Power moments of automorphic L-function on the critical line

Deyu Zhang

Shandong Normal University, Jinan

2022 Colloquium of Arithmetic, Harbin Institute of Technology

2022.01.10

Outline

Arithmetic function:

$$f : \mathbb{N} \to \mathbb{C}, \quad n \mapsto f(n).$$

The generating function—Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

The set $\mathcal{A}$ of arithmetic functions forms a commutative ring with respect to the standard addition

$$(f + g)(n) := f(n) + g(n)$$

and

$$(f * g)(n) := \sum_{d|n} f(d)g(\frac{n}{d}).$$

Arithmetic function:

$$f : \mathbb{N} \to \mathbb{C}, \quad n \mapsto f(n).$$

The generating function—Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

The set $\mathcal{A}$ of arithmetic functions forms a commutative ring with respect to the standard addition

$$(f + g)(n) := f(n) + g(n)$$

and

$$(f * g)(n) := \sum_{d|n} f(d)g(\frac{n}{d}).$$

The addition and the multiplication of the associated Dirichlet series:

$$L_{f+g}(s) = L_f(s) + L_g(s), \quad L_{f*g}(s) = L_f(s) \cdot L_g(s).$$

Examples:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \qquad L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

$$L(s, \mathrm{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym^2f}}(n)}{n^s} \cdots$$

The addition and the multiplication of the associated Dirichlet series:

$$L_{f+g}(s) = L_f(s) + L_g(s), \quad L_{f*g}(s) = L_f(s) \cdot L_g(s).$$

Examples:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \qquad L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

$$L(s, \mathrm{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym^2f}}(n)}{n^s} \cdots$$

The analytic properties of L-function:

- Analytic continuation
- Functional equation, Approximate Functional equation
- Zeros on the critical line
- Zero free region
- Zero density
- The order of $L(s,f)\cdots$

Outline

Lindelöf Hypothesis

Lindelöf Hypothesis

For $|t| \geq 1$ and any $\epsilon > 0$,

$$L(\frac{1}{2} + it, f) \ll |t|^{\epsilon}.$$

Convexity bound:
For $|t| \geq 1$ and any $\epsilon > 0$,

$$L(\frac{1}{2} + it, f) \ll_f |t|^{m/4+\epsilon}.$$

Subconvexity bound:
For $|t| \geq 1$ and any $\epsilon > 0$,

$$L(\frac{1}{2} + it, f) \ll_f |t|^{m/4-\delta+\epsilon}.$$

Known results

- Riemann zeta function:
  Weyl, 1921
  $$\zeta(\frac{1}{2}+it) \ll (1+|t|)^{\frac{1}{6}+\epsilon}.$$
  J. Bourgain, 2017
  $$\zeta(\frac{1}{2}+it) \ll (1+|t|)^{\frac{1}{6}-\frac{1}{84}+\epsilon}.$$

Known results

- Automorphic L-function of GL(2):

  A. Good

  $$L(\frac{1}{2}+it, f) \ll (1+|t|)^{\frac{1}{3}+\epsilon}.$$

  R. Munshi

  $$L(\frac{1}{2}+it, f) \ll (1+|t|)^{\frac{1}{3}-\frac{1}{1200}+\epsilon}.$$

  R. Holowinsky, R. Munshi, Z. Qi

  $$L(\frac{1}{2}+it, f) \ll (1+|t|)^{\frac{1}{3}-\rho+\epsilon}.$$

Known results

- Automorphic L-function of GL(3)
  R. Munshi, 2015
  $$L(\frac{1}{2}+it,f) \ll (1+|t|)^{\frac{3}{4}-\frac{1}{16}+\epsilon}.$$
  M. Makee, H. Sun, Y. Ye, 2019
  $$L(\frac{1}{2}+it,f) \ll (1+|t|)^{\frac{21}{32}+\epsilon},$$
  f is self-contragradient Hecke-Maass form of $SL(3,Z)$.
  K. Aggarwal, 2021
  $$L(\frac{1}{2}+it,\pi) \ll (1+|t|)^{\frac{3}{4}-\frac{3}{40}+\epsilon}.$$
- Symmetric L-function(Nunes, 2017)
  $$L(\frac{1}{2}+it,sym^2 f) \ll (1+|t|)^{\frac{5}{8}+\epsilon}.$$

