Counting rational points in arithmetic varieties by determinant method

K : number field, $\mathcal{O}_K$: ring of integers of K, $M_K = M_{K,f} \sqcup M_{K,\infty}$.

$X \hookrightarrow \mathbb{P}^n_K$: ~~variety~~ integral, $\dim(X) = d$, $\deg(X) = \delta$. $\xi \in X(K)$, $\xi = [x_0 : \cdots : x_n]$

Def (Weil height).

$$H_K(\xi) := \prod_{v \in M_K} \max_{0 \le i \le n} \{|x_i|_v\}^{[K_v:\mathbb{Q}_v]}, \quad h(\xi) = \frac{1}{[K:\mathbb{Q}]}\log(H_K(\xi)) \text{ independent of } K.$$

$S(X;B) = \{\xi \in X(K) \mid H_K(\xi) \le B\}$. $N(X;B) = \# S(X;B) < +\infty$ for B fixed.
 └ Northcott

Key point. understand $N(X;B)$, ~~we~~ in order to understand the density of rational points of X. usually, $N(X;B) \le$????? orally

Prop (Schanuel) $N(\mathbb{P}^n_K; B) = \underbrace{\alpha(n,K)}_{\text{explicit}} B^{n+1} + o(B^{n+1})$

Trivial estimate. $N(X;B) \ll_{n,K,\delta} B^{d+1}$. $N(X;B) \sim_{n,K} B^{d+1}$ if X is linear.

Conj-Thm For, $n \gg 0$, $\delta \geq 2$, $d \geq 2$. $\forall \varepsilon > 0$ we have.

$$N(X;B) \ll_{n,K,\varepsilon} B^{d+\varepsilon}$$

~~or~~ $N(X;B) \ll_{n,K,\varepsilon,\delta} B^{d+\varepsilon}$ (weaker version); ——, $\mathbb{N}$ ...

Idea of determinant method (Bombieri-Pila (1989), Heath-Brown (2002)).

Construct $\{H_i\}_{i=1}^r$ auxiliary hypersurfaces, s.t.

$$\eta_X \notin \bigcup_{i=1}^r H_i \supseteq S(X;B)$$

└ generic point

Target. bounded r and $\max_i\{\deg(H_i)\}$.

Reformulated by Arakelov geometry.

$\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$. Zariski closure of $X \hookrightarrow \mathbb{P}^n_K$ in $\mathbb{P}^n_{\mathcal{O}_K}$.

$$\begin{tikzcd}
P_\xi^*\overline{\mathcal{O}_{\mathcal{X}}(1)} \arrow[r] \arrow[d] & \overline{\mathcal{O}_{\mathcal{X}}(1)} \arrow[d] & \\
\operatorname{Spec}\mathcal{O}_K \arrow[r, "P_\xi"] & \mathcal{X} \arrow[r] & \operatorname{Spec}\mathcal{O}_K
\end{tikzcd}$$

$h_{\overline{\mathcal{O}(1)}}(\xi) := \widehat{\deg}_n (P_\xi^*(\overline{\mathcal{O}_{\mathcal{X}}(1)}))$ Normalized Arakelov degree.

If we equip $\mathcal{O}(1)$ with l^2-norm over all $v \in M_{K,\infty}$, we have.

$$h_{\overline{\mathcal{O}(1)}}(\xi) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_{K,f}} \log \max_{0 \le i \le n} \{|x_i|_v\}^{[K_v:\mathbb{Q}_v]} + \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_{K,\infty}} \log \sqrt{|x_0|_v^2 + \cdots + |x_n|_v^2}.$$

Then we have $h(\xi) \le h_{\overline{\mathcal{O}(1)}}(\xi) \le h(\xi) + \frac{1}{2}\log(n+1)$. almost same. orally

We write $h(\xi)$ instead $h_{\overline{\mathcal{O}(1)}}(\xi)$.

(*) <u>Thm</u>. (P. Salberger, H. Chen).

$(R_i)_{i \in I} \in X(K)$ ($\mathcal{X}(\mathcal{O}_K)$) rational points.

$(\mathfrak{P}_j)_{j \in J} \in \mathrm{Spm}\, \mathcal{O}_K$.

Suppose that $(R_i)_{i \in I} \bmod \mathfrak{P}_j = \xi_j \in \mathcal{X}_{\mathfrak{P}_j}(\mathbb{F}_{\mathfrak{P}_j})$.

