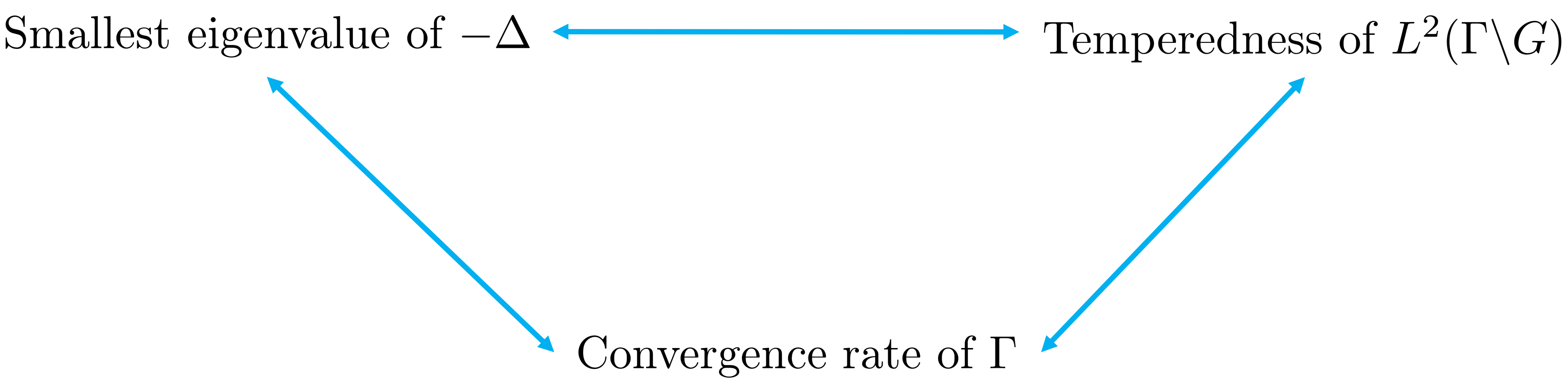
IASM of HIT, Harbin, 10 April 2024

L^2-spectrum, convergence exponents, and temperedness on locally symmetric spaces

Hong-Wei Zhang (Paderborn University)

PADERBORN UNIVERSITY

Objects to Study



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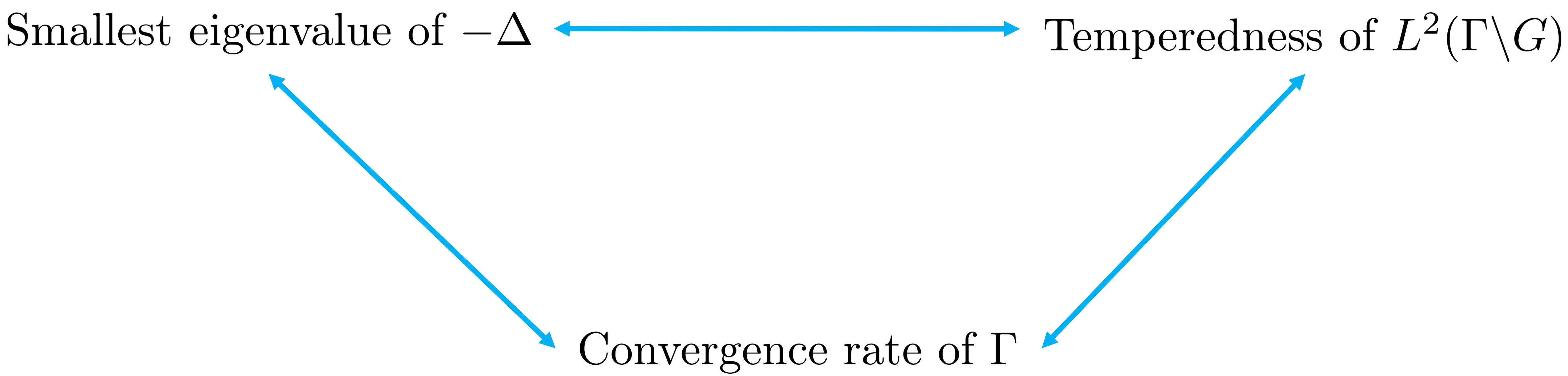

Anosov subgroup conjecture

Strichartz inequality

Notation: Hyperbolic Space

Hyperbolic plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\} \qquad \text{(upper half-plane)}$$

$$\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$$

Real hyperbolic space

$$\mathbb{H}^n = \{x \in \mathbb{R} \times \mathbb{R}^n \,|\, -x_0^2 + x_1^2 + \cdots x_{n+1}^2 = -1,\ x_0 \geq 1\} \qquad \text{(hyperboloid)}$$

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Non-compact symmetric space of rank 1

$\mathbb{H}^n = \mathbb{H}^n(\mathbb{R})$ $\quad$ $\mathbb{H}^n(\mathbb{C})$ $\quad$ $\mathbb{H}^n(\mathbb{H})$ $\quad$ $\mathbb{H}^2(\mathbb{O})$

L^2-Spectrum

- $-\Delta f = \lambda f$
- $\sigma(-\Delta) = \{\lambda \in \mathbb{C} \,|\, (-\Delta - \lambda)^{-1} : L^2 \to L^2 \text{ does not exist}\}$

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$\mathbb{R}^n$: $\sigma_c(-\Delta) = [0, \infty)$

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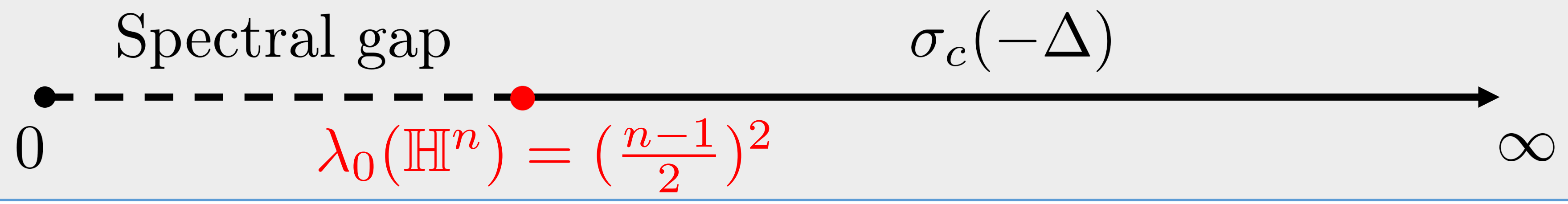
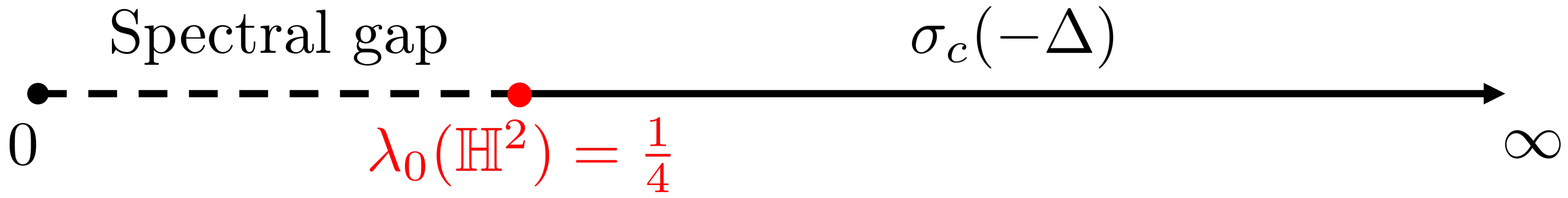


L^2-Spectrum on hyperbolic surfaces

Hyperbolic plane $\mathbb{H}^2$:



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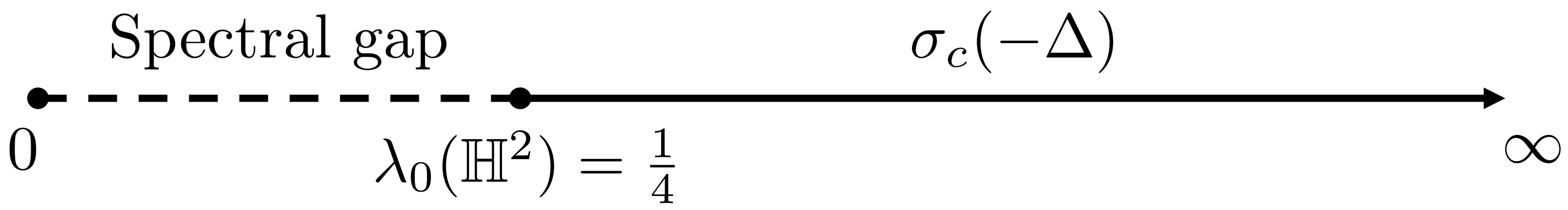

Modular curve $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ (non-compact and finite area):