Known results

- Automorphic L-function of GL(3)
  R. Munshi, 2015
  $$L(\frac{1}{2}+it,f) \ll (1+|t|)^{\frac{3}{4}-\frac{1}{16}+\epsilon}.$$
  M. Makee, H. Sun, Y. Ye, 2019
  $$L(\frac{1}{2}+it,f) \ll (1+|t|)^{\frac{21}{32}+\epsilon},$$
  f is self-contragradient Hecke-Maass form of $SL(3,Z)$.
  K. Aggarwal, 2021
  $$L(\frac{1}{2}+it,\pi) \ll (1+|t|)^{\frac{3}{4}-\frac{3}{40}+\epsilon}.$$
- Symmetric L-function(Nunes, 2017)
  $$L(\frac{1}{2}+it,sym^2 f) \ll (1+|t|)^{\frac{5}{8}+\epsilon}.$$

Lindelöf Hypothesis

Lindelöf Hypothesis

For any $\epsilon > 0$,

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^{2k} dt \ll T^{1+\epsilon}.$$

holds for all positive integers k.

- Riemann zeta function

$$\int_1^T \left| \zeta(\frac{1}{2} + it) \right|^{4} dt \ll T \log^4 T.$$

- Automorphic L-function of GL(m)

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^{2} dt \ll T^{\frac{m}{2}+\epsilon}.$$

Lindelöf Hypothesis

Lindelöf Hypothesis

For any $\epsilon > 0$,

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^{2k} dt \ll T^{1+\epsilon}.$$

holds for all positive integers k.

- Riemann zeta function

$$\int_1^T \left| \zeta(\frac{1}{2} + it) \right|^{4} dt \ll T \log^4 T.$$

- Automorphic L-function of GL(m)

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^{2} dt \ll T^{\frac{m}{2}+\epsilon}.$$

Outline

Definition of $m(\sigma)$

- For $\frac{1}{2} < \sigma < 1$ fixed, we define $m(\sigma)$ as the supremum of all numbers m such that

$$\int_1^T |L(\sigma + it, f)|^m \, dt \ll T^{1+\epsilon},$$

for any $\epsilon > 0$.

Remark 1. Our aim is to seek the lower bound of $m(\sigma)$.

Remark 2. The results of $m(\sigma)$ can be used in the mean value of arithmetic functions.

Definition of $m(\sigma)$

- For $\frac{1}{2} < \sigma < 1$ fixed, we define $m(\sigma)$ as the supremum of all numbers m such that

$$\int_1^T |L(\sigma + it, f)|^m \, dt \ll T^{1+\epsilon},$$

for any $\epsilon > 0$.

Remark 1. Our aim is to seek the lower bound of $m(\sigma)$.

Remark 2. The results of $m(\sigma)$ can be used in the mean value of arithmetic functions.

- Riemann zeta function (C. Tirchmarsh, A. Ivić)

$$
\begin{aligned}
m(\sigma) &\geq 4/(3-4\sigma), && 1/2 < \sigma \leq 5/8, \\
m(\sigma) &\geq 10/(5-6\sigma), && 5/8 \leq \sigma \leq 35/54, \\
m(\sigma) &\geq 19/(6-6\sigma), && 35/54 \leq \sigma \leq 41/60, \\
m(\sigma) &\geq 2112/(859-948\sigma), && 41/60 \leq \sigma \leq 3/4, \\
m(\sigma) &\geq 12408/(4537-4890\sigma), && 3/4 \leq \sigma \leq 5/6, \\
m(\sigma) &\geq 4324/(1031-1044\sigma), && 5/6 \leq \sigma \leq 7/8, \\
m(\sigma) &\geq 98/(31-32\sigma), && 7/8 \leq \sigma \leq 0.91591\ldots, \\
m(\sigma) &\geq (24\sigma-9)/(4\sigma-1)(1-\sigma), && 0.91591\ldots \leq \sigma < 1.
\end{aligned}
$$