($R_i \bmod \mathfrak{P}_j$, in $\mathcal{X}$). $D \in \mathbb{N}_+$.

$\boxed{\text{If}}$ $\sup_{i \in I} h(R_i) < \frac{\hat{\mu}(\overline{F}_D)}{D} - \frac{\log r_1(D)}{D} + \frac{1}{[K:\mathbb{Q}]} \sum_{j \in J} \frac{Q_{\xi_j}(r_1(D))}{D\, r_1(D)} \log N_{\mathfrak{P}_j}$.

where: $\hat{\mu}(\cdot)$: slope of hermitian vector bundle.

$\hat{\mu}(\overline{E}) := \widehat{\deg}_n(\overline{E}) / \mathrm{rk}(E)$.

- $\overline{F}_D$: $\mathcal{O}_{\mathbb{P}^n}(1) — \mathbb{P}^n$ hermitian line bundle.

<u>induced</u> a metric over $E_D = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(D))$, $\forall v \in M_{K,\infty}, D \in \mathbb{N}^+$.

induced by the sup norm.

<u>induced</u>. $\overline{E}_D$ a hermitian vector bundle / $\mathrm{Spec}\,\mathcal{O}_K$.

Consider $H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n}(D)) \xrightarrow{\eta_{X,D}} H^0(X, \mathcal{O}_{\mathbb{P}^n}(D)|_X) \to H^0(\mathcal{X}, \mathcal{O}(D))$

F_D: the largest saturated sub-$\mathcal{O}_K$-module ~~of E_D~~, s.t. $\mathrm{Im}(\eta_{X,D}) = F_D \otimes_{\mathcal{O}_K} K$.

- $\overline{F}_D$: equipped with ~~norms~~ quotient norms from $\overline{E}_D$, $\hat{\mu}(\overline{F}_D) \ge -\frac{1}{2} D \log(n+1)$.
- $r_1(D) = \mathrm{rk}_{\mathcal{O}_K}(F_D)$.
- $Q_\xi(r)$: an increasing function constructed from induced via the local Hilbert-Samuel function of $\mathcal{X}_{\mathfrak{P}_j}$ at ξ_j.
- $N_{\mathfrak{P}_j} = \#(\mathcal{O}_K / \mathfrak{P}_j)$.

$\boxed{\text{Then}}$ $\exists$ one hypersurface H of degree D, s.t. $(R_i)_{i\in I}\in H$, $\eta_x\notin H$.

What we need: upper bound of $r_1(D)$

lower bound of $r_1(D)$, $\hat{\mu}(\overline{F}_D)$, $Q_{\xi_i}(r)$.

uniform

$\square$ color

~~in the statement~~

Very difficult for general X.

Thm (Browning, Heath-Brown, Salberger).

$\forall\ X'\hookrightarrow \mathbb{P}^{d+1}_K$ integral hypersurface, $\deg(X')=\delta$.

$\boxed{\text{If}}$ $N(X';B)\ll_{d,\delta,\varepsilon} B^{\theta(d,\delta)+\varepsilon}$ $\forall\,\varepsilon>0$.

$\boxed{\text{Then}}$ $\forall\ X\hookrightarrow \mathbb{P}^n_K$ arithmetic variety, $\dim(X)=d$, $\deg(X)=\delta$.

$$N(X;B)\ll_{n,\delta,\varepsilon} B^{\theta(d,\delta)+\varepsilon}\quad \forall\,\varepsilon>0.$$

Only need to deal with the $\boxed{\text{case } \cancel{\text{of}}}$ $\boxed{\text{hypersurface}}$ ~~since the Hilbert-Samuel functions are simpler~~. $d=n-1$

①. lower and upper bounds of $r_1(D)$:

$$r_1(D)=\binom{n+D}{D}-\binom{n+D-\delta}{n}.$$

②. uniform lower bound of $\hat{\mu}(\overline{F}_D)$:

(asymptotic estimate) $\widehat{\deg}_n(\overline{F}_D):=\dfrac{D^{d+1}}{(d+1)!}h_{\overline{\mathcal{L}}}(X)+o(D^{d+1})$

$$h_{\overline{\mathcal{L}}}(X)=\widehat{\deg}_n\left([\mathcal{X}]\cdot\hat{c}_1(\overline{\mathcal{L}}^{\otimes(d+1)})\right).$$

uniform version (S. David, P. Philippons, 1998).

$$\hat{\mu}(\overline{F}_D)\geqslant\frac{(D-(n-d)(\delta-1)-1)^{d+1}}{r_1(D)(d+1)^{d+1}}\left[h_{\overline{\mathcal{L}}}(X)-\delta(d+1)\right]+\underbrace{C_0(n,\delta,d,D)}_{o(D^{d+1})}.$$

explicit.

$$\approx\frac{d!}{\delta(2d+2)^{d+1}}h_{\overline{\mathcal{L}}}(X)+C_0'(n,\delta,d,D).$$

uniform upper bound: $\hat{\mu}(\overline{F}_D)\leq\frac{D}{\delta}h_{\overline{\mathcal{L}}}(X)+\frac{D}{2}\log(n+1)$.

