*-algebra and $\pi, \psi : \mathcal{A} \to B(\mathcal{L})$ be unital representations. We say that π is approximately equivalent to ψ,
denoted by $\pi \sim_{K(\mathcal{L})} \psi$,
if $\exists$ a sequence of unitaries $(U_k)_{k\geq 1} \subset B(\mathcal{L})$ s.t.
    1. $\pi(A) - U_k\psi(A)U_k^* \in K(\mathcal{L}), \quad \forall A \in \mathcal{A},\ k \geq 1.$
    2. $\lim_{k\to\infty} \|\pi(A) - U_k\psi(A)U_k^*\| = 0, \quad \forall A \in \mathcal{A}.$

Voiculescu's theorem

Theorem (Voiculescu's theorem, 1976)

Suppose $\mathcal{A} \subset B(\mathcal{H})$ *is a unital separable* C^**-algebra and* $\rho : \mathcal{A} \to B(\mathcal{L})$ *is a unital representation s.t.* $\rho(\mathcal{A} \cap K(\mathcal{H})) = 0$.
$\exists$ *a sequence of isometries* $V_k : \mathcal{K} \to \mathcal{H}$ *s.t.*
$V_k \rho(A) - AV_k$ *is compact,* $\quad \forall k \geq 1,\ \forall A \in \mathcal{A}$,
and $\lim_{k\to\infty} \|V_k \rho(A) - AV_k\| = 0, \quad \forall A \in \mathcal{A}$.

Corollary

Suppose $\rho : \mathcal{A} \to B(\mathcal{L})$ *is a unital representation s.t.* $\rho(\mathcal{A} \cap \mathcal{K}(H)) = 0$.
Then $\mathrm{id} \sim_{\mathcal{K}} \mathrm{id} \oplus \rho$.

Quasicentral modulus

$\alpha = (A_1, \ldots, A_n) \in B(\mathcal{H})^n, A \in B(\mathcal{H})$.
$\beta = (B_1, \ldots, B_n)$.
$\alpha + \beta := (A_1 + B_1, \ldots, A_n + B_n)$.
$[A, \alpha] := ([A, A_1], \ldots, [A, T_n])$.
$\|\alpha\|_\Phi := \max_{1 \le j \le n} \|A_j\|_\Phi$.

Definition

For $\alpha \in B(\mathcal{H})^n$ and a symmetric norm Φ,

$$k_\Phi(\alpha) := \inf\{\liminf_{k\to\infty} \|[A_k, \alpha]\|_\Phi : \mathrm{rank}(A_k) < \infty, 0 \le A_k, A_k \uparrow \mathbf{1}\}.$$

$k_p(\alpha) := k_{\Phi_p}(\alpha)$.
$k_{p,1}(\alpha) := k_{\Phi_{p,1}}(\alpha)$.

Theorem (Voiculescu '79)

For every commuting $\alpha \in (B(\mathcal{H})_{sa})^n$ *and a symmetric norm* Φ, *T.F.A.E.*

1. $k_\Phi(\alpha) = 0$.
2. $\exists$ *commuting diagonal* $\beta \in (B(\mathcal{H})_{sa})^n$ *s.t.* $\alpha - \beta \in (S_\Phi^{(0)})^n$.
3. $\forall \varepsilon > 0$, $\exists$ *diagonal commuting* $\beta \in (B(\mathcal{H})_{sa})^n$ *s.t.* $\alpha - \beta \in (S_\Phi^{(0)})^n$ *and* $\|\alpha - \beta\|_\Phi < \varepsilon$.

Theorem (Voiculescu '79)

Assume $\alpha \in B(\mathcal{H})^n$,

1. *For* $1 < p < \infty$, $k_p(\alpha)$ *is either* 0 *or* ∞.
2. *If* α *is a commuting self-adjoint* n*-tuple with* $n \geq 2$, *then* $k_n(\alpha) = 0$.

- Question: k_p is not sharp, how about other normed ideals?