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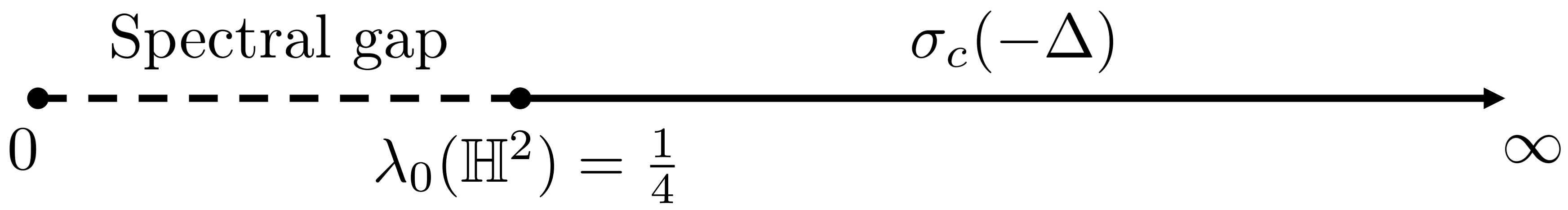

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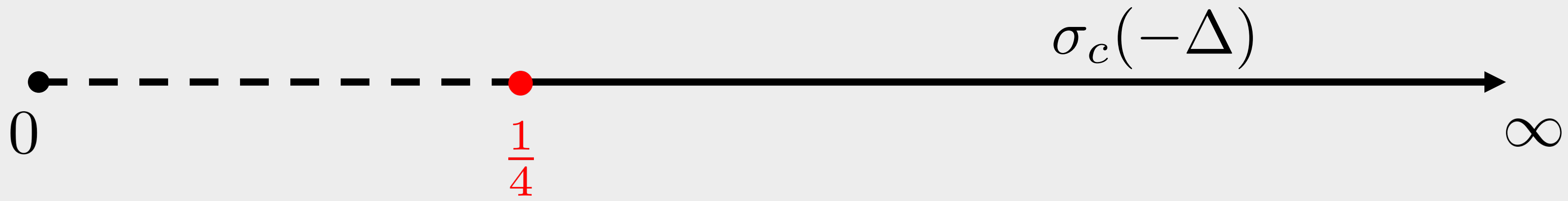

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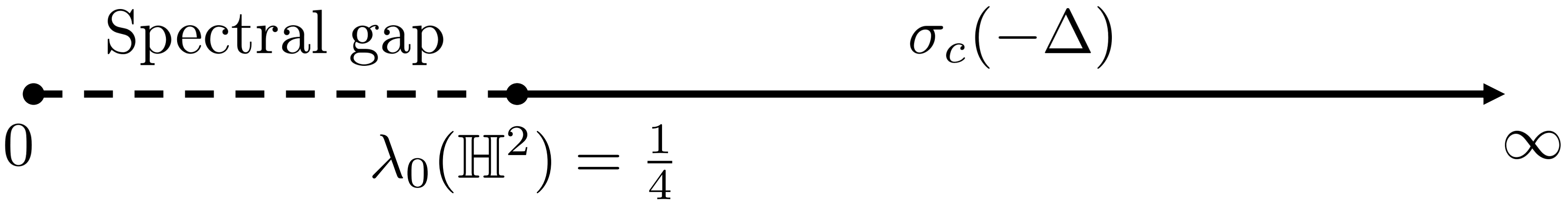

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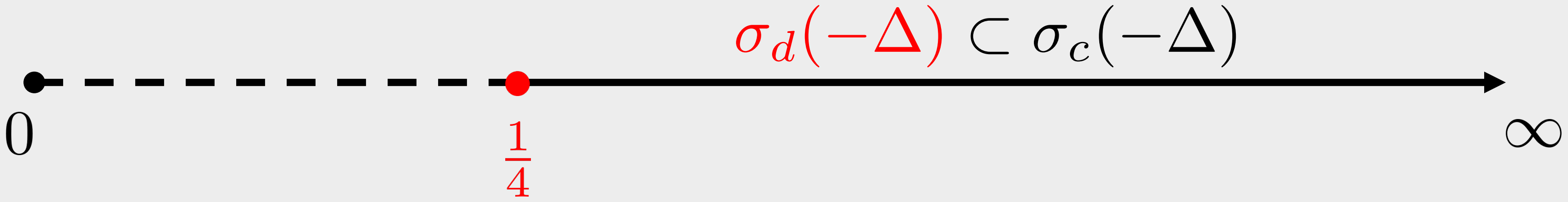

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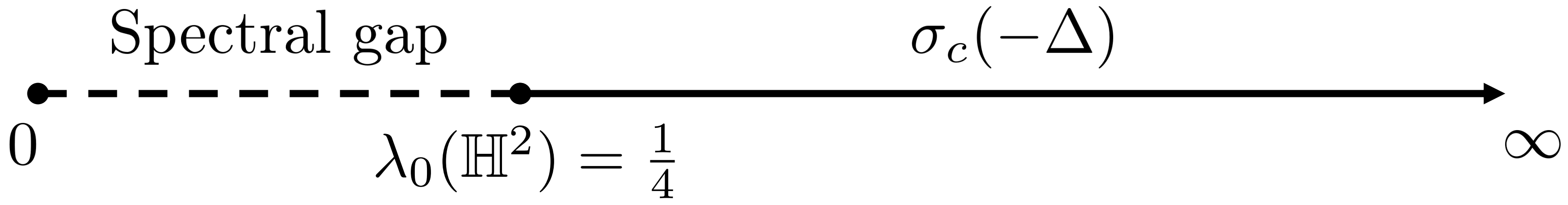

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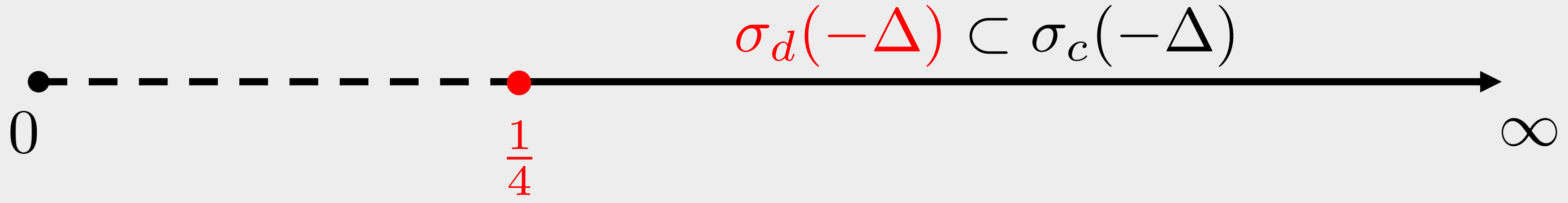

i.e., there are

- infinitely many embedded eigenvalues
- no exceptional eigenvalues

(Selberg's 1/4 Conjecture for general Riemann surface)

L^2-Spectrum on hyperbolic surfaces

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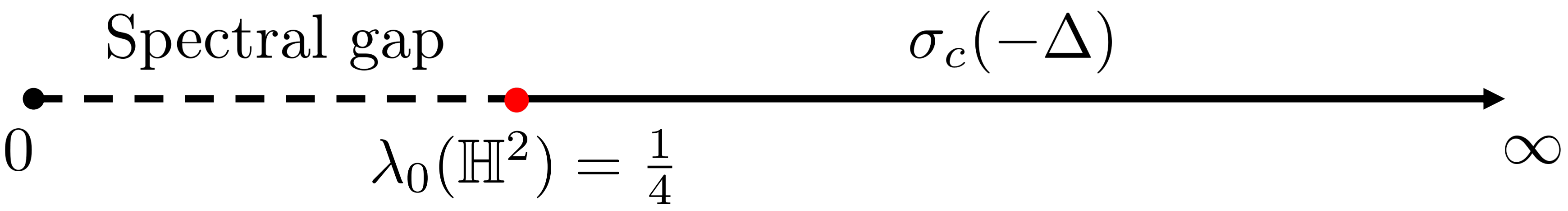

Thin group: $\Gamma \leq \mathrm{SL}(2,\mathbb{R})$ s.t. $\mathrm{Vol}(\Gamma \backslash \mathbb{H}^2) = \infty$

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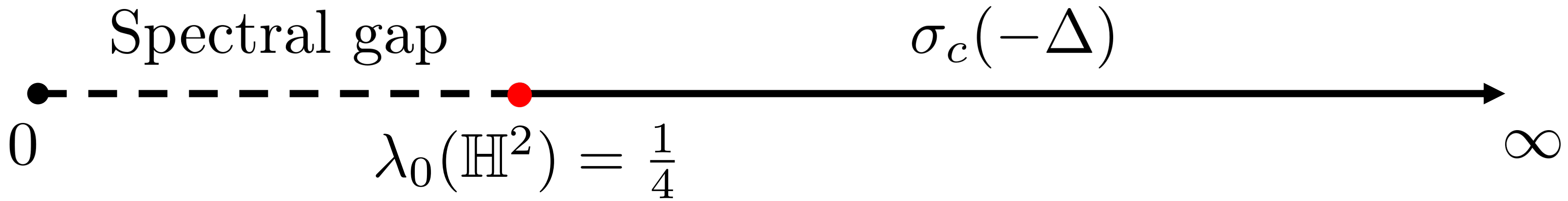