Power moments of Riemann zeta function

- The lower bounds of $m(\sigma)$ implies

$$\int_1^T |\zeta(\sigma+it)|^6 \, dt \ll T^{1+\epsilon}, \qquad 7/12 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma+it)|^8 \, dt \ll T^{1+\epsilon}, \qquad 5/8 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma+it)|^{10} \, dt \ll T^{1+\epsilon}, \qquad 41/60 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma+it)|^{12} \, dt \ll T^{1+\epsilon}, \qquad 5/7 < \sigma < 1.$$

Automorphic L-function of GL(2)

- A. Ivić(1989)

$$m(\sigma) \geq \frac{2}{3-4\sigma}, \quad 1/2 < \sigma \leq 5/8.$$

- H. Liu, S. Li and D. Zhang (2016)

$$m(\sigma) \geq \frac{5-2\sigma}{2(3-2\sigma)(1-\sigma)}, \quad 5/8 \leq \sigma \leq 1-\epsilon.$$

which implies

$$\int_1^T |L(\sigma+it,f)|^4\,dt \ll T^{1+\varepsilon}, \quad 5/8 < \sigma < 1.$$

$$\int_1^T |L(\sigma+it,f)|^6\,dt \ll T^{1+\varepsilon}, \quad 0.797\ldots < \sigma < 1.$$

Automorphic L-function of GL(2)

- A. Ivić(1989)

$$m(\sigma) \geq \frac{2}{3-4\sigma}, \quad 1/2 < \sigma \leq 5/8.$$

- H. Liu, S. Li and D. Zhang (2016)

$$m(\sigma) \geq \frac{5-2\sigma}{2(3-2\sigma)(1-\sigma)}, \quad 5/8 \leq \sigma \leq 1-\epsilon.$$

which implies

$$\int_1^T |L(\sigma+it,f)|^4 \, dt \ll T^{1+\varepsilon}, \quad 5/8 < \sigma < 1.$$

$$\int_1^T |L(\sigma+it,f)|^6 \, dt \ll T^{1+\varepsilon}, \quad 0.797\ldots < \sigma < 1.$$

Automorphic L-function of GL(3)

J. Huang, H. Liu and D. Zhang (2021)

$$m(\sigma) \geq \frac{4(3-2\sigma)}{5(4-3\sigma)(1-\sigma)}, \quad \frac{2}{3} < \sigma < 1.$$

which implies

$$\int_1^T |L(\sigma+it,f)|^2 dt \ll T^{1+\varepsilon} \qquad \frac{2}{3} \leq \sigma \leq 1.$$

$$\int_1^T |L(\sigma+it,f)|^4 dt \ll T^{1+\varepsilon} \qquad 0.823\ldots \leq \sigma \leq 1.$$

$$\int_1^T |L(\sigma+it,f)|^6 dt \ll T^{1+\varepsilon} \qquad 0.877\ldots \leq \sigma \leq 1.$$

Automorphic L-function of GL(3)

J. Huang, H. Liu and D. Zhang (2021)

$$m(\sigma) \geq \frac{4(3-2\sigma)}{5(4-3\sigma)(1-\sigma)}, \quad \frac{2}{3} < \sigma < 1.$$

which implies

$$\int_1^T |L(\sigma+it,f)|^2 dt \ll T^{1+\varepsilon} \qquad \frac{2}{3} \leq \sigma \leq 1.$$

$$\int_1^T |L(\sigma+it,f)|^4 dt \ll T^{1+\varepsilon} \qquad 0.823\ldots \leq \sigma \leq 1.$$

$$\int_1^T |L(\sigma+it,f)|^6 dt \ll T^{1+\varepsilon} \qquad 0.877\ldots \leq \sigma \leq 1.$$

Definition of $M(A)$

- For any fixed number A, the number $M(A)$ is defined as the infimum of all numbers M such that

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^A dt \ll T^{M+\epsilon},$$

for any $\epsilon > 0$.

Remark 1. Our aim is to seek the upper bound of $M(A)$.

Remark 2. $M(A) = 1$, for $A \geq 1$ is equivalent to the Lindelöf Hypothesis.

Remark 3. The results of $M(A)$ can be used in proving zero density estimates of L-functions.

Definition of $M(A)$

- For any fixed number A, the number $M(A)$ is defined as the infimum of all numbers M such that

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^A dt \ll T^{M+\epsilon},$$

for any $\epsilon > 0$.

Remark 1. Our aim is to seek the upper bound of $M(A)$.