Case of hypersurface.

i). estimate of $\hat{\mu}(\overline{E}_D)$ $\quad E_D \cong \mathrm{Sym}^D_{O_K}(O_K^{\oplus(n+1)})$ ⟵ symmetric norm.

$$\widehat{\deg}_n(\overline{E}_{D,\mathrm{sym}}) = \frac{1}{2}\sum_{i_0+\cdots+i_n=D}\log\left(\frac{i_0!\cdots i_n!}{D!}\right).$$

$$= \frac{1-\mathcal{H}_{n+1}}{2n!}D^{n+1} - \frac{n-2}{4n!}D^n\log D + \underset{\substack{\uparrow\\ \text{explicit.}}}{C(n)}D^n + o(D^n)$$

$\mathcal{H}_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$, compare with the sup. norm.

ii) estimate of $\hat{\mu}(\overline{F}_D)$, $0\longrightarrow E_{D-\delta}\longrightarrow E_D\longrightarrow F_D\longrightarrow 0$.

$\widehat{\deg}_n(\overline{F}_D)\approx \widehat{\deg}_n(\overline{E}_D) - \widehat{\deg}_n(\overline{E}_{D-\delta}) + \boxed{\text{comparing the norms}}$

$$\Longrightarrow \frac{\hat{\mu}(\overline{F}_D)}{D} \geq \frac{\check{h}_{\xi}(X)}{\delta} + \underline{B_0(n,\delta)}$$

⟶ explicit.

③ Lower bound of $Q_\xi(r)$.

$\xi\in X(k)$. $\mu_\xi(X)$: multplicity of ξ in X via the local Hilbert-Samuel function.

$$H_\xi(s) = \dim_k(\mathfrak{m}_{X,\xi}^s/\mathfrak{m}_{X,\xi}^{s+1}) = \binom{n+s-1}{s} - \binom{n+s-\mu_\xi(X)-1}{s-\mu_\xi(X)}.$$

General case: <u>we have upper bounds for each coefficients</u>. but too large.

$$\rightsquigarrow Q_\xi(r) > \underbrace{\left(\frac{(n-1)!}{\mu_\xi(X)}\right)^{\frac{1}{n-1}}\left(\frac{n-1}{n}\right)r^{\frac{n}{n-1}}}_{\text{P. Salberger.}} - \underset{\substack{\uparrow\\ \text{explicit.}}}{O(n)}\,r \qquad (\text{uniform}). \text{ oral}$$

Combine ①, ②, ③, we have:

(**) <u>Thm</u>. $\boxed{\text{If}}$ $\displaystyle\sum_{i\in I}\frac{\log N_{\mathfrak{p}_i}}{\mu_{\xi_i}(\mathcal{X}_{\mathfrak{p}_i})^{\frac{1}{n-1}}} \geq (1+\varepsilon)(\log B + [K:\mathbb{Q}]\log(n+1))\,\delta^{-\frac{1}{n-1}}\frac{n}{n-1}.$

where $\max_{i\in I} H_K(R_i) = B$, $\varepsilon>0$. ⟵ explicit

$\boxed{\text{Then}}$ $\exists$ <u>one</u> hypersurface H of degree $>(1+\varepsilon^{-1})(O(n)+\delta)$

s.t. $(R_i)_{i\in I}\in H$, $\eta_X\notin H$.

In order to apply (**) Thm. we consider.

Ob I. If $\mathcal{X}_{p_i}$ ~~is~~ reduced, there aren't too many points with large multiplicity

$\mathcal{X}_{p_i}^{sing}$ cannot be too complex. oral. or abbreviate

II. $Q(\mathcal{X}) = \{\mathfrak{p} \in \mathrm{Spm}\,\mathcal{O}_K \mid \mathcal{X}_{\mathfrak{p}}$ ~~is~~ not reduced$\}$

$\boxed{\text{Then}}$ $\# Q(\mathcal{X}) < +\infty$.