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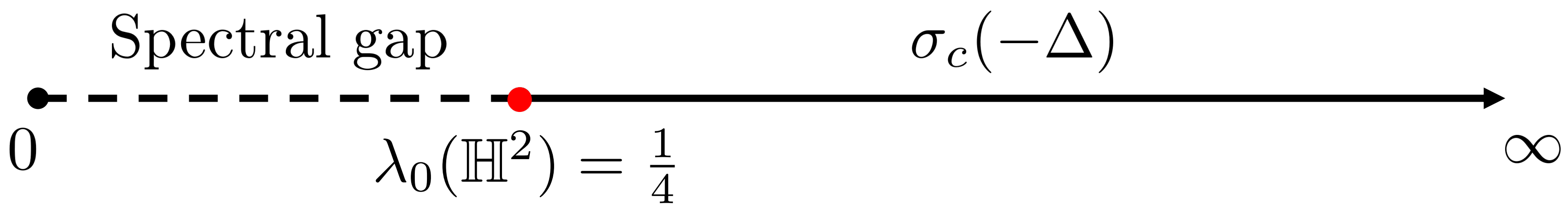

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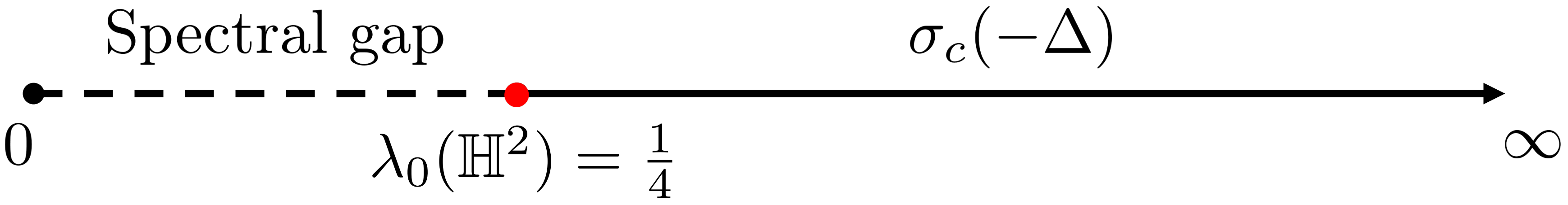

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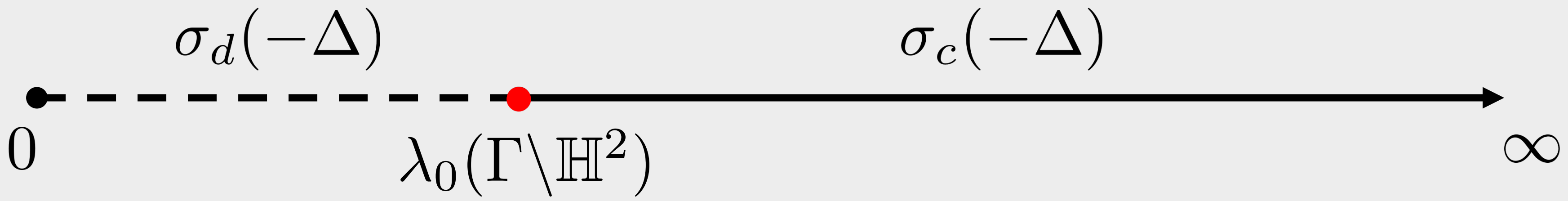

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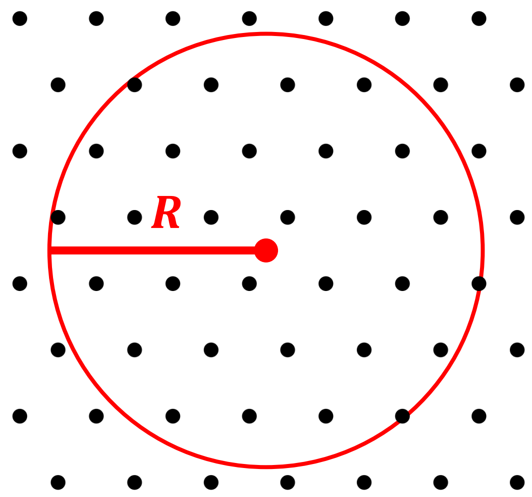
Characterize $\lambda_0(\Gamma\backslash X)$

$$\lambda_0(\Gamma\backslash X) := \inf_{f \in C_c^\infty(\Gamma\backslash X)} \frac{\int_{\Gamma\backslash X} \|\mathrm{grad} f\|^2 \, dvol}{\int_{\Gamma\backslash X} \|f\|^2 \, dvol} = \inf \sigma_c(-\Delta)$$

Convergence/Critical Exponent

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{\log(\#\{\gamma \in \Gamma \mid d(e,\gamma e) \le R\})}{R}$$



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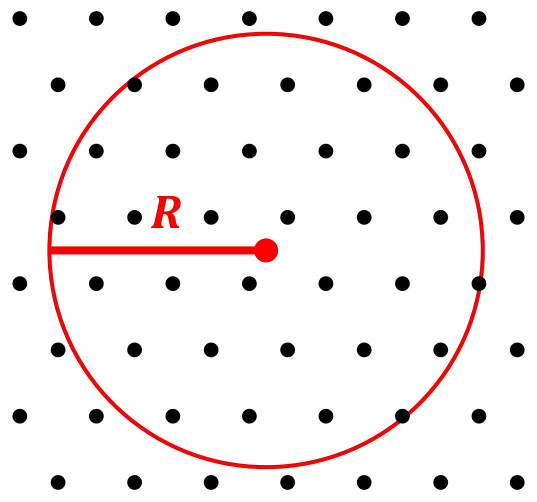

$$\delta_\Gamma = \inf \left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma} e^{-sd(e,\gamma e)} < \infty \right\}$$

Poincaré Series: $\displaystyle \sum_{\gamma\in\Gamma} e^{-sd(e,\gamma e)} \begin{cases} < \infty & \text{if} \quad s > \delta_\Gamma \\ = \infty & \text{if} \quad s < \delta_\Gamma \end{cases}$

e.g. In $\mathbb{H}^2$: $0 \le \delta_\Gamma \le 1$ In $\mathbb{H}^n$: $0 \le \delta_\Gamma \le n-1$

Characterization in Rank 1

$\lambda_0(\Gamma\backslash\mathbb{H}^2) = \frac{1}{4}$

$L^2(\Gamma\backslash G)$ is tempered

$\delta(\Gamma) \leq \frac{1}{2}$

Theorem [Elstrodt '73, Patterson '76]

$$\lambda_0(\Gamma\backslash\mathbb{H}^2) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq \delta_\Gamma \leq \frac{1}{2} \\ \frac{1}{4} - (\delta_\Gamma - \frac{1}{2})^2 & \text{if } \frac{1}{2} \leq \delta_\Gamma \leq 1 \end{cases}$$

Characterization in Rank 1

$\lambda_0(\Gamma\backslash X) = \rho^2$

$L^2(\Gamma\backslash G)$ is tempered

$\delta(\Gamma) \leq \rho$

Theorem [Elstrodt '73, Patterson '76, Sullivan '87, Corlette '90]

$$\lambda_0(\Gamma\backslash X) = \begin{cases} \rho^2 & \text{if } 0 \leq \delta_\Gamma \leq \rho \\ \rho^2 - (\delta_\Gamma - \rho)^2 & \text{if } \rho \leq \delta_\Gamma \leq 2\rho \end{cases}$$

where $\rho = \frac{n-1}{2}$ on $\mathbb{H}^n(\mathbb{R})$, n on $\mathbb{H}^n(\mathbb{C})$, $2n+1$ on $\mathbb{H}^n(\mathbb{H})$, 11 on $\mathbb{H}^2(\mathbb{O})$

Temperedness

G connected semisimple Lie group $\implies$ direct integrals:

$$L^2(\Gamma\backslash G) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \, \mathrm{d}\nu(\pi) \qquad \text{and} \qquad L^2(\Gamma\backslash X) \cong \int_{\widehat{G}_K}^{\oplus} (\mathcal{H}_\pi)^K \, \mathrm{d}\nu(\pi)$$

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In rank 1, $\widehat{G}_K$ consists of

- the unitary spherical principal series $\pi_{\pm\lambda}$ $(\lambda \in \mathbb{R}/\pm 1)$
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- the unitary spherical principal series $\pi_{\pm\lambda}$ $(\lambda \in \mathbb{R}/\pm 1)$
- the trivial representation $\pi_{\pm i\rho} = 1$
- the complementary series $\pi_{\pm i\lambda}$ $(\lambda \in I)$, where

$$I = \begin{cases} (0,\rho) & \text{if } X = \mathbb{H}^n(\mathbb{R}) \text{ or } \mathbb{H}^n(\mathbb{C}) \\ \left(0, \frac{m_\alpha}{2}+1\right] & \text{if } X = \mathbb{H}^n(\mathbb{H}) \text{ or } \mathbb{H}^2(\mathbb{O}) \end{cases}$$

(no higher rank analogue)

Equivalent conditions in rank 1

$\lambda_0(\Gamma \backslash X) = \rho^2$

$L^2(\Gamma \backslash G)$ is tempered

$\delta(\Gamma) \leq \rho$

By definition

- $L^2(\Gamma \backslash G)$ is called tempered if $\widehat{G}_K$ does not involve complementary series
- $-\Delta$ acts on $(\mathcal{H}_\pi)^K$ by multiplication by $\lambda^2 + \rho^2$