Remark 2. $M(A) = 1$, for $A \geq 1$ is equivalent to the Lindelöf Hypothesis.

Remark 3. The results of $M(A)$ can be used in proving zero density estimates of L-functions.

Riemann zeta function

A. Ivić

If $A \geq 4$ is a fixed number. Then

$$M(A) \leq \begin{cases} 1 + \frac{A-4}{8}, & 4 \leq A \leq 12, \\ 2 + \frac{3(A-12)}{22}, & 12 \leq A \leq \frac{178}{13}, \\ 1 + \frac{35(A-6)}{216}, & A \geq \frac{178}{13}. \end{cases}$$

which implies that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^6 dt \ll T^{\frac{5}{4}+\epsilon},$$

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^8 dt \ll T^{\frac{3}{2}+\epsilon}.$$

Riemann zeta function

A. Ivić

If $A \geq 4$ is a fixed number. Then

$$M(A) \leq \begin{cases} 1+\frac{A-4}{8}, & 4 \leq A \leq 12, \\ 2+\frac{3(A-12)}{22}, & 12 \leq A \leq \frac{178}{13}, \\ 1+\frac{35(A-6)}{216}, & A \geq \frac{178}{13}. \end{cases}$$

which implies that

$$\int_1^T \left|\zeta(\frac{1}{2}+it)\right|^6 dt \ll T^{\frac{5}{4}+\epsilon},$$

$$\int_1^T \left|\zeta(\frac{1}{2}+it)\right|^8 dt \ll T^{\frac{3}{2}+\epsilon}.$$

Automorphic L-function of GL(2)

R. Zhang, and D. Zhang

If $A \geq 2$ is a fixed number, then

$$M(A) \leq \begin{cases} 1 + \frac{A-2}{4}, & 2 \leq A \leq 6; \\ 1 + \frac{3A-7}{11}, & 6 \leq A \leq \frac{89}{13}; \\ 1 + \frac{A-3}{3}, & A > \frac{89}{13}. \end{cases}$$

which implies that

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^4 dt \ll T^{\frac{3}{2}+\epsilon},$$

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^8 dt \ll T^{\frac{8}{3}+\epsilon}.$$

Automorphic L-function of GL(2)

R. Zhang, and D. Zhang

If $A \geq 2$ is a fixed number, then

$$M(A) \leq \begin{cases} 1 + \frac{A-2}{4}, & 2 \leq A \leq 6; \\ 1 + \frac{3A-7}{11}, & 6 \leq A \leq \frac{89}{13}; \\ 1 + \frac{A-3}{3}, & A > \frac{89}{13}. \end{cases}$$

which implies that

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^4 dt \ll T^{\frac{3}{2}+\epsilon},$$

$$\int_1^T \left| L(\frac{1}{2} + it, f) \right|^8 dt \ll T^{\frac{8}{3}+\epsilon}.$$

Symmetric square L-function

J. Huang, W. Zhai and D. Zhang

If $A \geq 2$ is a fixed number, then

$$M(A) \leq \begin{cases} \frac{3}{2} + \frac{3(A-2)}{10}, & \text{if } 2 \leq A \leq 7, \\ 3 + \frac{3(A-7)}{7}, & \text{if } 7 \leq A \leq \frac{105}{13}, \\ 2 + \frac{5(A-4)}{8}, & \text{if } A \geq \frac{105}{13}. \end{cases}$$

which implies that

$$\int_1^T \left|L(1/2+it, \mathrm{sym}^2 f)\right|^4 dt \ll T^{\frac{21}{10}+\epsilon},$$

$$\int_1^T \left|L(1/2+it, \mathrm{sym}^2 f)\right|^8 dt \ll T^{\frac{24}{7}+\epsilon}.$$

Symmetric square L-function

J. Huang, W. Zhai and D. Zhang

If $A \geq 2$ is a fixed number, then

$$M(A) \leq \begin{cases} \frac{3}{2} + \frac{3(A-2)}{10}, & \text{if } \ 2 \leq A \leq 7, \\ 3 + \frac{3(A-7)}{7}, & \text{if } \ 7 \leq A \leq \frac{105}{13}, \\ 2 + \frac{5(A-4)}{8}, & \text{if } \ A \geq \frac{105}{13}. \end{cases}$$

which implies that

$$\int_1^T \left|L(1/2+it, \mathrm{sym}^2 f)\right|^4 dt \ll T^{\frac{21}{10}+\epsilon},$$

$$\int_1^T \left|L(1/2+it, \mathrm{sym}^2 f)\right|^8 dt \ll T^{\frac{24}{7}+\epsilon}.$$

Outline

Zero density of L-functions

- All the nontrivial zeros of $L(s,f)$ lie in the critical strip $0 \le \Re s \le 1$.