Spectral multiplicity theory

For a commuting $\alpha \in (B(\mathcal{H})_{sa})^n$,

- $\sigma(\alpha) \subset \mathbb{R}^n$, here $\sigma(\alpha)$ is the support of the spectral measure of α.
  $(\mathcal{H}, \alpha) \simeq \oplus_i (L^2(\sigma(\alpha), \mu_i), \alpha_i)$
  $\alpha_i = (M_{x_1}, M_{x_2} \dots, M_{x_n})$ on $L^2(\sigma(\alpha), \mu_i)$.
- $(\mathcal{H}, \alpha) \simeq (\mathcal{H}_{ac}, \alpha_{ac}) \oplus (\mathcal{H}_s, \alpha_s)$ w.r.t. the n-dimensional Lebesgue measure λ_n.
- $(\mathcal{H}_{ac}, \alpha_{ac}) \simeq \oplus_{k=1}^{\infty} (L^2(X_k, \lambda_n), \alpha_k)$
  $\sigma(\alpha) = X_0 \supset X_1 \supset X_2 \supset \cdots$
  The mutiplicity function $m : \sigma(\alpha) \to \{0, 1, 2, \dots, \infty\}$ is defined by

$$m(x) := \begin{cases} k & \text{if } x \in X_k \setminus X_{k+1} \\ \infty & \text{if } x \in \cap_{k=1}^{\infty} X_k. \end{cases}$$

Theorem (Voiculescu '79)

$\exists$ *a universal constant* $0 < \gamma_n < \infty$ *s.t.* $\forall$ *commuting* $\alpha \in (B(\mathcal{H})_{sa})^n$, $(k_{n,1}(\alpha))^n = \gamma_n \int_{\sigma(\alpha)} m(x) d\lambda_n(x)$.

$\gamma_1 = \frac{1}{\pi}, \gamma_n$ for $n \geq 2$ is unknown.

Corollary (Voiculescu '79)

$\forall \alpha \in (B(\mathcal{H})_{sa})^n$, $\alpha = \alpha_s \Leftrightarrow k_{n,1}(\alpha) = 0 \Leftrightarrow \exists$ *commuting diagonal* $\beta \in (B(\mathcal{H})_{sa})^n$ *s.t.* $\alpha - \beta \in (\mathcal{C}_{n,1})^n$.

- A normed ideal S_Φ is called an n-diagonalization ideal if $\forall$ commuting self-adjoint n-tuple α is diagonal modulo $(S_\Phi)^n$.
  A normed ideal which is not an n-diagonalization ideal will be called an n-obstruction ideal.

Theorem (Bercovici-Voiculescu '89)

Let $n \geq 1$. A symmetric normed ideal S_Φ is an n-obstruction ideal if and only if $S_\Phi \subset \mathcal{C}_{n,1}$.

Generalizations to hybrid normed ideals

Theorem (Voiculescu, 2018)

Let $n \geq 1$. *For every commuting self-adjoint* $\alpha \in (B(\mathcal{H})_{sa})^n$ *and symmetric normed ideals* $S^{(0)}_{\Phi_1}, \ldots, S^{(0)}_{\Phi_n}$, $\Phi = S^{(0)}_{\Phi_1} \times \cdots \times S^{(0)}_{\Phi_n}$, *T.F.A.E.*

1. $k_{\Phi}(\alpha) = 0$.
2. $\forall \varepsilon > 0$, $\exists$ *commuting diagonal* $\delta \subset (B(\mathcal{H})_{sa})^n$ *s.t.* $\alpha - \delta \in S^{(0)}_{\Phi_1} \times \cdots \times S^{(0)}_{\Phi_n}$ *and* $\|\alpha - \delta\|_{\Phi} < \varepsilon$.

Theorem (Voiculescu, 2018)

Let $p_j > 1, 1 \leq j \leq n$ *s.t.* $\sum_{j=1}^{n} \frac{1}{p_j} = 1$ *and* $\Phi_j = \Phi_{p_j,1}, 1 \leq j \leq n$.
$\exists$ *a universal constant* $0 < \gamma_n < \infty$ *depending only on* $p_j, 1 \leq j \leq n$ *s.t.* $\forall$ *commuting* $\alpha \in (B(\mathcal{H})_{sa})^n$,
$(k_{\Phi}(\alpha))^n = \gamma_n \int m(x)dx$,
where m *is the multiplicity function of* α_{ac}.

Unbounded Fredholm modules

Unbounded Fredholm modules is an object studied in non-commutative geometry.

- Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{A} \subset B(\mathcal{H})$ a unital C^*-algebra, $\mathcal{J}$ a normed ideal.
  An unbounded $\mathcal{J}$-Fredholm module over $\mathcal{A}$ is a pair $(\mathcal{H}, D)$, where D is an unbounded densely defined self-adjoint operator on $\mathcal{H}$ s.t.
  a. the set $\mathfrak{A} := \{a \in \mathcal{A} : [D, a]$ can be extended to a bounded operator$\}$ is dense in $\mathcal{A}$.
  b. $|D|^{-1} \in \mathcal{J}$.

  Here $|D|^{-1}$ is the pseudoinverse of $|D|$.
- If $\mathcal{J} = K(\mathcal{H})$, an unbounded $\mathcal{J}$-Fredholm module is simply called an unbounded Fredholm module.

- (p,∞)-Lorentz ideals, $1 < p < \infty$
  $\Phi_{p,\infty}(a) = \sup_{k\geq 1} \frac{\sum_{j=1}^{k} a_j}{\sum_{j=1}^{k} (\frac{1}{j})^{1/p}}$
  $\mathcal{C}_{p,\infty} := \{T \in K(\mathcal{H}) : \|T\|_{p,\infty} < \infty\}$.
  $(\mathcal{C}_{q,1})^{dual} = \mathcal{C}_{p,\infty}$ with $q = \frac{p}{p-1}$, where the dual is with respect to the coupling $\langle A, B\rangle = \mathrm{Tr}(AB)$.
- Dixmier trace
  $\mathrm{Tr}_\omega(A) := \omega\big(\{\frac{1}{\log(N+1)} \sum_{k=1}^{N} \mu_k(A)\}_{N\geq 1}\big), \quad A \geq 0, A \in \mathcal{C}_{1,\infty}.$
  where $\omega \in \ell_\infty^*$, s.t.
  1. ω is a singular: i.e. vanishes for any finite sequence
  2. dialation-invariant: $\omega(a_1, a_2, \ldots,) = \omega(a_1, a_1, a_2, a_2, \ldots,)$
  3. $\omega(1, 1, 1, \ldots) = 1$

Dixmier trace as an estimate of $k_{p,1}$

Theorem (Connes, '88)

Let D *be an unbounded self-adjoint operator in* $\mathcal{H}$*, such that* $|D|^{-1} \in \mathcal{C}_{p,\infty}$ *(where* $1 < p < \infty$*).*
$\forall$ *finite subset* X *of* $\mathfrak{A} = \{T \in B(H) : [T, D]$ *bounded*$\}$,
$k_{p,1}(X) \leq \beta_p(\sup_{T \in X} \|[D, X]\|)(\mathrm{Tr}_\omega(|D|^{-p}))^{1/p}$,
where β_p *is a universal constant and* Tr_ω *is the Dixmier trace.*

Problems

What happens if instead of $B(\mathcal{H})$, we consider von Neumann algebras $\mathcal{M} \subset B(\mathcal{H})$, normed ideals $\mathcal{J}_1, \ldots, \mathcal{J}_n$ of $\mathcal{M}$ and commuting self-adjoint tuple $\alpha \in (\mathcal{M}_{sa})^n$?

For convenience, we denote $\Phi = \mathcal{J}_1 \times \cdots \times \mathcal{J}_n$.

If $\exists$ commuting diagonal n-tuple $\delta = (D_1, \ldots, D_n) \in (\mathcal{M}_{sa})^n$ s.t. $A_i - D_i \in \mathcal{J}_i$, we say that α is diagonal modulo Φ.

If $\mathcal{J}_1 = \ldots = \mathcal{J}_n = \mathcal{J}$, we say α is diagonal modulo $\mathcal{J}$.

- Can we define analogously quasicentral modulus $k_\Phi(\alpha)$ for $\alpha \in \mathcal{M}^n$?
- Is is true that $k_\Phi(\alpha) = 0 \Leftrightarrow \alpha$ is diagonal modulo Φ?
- What can we say about α_s and α_{ac}?
- Spectral multiplicity function does not work well for von Neumann algebras, what is the proper analogue of spectral multiplicity theory in von Neumann algebras?