Equivalent conditions in rank 1

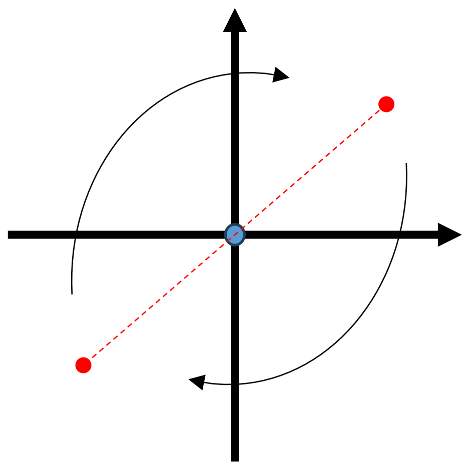
$\lambda_0(\Gamma\backslash X) = \rho^2 \quad \longleftrightarrow \quad L^2(\Gamma\backslash G)$ is tempered

$\delta(\Gamma) \leq \rho$

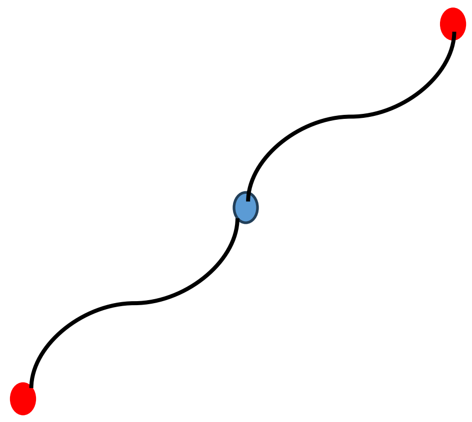
ⓘ When $\Gamma\backslash X$ is of higher rank and infinite volume?

(Riemannian) Symmetric Space

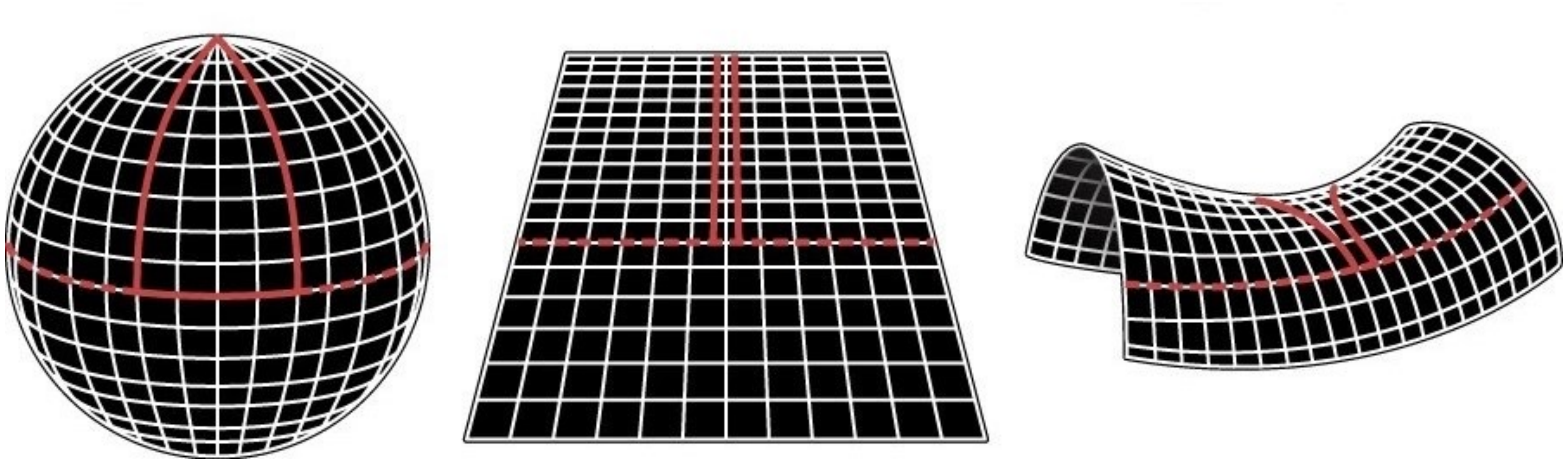
Extends the concept of central symmetry on geodesics



Reflection is an involutive isometry



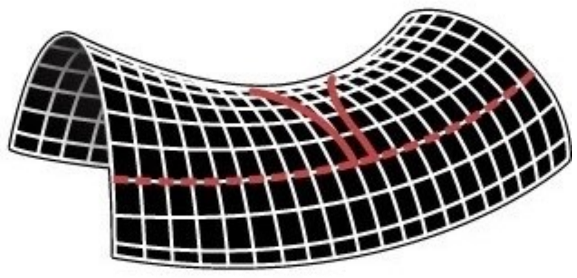
Noncompact Symmetric Space



- Compact type: nonnegative sectional curvature, e.g. sphere
- Euclidean type: vanishing curvature, isometric to Euclidean space
- Noncompact type: nonpositive curvature, e.g. real hyperbolic space

Fourier transform

Fourier transform



$$G/K \longrightarrow \mathfrak{a}$$

A_1

Real hyperbolic space

1-dimensional Cartan subspace

Non-compact Symmetric Space

$X = G/K$

- G noncompact semisimple Lie group (connected, finite center)
- K maximal compact subgroup of G

e.g., $\mathbb{H}^n(\mathbb{R}) = \mathrm{SO}(n,1)^\circ/\mathrm{SO}(n)$, $\mathrm{SPos}(n) = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$

X is a Cartan-Hadamard manifold

(complete, simply connected, with non-positive sectional curvature)

- with additional symmetric property
- which growths exponentially fast at infinity
- on which the Fourier transform is available

Non-compact Symmetric Space

$X = G/K$

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e.g., $\mathbb{H}^n(\mathbb{R}) = \mathrm{SO}(n,1)^\circ/\mathrm{SO}(n)$, $\mathrm{SPos}(n) = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$

- $\Gamma \leq G$: discrete and torsion-free subgroup of G
  - Γ is a lattice: $\mathrm{Vol}(\Gamma\backslash X) < \infty$
  - Γ has infinite covolume: $\mathrm{Vol}(\Gamma\backslash X) = \infty$

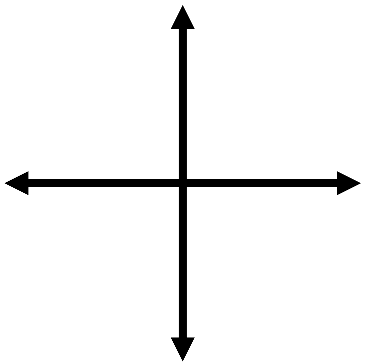
Cartan Subspace

Rank

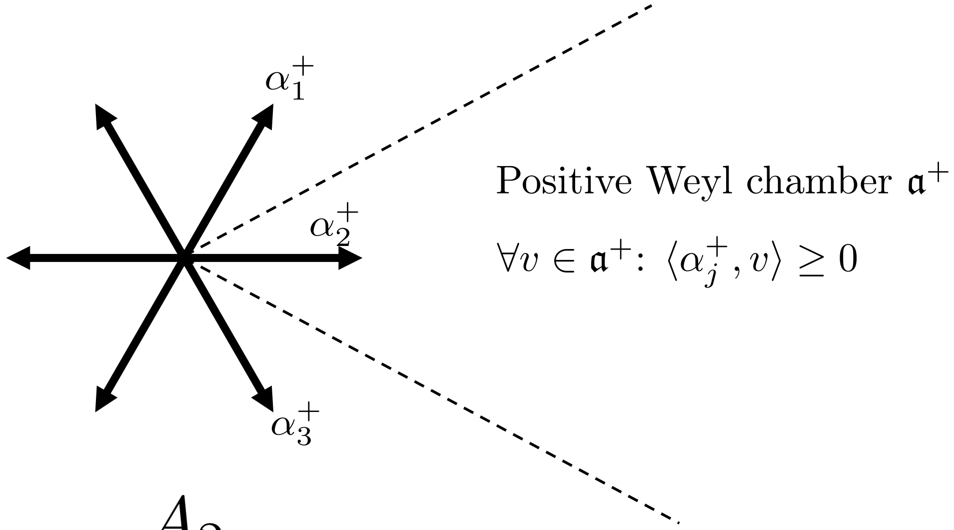
Cartan subspace $\mathfrak{a}$: maximal connected, totally geodesic, flat sub-manifold of $\mathbb{X}$

$$\mathfrak{a} \approx \mathbb{R}^\ell \qquad \text{and} \qquad \ell = \dim \mathfrak{a} = \operatorname{rank} G/K$$