Riemann Hypothesis

All the nontrivial zeros of $L(s,f)$ lie in the critical line $\Re s = 1/2$.

- Zero density: Define

$$N(\sigma, T, f) = \sharp \{\rho = \beta + i\gamma : L(\rho, f) = 0,\ \sigma \le \beta \le 1,\ |\gamma| \le T\},$$

where $1/2 \le \sigma \le 1$.

- Aim:

$$N(\sigma, T, f) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}.$$

Zero density of L-functions

- All the nontrivial zeros of $L(s, f)$ lie in the critical strip $0 \leq \Re s \leq 1$.

Riemann Hypothesis

All the nontrivial zeros of $L(s, f)$ lie in the critical line $\Re s = 1/2$.

- Zero density: Define

$$N(\sigma, T, f) = \sharp \{\rho = \beta + i\gamma : L(\rho, f) = 0,\ \sigma \leq \beta \leq 1,\ |\gamma| \leq T\},$$

where $1/2 \leq \sigma \leq 1$.

- Aim:

$$N(\sigma, T, f) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}.$$

Zero density of L-functions

- All the nontrivial zeros of $L(s,f)$ lie in the critical strip $0 \le \Re s \le 1$.

Riemann Hypothesis

All the nontrivial zeros of $L(s,f)$ lie in the critical line $\Re s = 1/2$.

- Zero density: Define

$$N(\sigma, T, f) = \sharp\{\rho = \beta + i\gamma : L(\rho, f) = 0,\ \sigma \le \beta \le 1,\ |\gamma| \le T\},$$

where $1/2 \le \sigma \le 1$.

- Aim:

$$N(\sigma, T, f) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}.$$

Zero Density Conjecture

$$N(\sigma, T, f) \ll T^{2(1-\sigma)+\varepsilon}.$$

- Riemann zeta function(M. N. Huxley and A. E. Ingham)

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}+\varepsilon}, \quad \text{for} \quad 1/2 \le \sigma \le 3/4$$

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{3\sigma-1}+\varepsilon}, \quad \text{for} \quad 3/4 \le \sigma \le 1$$

- $$A(\sigma) \le \frac{12}{5}.$$

Zero Density Conjecture

$$N(\sigma, T, f) \ll T^{2(1-\sigma)+\varepsilon}.$$

- Riemann zeta function(M. N. Huxley and A. E. Ingham)

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}+\varepsilon}, \quad \text{for} \quad 1/2 \le \sigma \le 3/4$$

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{3\sigma-1}+\varepsilon}, \quad \text{for} \quad 3/4 \le \sigma \le 1$$

- $$A(\sigma) \le \frac{12}{5}.$$

Zero density of automorphic L-function of GL(2)

- Ivić proved

$$N(\sigma, T, f) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}, \quad \text{for} \quad 1/2 \le \sigma \le 3/4$$

and

$$N(\sigma, T, f) \ll T^{\frac{2(1-\sigma)}{\sigma}+\varepsilon}, \quad \text{for} \quad 3/4 \le \sigma \le 1.$$

- $$A(\sigma) \le \frac{8}{3}.$$

Zero density of automorphic L-function of GL(2)

- Ivić proved

$$N(\sigma, T, f) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}, \quad \text{for} \quad 1/2 \leq \sigma \leq 3/4$$

and

$$N(\sigma, T, f) \ll T^{\frac{2(1-\sigma)}{\sigma}+\varepsilon}, \quad \text{for} \quad 3/4 \leq \sigma \leq 1.$$