I. Thm $\boxed{\text{If}}$ $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ reduced hypersurface, $\deg(X)=\delta$, $\dim(X^{sing})=s$.

$\boxed{\text{Then}}$ $$\sum_{\xi \in X(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1} \ll_n \delta^{n-s} \max\{q, \delta-1\}^s.$$

Rmk The orders of δ and $\max\{q,\delta-1\}^s$ are ~~both~~ optimal when $q \geq \delta-1$.

Idea: suppose $q \gg 0$ (or we can take an extension).

$X = V(f)$. g_i : directional derivatives of f.

$\exists g_1, \ldots, g_{n-s-1}$. s.t. $\dim(V(f) \cap V(g_1) \cap \cdots \cap V(g_{n-s-1})) = s$

complete intersection

$$\begin{tikzcd}[column sep=small]
 & & V(f)\cap V(g_1)\cap\cdots\cap V(g_{n-s-1}). \arrow[dll] \arrow[dl] \arrow[d] & \\
Y_1 \arrow[d] & Y_2 & \cdots\ Y_l \arrow[d] \arrow[dr] & \\
Y_{11}\ \cdots & & Y_{l,1}\ \cdots & Y_{l,i_l}
\end{tikzcd}$$

$\leftarrow$ irreducible components

......

$\forall\ Y_l(\mathbb{F}_q)$

$\|$

$Y_l^{(a)}(\mathbb{F}_q) \sqcup Y_l^{(b)}(\mathbb{F}_q)$.

$Y_l^{(a)}(\mathbb{F}_q)$	$Y_l^{(b)}(\mathbb{F}_q)$
mult $= \mu_{Y_l}(X)$	mult $\geq \mu_{Y_l}(X)+1$
open dense	codim ≥ 1
in Y_l	in Y_l.

- If $Y_l^{(b)}(\mathbb{F}_q) = \phi$. it's a leaf in this tree.
- If not, $\exists\ \partial^I f$. $|I| = \mu_{Y_l}(X)$. s.t. $Y_l \cap V(\partial^I f)$ is proper.

We can construct a tree as above.

Prop (Taylor expansion) $f(T_0, \ldots, T_n) = \sum_I f^I(\xi_0, \ldots, \xi_n)(T_0-\xi_0)^{i_0}\cdots(T_n-\xi_n)^{i_n}$.

$$\mu_\xi(X) = r \iff \begin{cases} \forall\ |I| < r, & f^I(\xi) = 0. \\ \exists\ |I| = r, & \text{s.t. } f^I(\xi) \neq 0. \end{cases}$$

Depth 1. $\sum_i \mu_{Y_i}(X)(\mu_{Y_i}(X)-1)^{n-s-1}\deg(Y_i) \le \delta(\delta-1)^{n-s-1}$.

$$\Rightarrow \sum_i \sum_{\xi\in Y_i^{(a)}(\mathbb{F}_q)} \mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1} \le \delta(\delta-1)^{n-s-1}(q^s+\cdots+1).$$

Already consider almost all points.

Depth 2. Consider the remaining terms.

Consider these vertices.

$Z=\{M \mid \text{If } M\subsetneq Y_i,\text{ then } M \text{ is a descendent of } Y_i\}$.

Prop. $\forall M\in Z$. $\boxed{\text{Then}}$ $\exists M' \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$, s.t.

① $\forall i=1,\dots,l$. Y_i intersects M' properly at M. ② $\mu_M(Y_i)=i(M;Y_i,M')$

Construction of M'.

By the associativity of proper intersection.

$$\sum_{Y\in Z}\mu_Y(X)(\mu_Y(X)-1)^{n-s-1}\deg(Y)\le\delta(\delta-1)^{n-s}$$

Then. $$\sum_{Y\in Z}\sum_{\xi\in Y^{(a)}(\mathbb{F}_q)}\mu_\xi(X)(\mu_\xi(X)-1)^{n-s-1}\le\delta(\delta-1)^{n-s}(q^{s-1}+\cdots+1)$$

Repeat it again and again ......