A_1



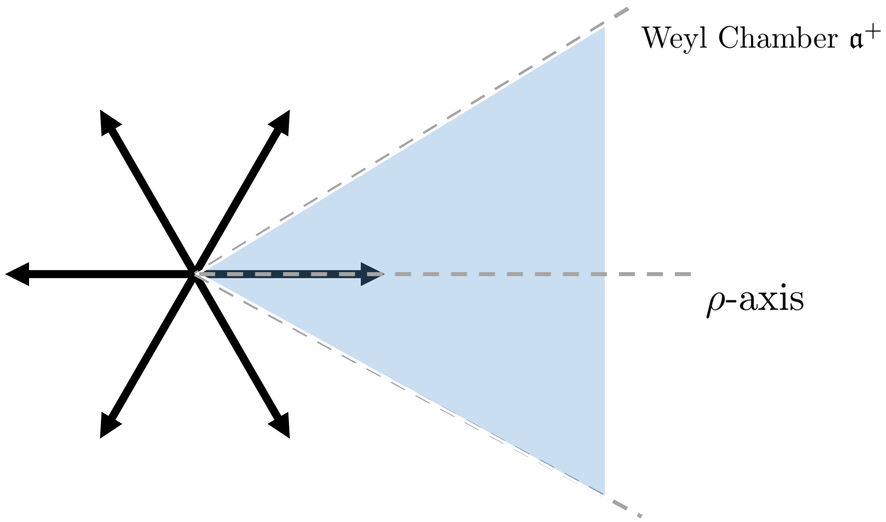
$A_1 \times A_1$



A_2

Preliminary: From Rank 1 to Higher Rank

- Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$



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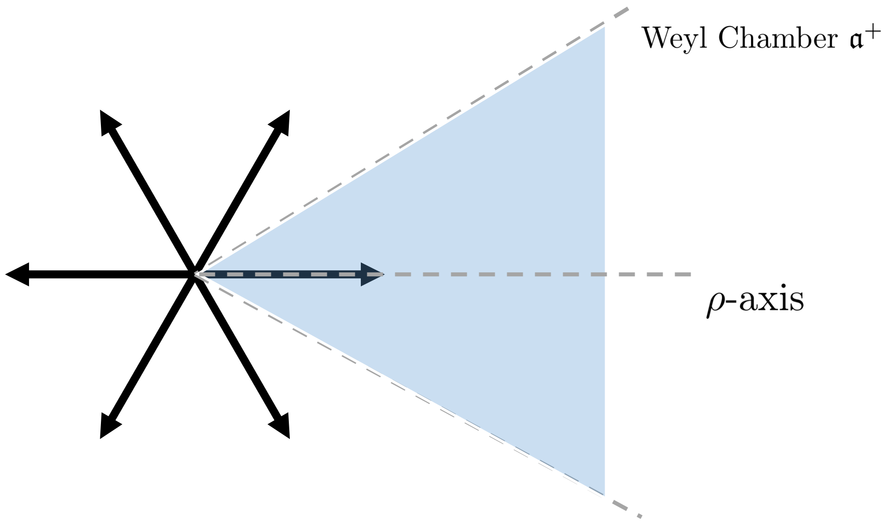
- Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$
- Cartan projection $\mu : G \longrightarrow \overline{\mathfrak{a}^+}$ such that

$$g \in K e^{\mu(g)} K$$

- $d(e, \gamma e) = \|\mu(\gamma)\| \qquad \forall \gamma \in \Gamma$



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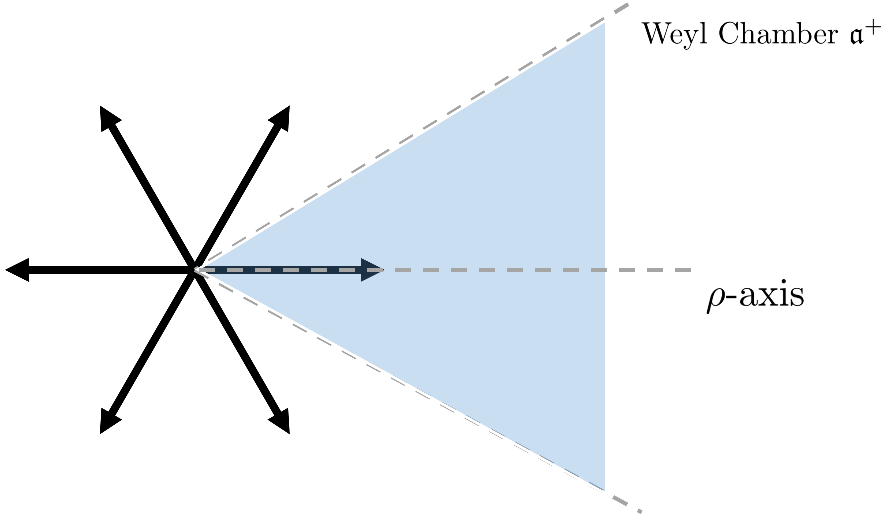


- In rank 1: $\dim \mathfrak{a}^+ = 1$, $\rho \in \mathfrak{a}^+$ is a positive number

  In higer rank: $\rho \in \mathfrak{a}^+$ is a vector, known as $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$

Spectrum Bottom and Temperedness

$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \longleftarrow L^2(\Gamma\backslash G) \text{ is tempered}$$

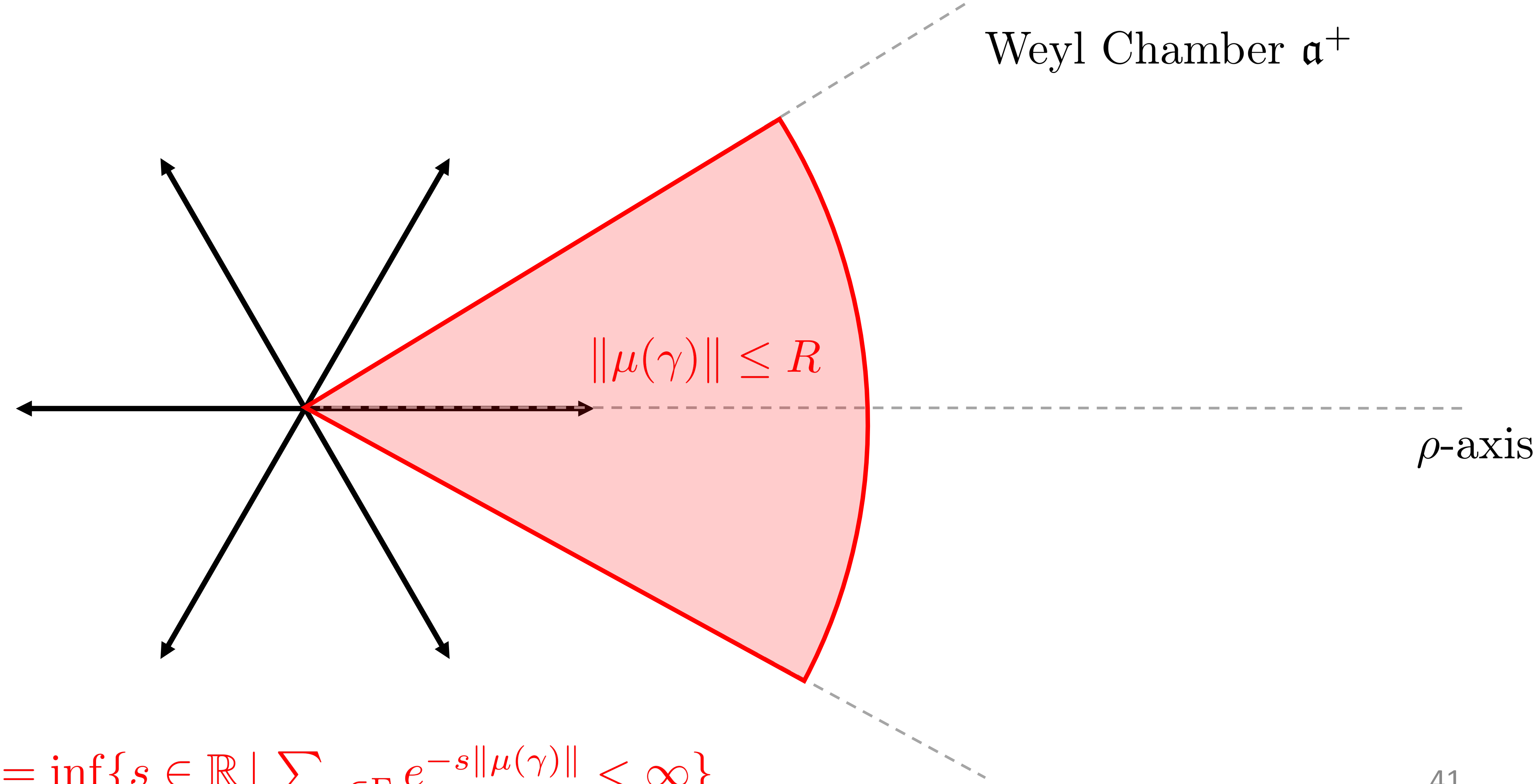
$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \overset{?}{\dashrightarrow} L^2(\Gamma\backslash G) \text{ is tempered}$$

(modified) Poincaré Series

- [Leuzinger '03]: Lower and upper bounds of $\lambda_0(\Gamma\backslash X)$ in terms of

$$\delta_\Gamma = \inf \left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma \in \Gamma} e^{-s\|\mu(\gamma)\|} < \infty \right\}$$