- $$A(\sigma) \leq \frac{8}{3}.$$

Zero density of symmetric square L-function

- Y. Ye and D. Zhang, 2013

$$N(\sigma, T, f) \ll \begin{cases} T^{\frac{5(1-\sigma)}{3-2\sigma}+\epsilon}, & \text{for } \frac{1}{2} \le \sigma < \frac{3}{4}; \\ T^{3(1-\sigma)+\epsilon}, & \text{for } \frac{3}{4} \le \sigma < 1. \end{cases}$$

- J. Huang, W. Zhai and D. Zhang

$$N(\sigma, T, f) \ll \begin{cases} T^{\frac{5(1-\sigma)}{3-2\sigma}+\epsilon}, & \frac{1}{2} \le \sigma < \frac{23}{32}; \\ T^{\frac{26(1-\sigma)}{11-4\sigma}+\epsilon}, & \frac{23}{32} \le \sigma < \frac{3}{4}; \\ T^{\frac{2(1-\sigma)}{\sigma}+\epsilon}, & \frac{3}{4} \le \sigma < 1. \end{cases}$$

- $$A(\sigma) \le \frac{13}{4}.$$

Zero density of symmetric square L-function

- Y. Ye and D. Zhang, 2013

$$N(\sigma, T, f) \ll \begin{cases} T^{\frac{5(1-\sigma)}{3-2\sigma}+\epsilon}, & \text{for } \frac{1}{2} \le \sigma < \frac{3}{4}; \\ T^{3(1-\sigma)+\epsilon}, & \text{for } \frac{3}{4} \le \sigma < 1. \end{cases}$$

- J. Huang, W. Zhai and D. Zhang

$$N(\sigma, T, f) \ll \begin{cases} T^{\frac{5(1-\sigma)}{3-2\sigma}+\epsilon}, & \frac{1}{2} \le \sigma < \frac{23}{32}; \\ T^{\frac{26(1-\sigma)}{11-4\sigma}+\epsilon}, & \frac{23}{32} \le \sigma < \frac{3}{4}; \\ T^{\frac{2(1-\sigma)}{\sigma}+\epsilon}, & \frac{3}{4} \le \sigma < 1. \end{cases}$$

- $$A(\sigma) \le \frac{13}{4}.$$

Outline

For $M = M(A)$,

$$\int_1^T |L(1/2 + it, \mathrm{sym}^2 f)|^A dt \ll T^{M(A)+\epsilon}$$

is equivalent to

$$\sum_{r \leq R} |L(1/2 + it_r, \mathrm{sym}^2 f)|^A \ll T^{M(A)+\epsilon},$$

where

$t_r \in [T, 2T]$ *for* $r = 1, \cdots, R$; $|t_r - t_s| \geq \log^C T$ *for* $1 \leq r \neq s \leq R$

and $C \geq 0$ is fixed.

We suppose that $A \geq 2$ is fixed and

$$R \ll T^{M(A)+\epsilon} V^{-A},$$

we have

$$\left| L\left(1/2 + it_r, \mathrm{sym}^2 f\right) \right| \geq V \ (r = 1, 2, \cdots, R),$$

which is equivalent to

$$\sum_{r \leq R} \left| L\left(1/2 + it_r, \mathrm{sym}^2 f\right) \right|^A \ll T^{M(A)+\epsilon}.$$

(1) Letting $h = \log^2 T$, $Y = (3T)^{3/2}$, $M = (3T)^{3/2}$, $s = \frac{1}{2} + it_r$, and $\Re\omega + \sigma = \frac{3}{4}$, we obtain

$$L(s, \mathrm{sym}^2 f) = \sum_{n \le 2(3T)^{3/2}} \lambda_{\mathrm{sym}^2 f}(n) e^{-\left(\frac{n}{(3T)^{3/2}}\right)^h} n^{-\frac{1}{2} - it_r}$$

$$- (2\pi i)^{-1} \int_{-h^2}^{h^2} \chi\left(\frac{3}{4} + it_r + iv, \mathrm{sym}^2 f\right) \sum_{n \le (3T)^{3/2}} \lambda_{\mathrm{sym}^2 f}(n) n^{-\frac{1}{4} + it_r + iv}$$

$$\times \left((3T)^{3/2}\right)^{\frac{1}{4} + iv} \Gamma\left(1 + \frac{\omega}{h}\right) \frac{dv}{\frac{1}{4} + iv} + O(1).$$