II. $X\hookrightarrow\mathbb{P}^n_K$ reduced pure dimensional, $\deg(X)=\delta$. $\dim(X)=d$.

$\mathcal{X}\hookrightarrow\mathbb{P}^n_{\mathcal{O}_K}$. Zariski closure of X in $\mathbb{P}^n_{\mathcal{O}_K}$.

$$\begin{tikzcd} \mathcal{X}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_{\mathbb{F}_p} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathbb{P}^n_{\mathcal{O}_K} \end{tikzcd}$$

Thm. $$\frac{1}{[K:\mathbb{Q}]}\sum_{p\in Q(\mathcal{X})}\log N_p\le(2\delta-2)h_{\overline{\mathcal{O}(1)}}(X)+C(n,\delta).$$

$\hookrightarrow$ explicit.

Idea. i) Case of hypersurface. $X = V(f)$.

$\mathcal{X}_p$ is not reduced $\Longleftrightarrow \forall T_i$, $\mathrm{Res}(f, \frac{\partial f}{\partial T_i}) = 0$.

$\mathcal{X}_p$ is reduced $\Longleftrightarrow \exists$ one T_i, $\mathrm{Res}(f, \frac{\partial f}{\partial T_i}) \neq 0$

ii) General case. (only for pure dimensional case).

Chow (周) form of X: a hypersurface of $Gr(d+1, n)$ of degree δ, which parameters the dimension $n-d-1$ linear varieties intersecting X non empty.

If the fundamental cycle of X is $[X] = \sum_i n_i X_i$, X_i: irreducible componts.

$Gr(d+1, n) \xrightarrow{\text{Plücker embedding}} \mathbb{P}^{\binom{n+1}{d+1}-1}$

$\Phi_X = \prod_i \Phi_{X_i}^{n_i}$ Φ_{X_i}: irreducible polynomials.

X_p is not reduced $\Longleftrightarrow$ its Chow form is not.

+ Comparing heights. it's OK!

For different B, we use different method to attack $N(X;B)$.

- For B very small,

$\boxed{\text{If}}$ $\frac{\log B}{[K:\mathbb{Q}]} < \frac{1}{\delta} h_{\overline{\mathcal{L}}}(X) + B_0(n,\delta) - \frac{1}{2}\log(n+1) \leq \frac{\hat{\mu}(\overline{F}_D)}{D} - \frac{1}{2}\log(n+1)$

(The bound depends on the estimate of arithmetic Hilbert-Samuel function)

$\boxed{\text{Then}}$ $\exists$ one hypersurface H of degree D, s.t. $\mathcal{H}_X \notin H \supseteq S(X;B)$.

(evaluation map in slope method).

- For B larger, choose some non-reduced reduction, use I and II ....

Prop. (Heath-Brown, Broberg).

The number of auxillary hypersurface $r \ll_{n,\delta,\varepsilon} B^{(d+1)/(\delta^{\frac{1}{d}}+\varepsilon)}$

for $\varepsilon > 0$. (For general case). Idea is a little different.

Singular and regular, Singular is covered by one hypersurface.

(Reformulated by H. Chen, control its height).

$\cancel{X \subseteq \mathbb{P}^n_K}$, $Z = (P_i)_{i \in I} \subseteq X \hookrightarrow \mathbb{P}^n_K$.

$$\rightsquigarrow \quad
\begin{tikzcd}
\eta_{Z,D} : H^0(\mathbb{P}^n_K, \mathcal{O}(D)) \arrow[rr] \arrow[dr, "\eta_{X,D}"'] & & \bigoplus_{i \in I} P_i^* \mathcal{O}(D) \\
& F_{D,K} \arrow[ur, "\phi_{Z,D}"'] &
\end{tikzcd}$$

<u>Prop</u>. $\boxed{\text{If}}$ $\displaystyle\sup_{i \in I} h(P_i) < \frac{\hat{\mu}_{max}(\overline{F}_D)}{D} - \frac{1}{2D} \log r_1(D)$.

$\boxed{\text{Then}}$ $\phi_{Z,D}$ <u>cannot</u> be injective.

<u>Proof</u>. Assume $\phi_{Z,D}$ injective

$\exists\, I_0 \subset I$ with $\# I_0 = r_1(D)$ s.t. $pr_{I_0} \circ \phi_{Z,D}$ is injective

$$\xRightarrow{\text{slope inequality}} \quad \hat{\mu}_{max}(\overline{F}_D) \leq \max_{i \in I_0} D\, h(P_i) + \underbrace{h(pr_{I_0} \circ \phi_{Z,D})}_{\leq \frac{1}{2}\log r_1(D)}.$$

$$\lightning$$

$$\leq -\frac{1}{2}\log(n+1)^{D}.$$

With <u>Prop</u>, $\exists$ a hypersurface of degree D containing Z.

but not X.