Concepts of Different Convergence Exponents



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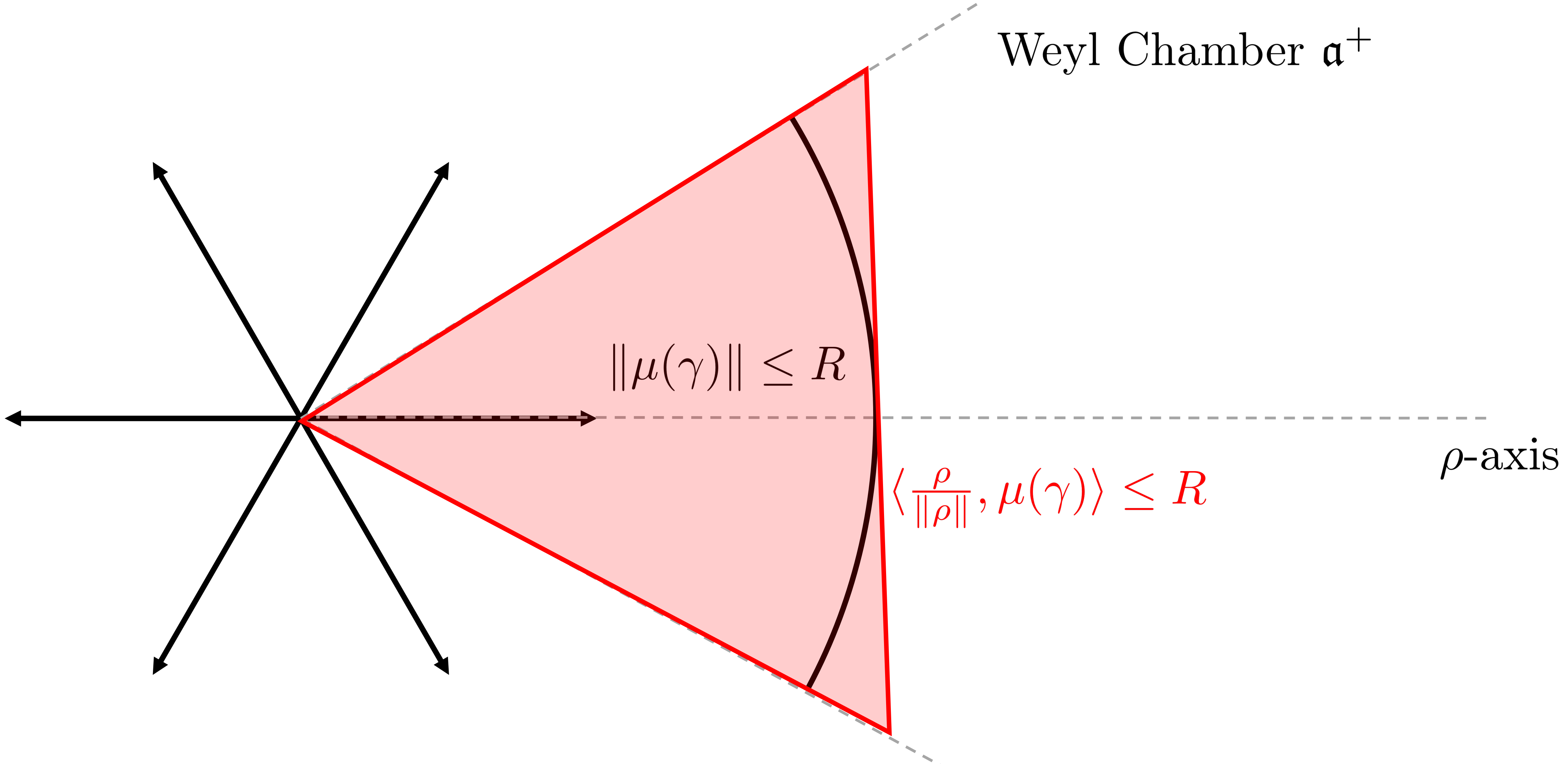
- [Carron-Pedon '04, Anker-Z. '22]: Introduce the modified critical exponent

$$\tilde{\delta}_\Gamma = \inf \left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma} e^{-\min\{s,\|\rho\|\}\langle\frac{\rho}{\|\rho\|},\mu(\gamma)\rangle - \max\{0,s-\|\rho\|\}\|\mu(\gamma)\|} < \infty \right\}$$

- $0 \le \delta_\Gamma \le \tilde{\delta}_\Gamma \le 2\|\rho\|$ and $\delta_\Gamma = \tilde{\delta}_\Gamma$ in rank 1

Concepts of Different Convergence Exponents



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Characterization in Higher Rank

$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \quad \longleftarrow \quad L^2(\Gamma\backslash G) \text{ is tempered}$$

?

$$\tilde{\delta}(\Gamma) \leq \|\rho\|$$

Theorem [Anker-Z. '22]

$$\lambda_0(\Gamma\backslash X) = \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \tilde{\delta}_\Gamma \leq \|\rho\| \\ \|\rho\|^2 - (\tilde{\delta}_\Gamma - \|\rho\|)^2 & \text{if } \|\rho\| \leq \tilde{\delta}_\Gamma \leq 2\|\rho\| \end{cases}$$

Growth Indicator Function

Patterson-Sullivan measure in rank 1

- [Patterson '76]: prob. measure ν over Λ_Γ s.t. $\gamma_*\nu = e^{-\delta_\Gamma b(\gamma e,\cdot)}\nu \quad \forall\gamma\in\Gamma$
- [Sullivan '79]: prob. measure ν over ∂X s.t. $\gamma_*\nu = e^{-\tau b(\gamma e,\cdot)}\nu \Rightarrow \tau \geq \delta_\Gamma$

Growth Indicator Function

Patterson-Sullivan measure in rank 1

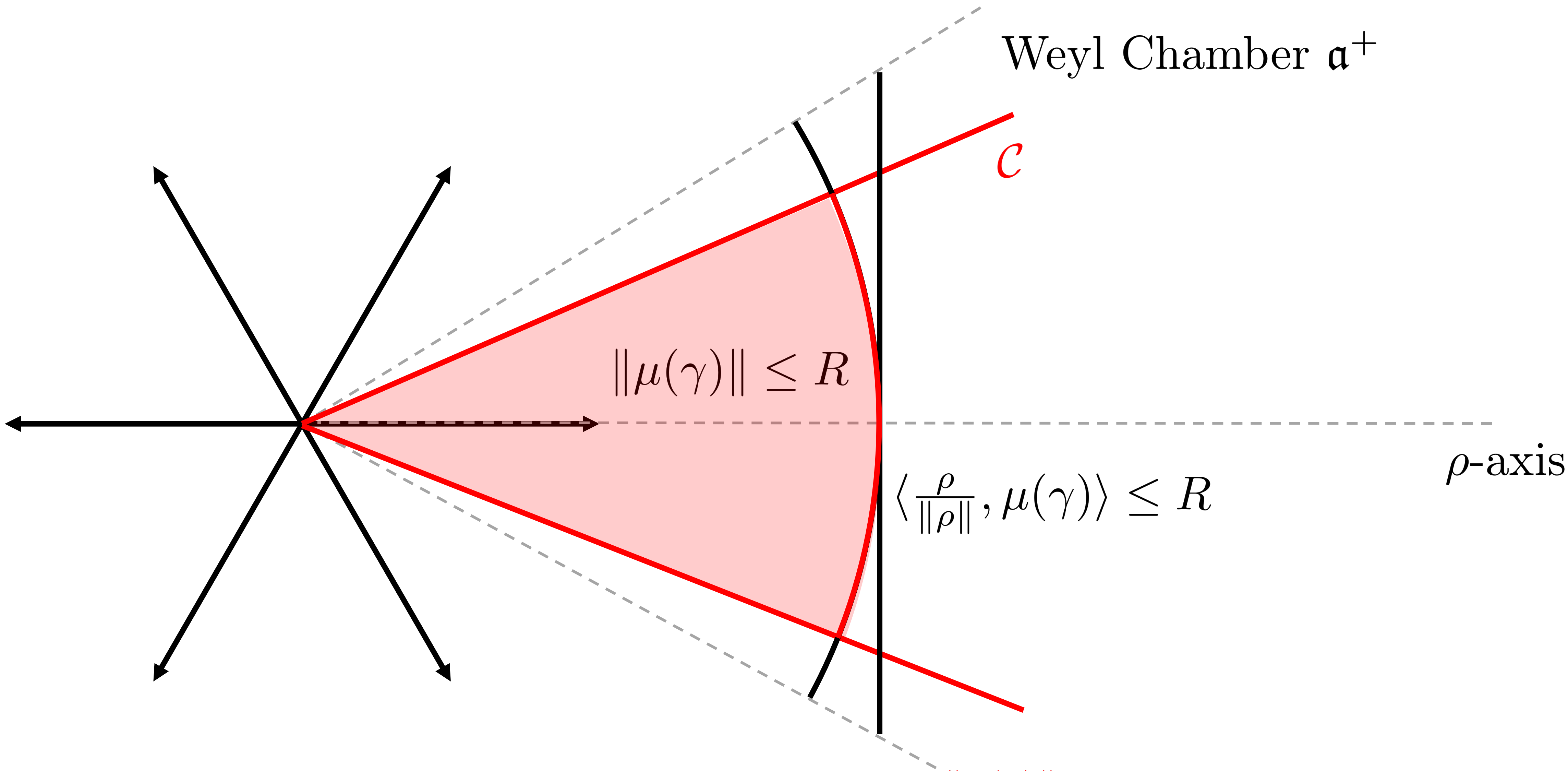
- [Patterson '76]: prob. measure ν over Λ_Γ s.t. $\gamma_*\nu = e^{-\delta_\Gamma b(\gamma e,\cdot)}\nu \quad \forall \gamma \in \Gamma$
- [Sullivan '79]: prob. measure ν over ∂X s.t. $\gamma_*\nu = e^{-\tau b(\gamma e,\cdot)}\nu \Rightarrow \tau \geq \delta_\Gamma$