Thus according to Cauchy-Schwarz inequality, we have

$$\sum_{r\le R}\left|L(1/2+it_r,\mathrm{sym}^2 f)\right|^2 \ll \sum_{r\le R}\left|\sum_{n\le 2(3T)^{3/2}}\lambda_{\mathrm{sym}^2 f}(n)n^{-\frac{1}{2}-it_r}\right|^2 + R$$

$$+T^{-\frac{3}{4}}\left(\max_{|v|\le h^2}\sum_{r\le R}\left|\sum_{n\le (3T)^{3/2}}\lambda_{\mathrm{sym}^2 f}(n)n^{-\frac{1}{4}+it_r+iv}\right|^2\right)\left(\int_{-h^2}^{h^2}\frac{e^{-|\frac{v}{h}|}}{|\frac{1}{4}+iv|}dv\right)^2$$

By using the mean value estimate for the sums over $r \leq R$, we have

$$RV^2 \leq \sum_{r \leq R} \left| L\left(1/2 + it_r, \mathrm{sym}^2 f\right)\right|^2$$
$$\ll T^{3/2} \log^3 T + R + T^{-3/4} (\log\log T)^2$$
$$\times \left(T \sum_{n \leq (3T)^{3/2}} \lambda^2_{\mathrm{sym}^2 f}(n) n^{-1/2} + \sum_{n \leq (3T)^{3/2}} \lambda^2_{\mathrm{sym}^2 f}(n) n^{1/2} \right) \log T$$
$$\ll T^{3/2+\epsilon}.$$

Then we get

$$R \ll V^{-2} T^{3/2+\epsilon}.$$

(2) For $T^{1/4+\epsilon} \leq G \leq T^{3/4-\epsilon}$, we have

$$\int_{T-G}^{T+G} \left| L\left(1/2+it, \mathrm{sym}^2 f\right)\right| dt \ll GT^{1/4} \log T +$$

$$G \sum_K (TK)^{-\frac{1}{3}} \left(|S(K)| + K^{-1} \int_0^K |S(x)| dx \right) \exp(-G^2 K/T),$$

where

$$\begin{aligned} S(x) &= S(x, K, T) = \sum_{K \leq n \leq K+x} \lambda_{\mathrm{sym}^2 f}(n) \exp(if(T)), \\ f(T) &= -T \log x_0 + 3T - 4\pi x_0, \\ x_0 &= \frac{T}{2\pi} + \left(-\frac{nT}{4\pi} - \left(\frac{n^2 T^2}{16\pi^2} + \frac{n^3}{27} \right)^{1/2} \right)^{1/3} \\ &\quad + \left(-\frac{nT}{4\pi} + \left(\frac{n^2 T^2}{16\pi^2} + \frac{n^3}{27} \right)^{1/2} \right)^{1/3} \end{aligned}$$

- The approximate Functional Equation of $L\left(s, \mathrm{sym}^2 f\right)$.
- Voronoi summation formula (Lau, Liu, Wu)

$$\sum_{a \le n \le b} \lambda_{\mathrm{sym}^2 f}(n) f(n) = \frac{2}{\sqrt{3}} (nx)^{-1/3} \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^2 f}(n) \int_a^b f(x) \alpha(nx) dx,$$

where

$$\alpha(nx) = \cos(6\pi (nx)^{1/3}) - (4\pi^3)^{-1/3} (nx)^{-1/3} \sin((6\pi (nx)^{1/3}) + O\left((nx)^{-2/3} \right).$$

- Estimation of the exponential sum

Lemma.

Let $\mathcal{A}$ be a set of real numbers t_r such that $T/2 \leq t_r \leq T$ and $\log^2 T \leq G \leq |t_r - t_s| \leq J$ for $r \neq s$. If $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$, then for $K \leq T/\log T$, $T \geq T_0$ and any exponent pair (κ, λ), one has

$$\sum_{t_r \in \mathcal{A}} |S(x, K, t_r)| \ll \Big((K + K^{5/6} T^{1/3} G^{-1/2} \log^{1/2} T) |\mathcal{A}|^{1/2} + J^{\kappa/2} T^{-\kappa/3} |\mathcal{A}| K^{(3+3\lambda-2\kappa)/6} \Big) \log^{3/2} T.$$