Patterson-Sullivan measure in higher rank

- [Albuquerque '99]: Peterson's construction under assumptions on Λ_Γ
- [Quint '02]: generalization of Patterson-Sullivan's results by introducing

$$\psi_\Gamma(H) = \|H\| \inf_{\mathcal{C}\ni H} \inf \left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma, \mu(\gamma)\in\mathcal{C}} e^{-s\|\mu(\gamma)\|} < \infty \right\}$$

Concepts of Different Convergence Exponents



$$\psi_\Gamma(H) = \|H\| \inf_{H \in \mathcal{C}} \inf\{s \in \mathbb{R} \mid \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|} < \infty\}$$

Spectrum Bottom and Critical Exponent

$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \longleftarrow L^2(\Gamma\backslash G)$ is tempered

?

$\tilde{\delta}(\Gamma) \leq \|\rho\|$

$\psi_\Gamma(H) \leq \rho(H) \quad \forall H \in \mathfrak{a}$

Theorem [Lee-Oh '23]

$$L^2(\Gamma\backslash G) \text{ is tempered} \iff \langle \exp(H)f, g\rangle \lesssim \|f\|_{L^2}\|g\|_{L^2}e^{-(1-\varepsilon)\rho(H)}$$

$$\implies \psi_\Gamma \leq \rho$$

Convergence Exponents Relation

$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \qquad L^2(\Gamma\backslash G) \text{ is tempered}$$

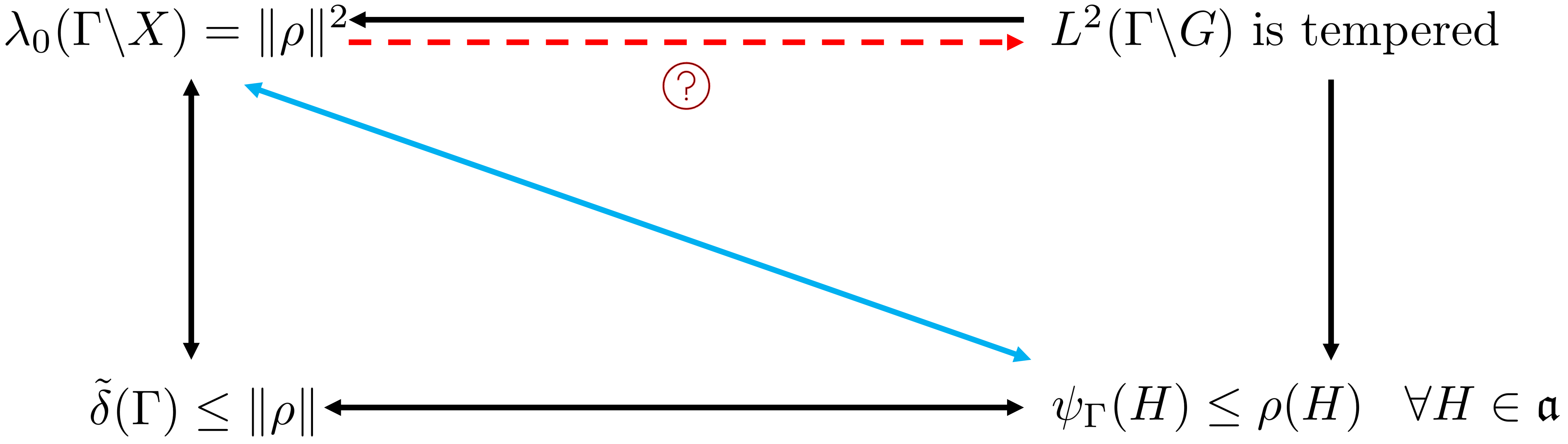
?

$$\tilde{\delta}(\Gamma) \le \|\rho\| \qquad \psi_\Gamma(H) \le \rho(H) \quad \forall H \in \mathfrak{a}$$

Theorem [Wolf-Z. '23]

$$\tilde{\delta}_\Gamma = \begin{cases} \sup_{H\in\overline{\mathfrak{a}^+}} \psi_\Gamma(H)\cdot\frac{\|\rho\|}{\rho(H)} & \text{if} \quad \psi_\Gamma \le \rho \\ \sup_{H\in\overline{\mathfrak{a}^+}} \frac{\psi_\Gamma(H)-\rho(H)}{\|H\|} + \|\rho\| & \text{otherwise} \end{cases}$$

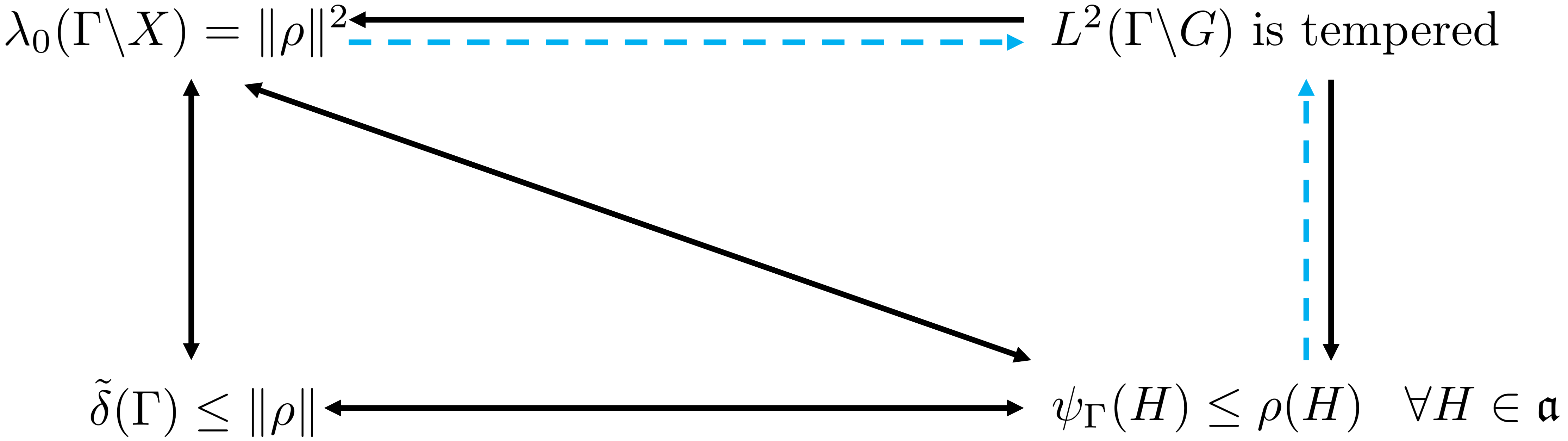
Another Characterization



Corollary [Wolf-Z. '23]

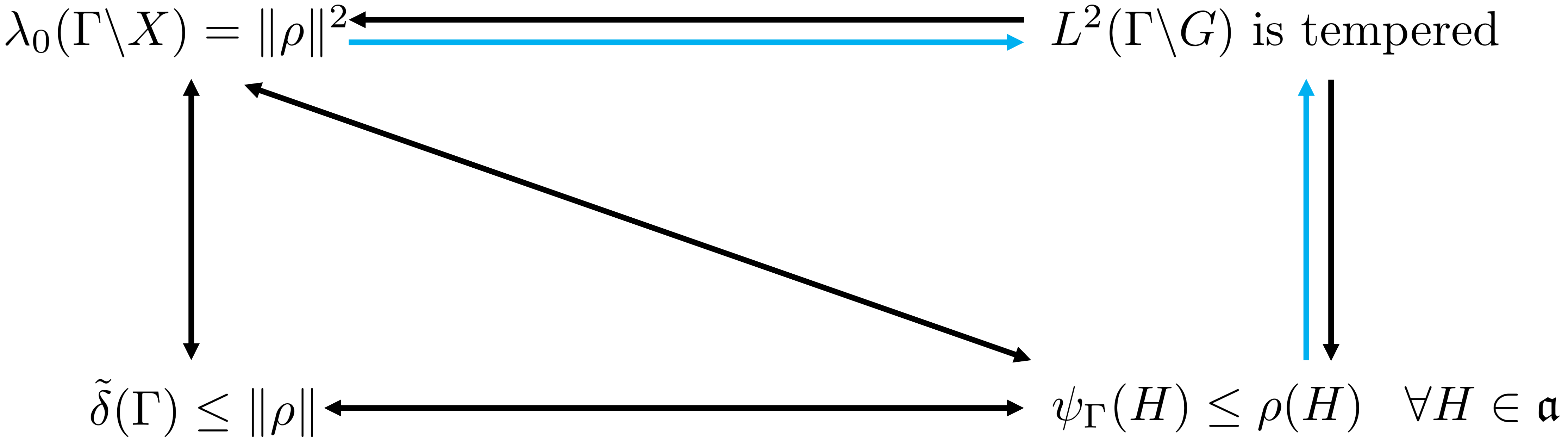
$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 - \max\left\{0, \sup_{H \in \overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H) - \rho(H)}{\|H\|}\right\}^2$$