Lemma

Let (κ, λ) be any exponent pair, and let $t_1 < \cdots < t_R$ satisfy

$$|t_r| \le T,\ \ r = 1, \cdots, R;\ \ |t_r - t_s| \ge 1,\ \ 1 \le r \ne s \le R,$$

and

$$\left| L\left(\frac{1}{2} + it_r, \mathrm{sym}^2 f\right)\right| \ge V > 0\ (r = 1, 2, \cdots, R).$$

Then we have

$$R \ll T^2 V^{-4} \log^3 T + T^{(2\lambda+\kappa)/\kappa} V^{-(2\kappa+3\lambda+1)/\kappa} (\log T)^{(4\kappa+6\lambda+1)/\kappa}. \tag{5.1}$$

Corollary

Choosing some special exponent pair, we can obtain

$$
\begin{aligned}
R &\ll T^2V^{-4}\log^3 T + T^{45/13}V^{-105/13}\log^{179/13} T,\\
R &\ll T^2V^{-4}\log^3 T + T^{4}V^{-37/4}\log^{63/4} T,\\
R &\ll T^2V^{-4}\log^3 T + T^{5}V^{-23/2}\log^{39/2} T,\\
R &\ll T^2V^{-4}\log^3 T + T^{7}V^{-79/5}\log^{134/5} T,\\
R &\ll T^2V^{-4}\log^3 T + T^{13/2}V^{-59/4}\log^{25} T.
\end{aligned}
$$

For $0 \le \alpha \le 1$, by (5.1) with $(\kappa, \lambda) = (1/2, 1/2)$, we have

$$\begin{aligned} R = R^{\alpha} R^{1-\alpha} &\ll T^{\epsilon} \left(T^{3/2} V^{-2}\right)^{\alpha} (T^3 V^{-7})^{1-\alpha} \\ &= T^{3-3\alpha/2+\epsilon} V^{5\alpha-7} \\ &= T^{1+\frac{3A-1}{10}+\epsilon} V^{-A}. \end{aligned}$$

for $\alpha = \frac{7-A}{5}$.

Similarly, from the first estimate in Corollary and (5.1) with $(\kappa, \lambda) = (1/2, 1/2)$, we have

$$\begin{aligned} R = R^{\alpha} R^{1-\alpha} &\ll T^{\epsilon} (T^3 V^{-7})^{\alpha} \left(T^{45/13} V^{-105/13} \right)^{1-\alpha} \\ &= T^{(45-6\alpha)/13+\epsilon} V^{(14\alpha-105)/13} \\ &= T^{3+\frac{3(A-7)}{7}+\epsilon} V^{-A}. \end{aligned}$$

for $\alpha = \frac{105-13A}{14}$.

The third estimate in Theorem will follow from a more general result. Let

$$S = \sum_{r \leq R} \left| L(1/2 + it, \mathrm{sym}^2 f) \right|^A \ll T^{c(A-4)+2+\epsilon}, \ A \geq 105/13, \ c \geq 19/53,$$

where $L(1/2 + it, \mathrm{sym}^2 f) \ll |t|^{c+\epsilon}$, thus from the subconvexity bound, the value $c = 5/8$ leads to

$$M(A) \leq 2 + 5(A-4)/8.$$

Therefore, for some $0 < V = 2^k \ll T^{5/8} \log^{3/2} T$, each t_r satisfies

$$V \le |L(1/2 + it_r, \mathrm{sym}^2 f)| \le 2V.$$

Denoting by R_V the number of corresponding $t_r's$ in each case, then we have $R_V \ll T^{2+\epsilon} V^{-4}$ by the first term in Corollary and $R_V \ll T^{45/13+\epsilon} V^{-105/13}$ by the second term in Corollary, thus we can obtain

$$\begin{aligned} S \ll & \sum_{V=2^k \ge T^{19/53}} R_V V^A + \sum_{V=2^k < T^{19/53}} R_V V^A \\ \ll & \; T^{2+\epsilon} \sum_{V=2^k \ge T^{19/53}} V^{A-4} + T^{45/13+\epsilon} \sum_{V=2^k < T^{19/53}} V^{A-105/13} \\ \ll & \; T^{c(A-4)+2+\epsilon}. \end{aligned}$$

THANK YOU FOR YOUR ATTENTION!