Anosov Case and Product Case



- [Edwards-Oh '23]: if Γ is Anosov
- [Weich-Wolf '23]: if $\Gamma \leq \mathbb{H}^n \times \mathbb{H}^n$

General Case



- [Edwards-Oh '23]: if Γ is Anosov
- [Weich-Wolf '23]: if $\Gamma \leq \mathbb{H}^n \times \mathbb{H}^n$
- [Lutsko-Weich-Wolf '24]: in general

Rank 1: Convex Cocompact Subgroup

Γ convex cocompact

$$\Updownarrow$$

$\Gamma \backslash \mathrm{Conv}(\Lambda_\Gamma)$ compact

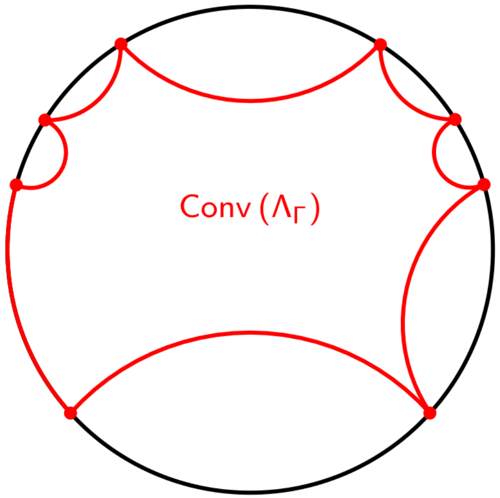


Figure: Convex hull of the limit set Λ_Γ in $\mathbb{H}^2$

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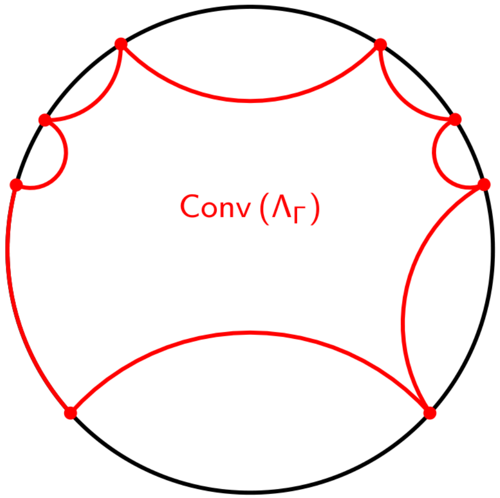

Figure: Convex hull of the limit set Λ_Γ in $\mathbb{H}^2$

Equivalent condition

$\Gamma \le G$ is convex cocompact if the orbit map $\gamma \mapsto \gamma e$ is a quasi-isometric embedding: there exist $C_1 \ge 1$, $C_2 \ge 0$ such that

$$\frac{1}{C_1}|\gamma|_S - C_2 \le d(e, \gamma e) \le C_1|\gamma|_S + C_2$$

for all $\gamma \in \Gamma$. Here $|\cdot|_S$ is the word metric.

(shortest length over S)

Higher Rank: Anosov Subgroup

[Kleiner-Leeb '06]

Γ acts on a closed convex subset cocompactly implies

$$\Gamma \text{ is a lattice} \qquad \text{or} \qquad \Gamma\backslash X \text{ is a product of rank one copies}$$

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Anosov subgroup

A representation $\pi : S \to G$ is Anosov if there exists $C > 0$ s.t. for all $\gamma \in S$ and every simple root α,

$$\alpha(\mu(\pi(\gamma))) \geq C|\gamma|_S - C$$

Γ is called an Anosov subgroup if it is the image by an Anosov representation

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Oh's Conjecture: Let G be a semisimple real algebraic group of higher rank

$$\Gamma \leq G \text{ is Anosov} \quad \Longrightarrow \quad \psi_\Gamma \leq \rho$$

Strichartz Inequality

e.g. Free Schrödinger equation: $(i\partial_t + \Delta_x)u(t,x) = 0, \quad u(0,x) = f(x)$

whose solution is given by $u(t,x) = e^{it\Delta} f(x)$

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Space-time mixed norm estimate

$$\|u\|_{L_t^p(\mathcal{I}, L_x^q(\mathcal{M}))} = \left(\int_{\mathcal{I}} \mathrm{d}t \, \|u\|_{L^q(\mathcal{M})}^p \right)^{1/p} \lesssim \|f\|_{H^s(\mathcal{M})}$$

- for all **admissible pairs** (p,q)
- $s = 0$: without loss; $s > 0$: with loss of derivatives
- $\mathcal{I}$ bounded: local-in-time; $\mathcal{I}$ unbounded: global-in-time

Euclidean Strichartz Inequality

e.g. Free Schrödinger equation: $(i\partial_t + \Delta_x)u(t,x) = 0, \quad u(0,x) = f(x)$

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In $\mathbb{R}^n$ [Segal '76, Strichartz '77, Ginibre-Velo '95, Keel-Tao '98]

Global-in-time Strichartz inequality **without loss**

$$\|u\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

holds for all **admissible** pairs (p,q), i.e.,

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \qquad p \geq 2 \qquad (p,q) \neq (2,\infty)$$

Strichartz Inequality on $X = G/K$

The solution to the Free Schrödinger equation satisfies

$$\|u\|_{L_t^p(\mathbb{R}, L_x^q(X))} \lesssim \|f\|_{L^2(X)}$$

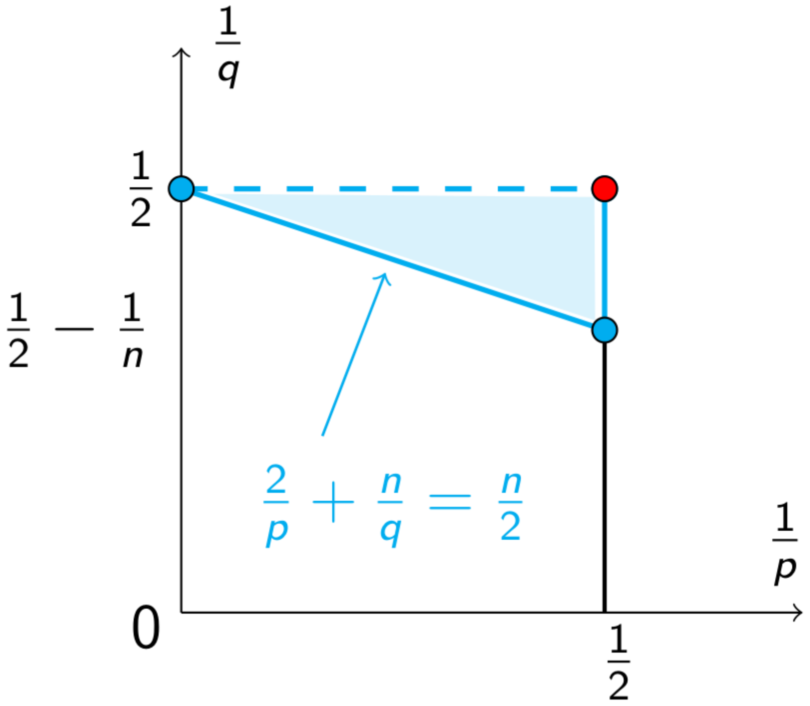
Strichartz Inequality on $X = G/K$

The solution to the Free Schrödinger equation satisfies

$$\|u\|_{L^p_t(\mathbb{R}, L^q_x(X))} \lesssim \|f\|_{L^2(X)}$$

for all (p, q) *admissible* in the sense that

$$\left\{\left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \left(0, \tfrac{1}{2}\right] \times \left(0, \tfrac{1}{2}\right) \,\middle|\, \tfrac{2}{p} + \tfrac{n}{q} \geq \tfrac{n}{2}\right\} \cup \left\{\left(0, \tfrac{1}{2}\right)\right\}$$



Strichartz Inequality on $\Gamma\backslash X$

Global-in-time [Burq-Guillarmou-Hassell '10, Fotiadis-Mandouvalos-Marias '18, Z. '20]

Global-in-time Strichartz inequality holds without lossing any derivatives for the large X-admissible set if the following conditions are met:

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- Γ is convex cocompact
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Remark.

- δ_Γ small enough $\implies$ Γ is convex cocompact [Liu-Wang '23]
- $\delta_\Gamma < \rho$ $\implies$ temperedness $\implies$ Kunze-Stein phenomenon [Z. '20]
- $\delta_\Gamma > \rho$ $\implies$ no more global-in-time Strichartz [BGH '10]