Perverse Sheaves on Semi-abelian Varieties

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The moduli space of rank 1 $\mathbb{C}$-local system can be defined as:

$$\mathrm{Char}(X) := \mathsf{Hom}(\pi_1(X), \mathbb{C}^*) \cong \mathsf{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*),$$

which is a complex affine torus $\mathrm{Char}(X) \cong (\mathbb{C}^*)^{b_1(X)}$.

jump loci

Definition

The *i-th cohomology jump loci of X* is defined as:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, L_\rho) \neq 0\},$$

where L_ρ is the rank-one local system on X associated to the representation $\rho \in \mathrm{Char}(X)$.

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$\chi(X) = \chi(X, L_\rho)$ for any $\rho \in \mathrm{Char}(G)$.

Example

Set $X = S^1$. Then $\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 1 \\ \emptyset & \text{else} \end{cases}$.

Example

Set $X = \Sigma_g$ with $g \geq 2$. Since $\chi(X) = 2 - 2g \neq 0$,

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 2, \\ \mathrm{Char}(X), & \text{if } i = 1, \\ \emptyset & \text{else} \end{cases}.$$

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Definition

Let X be a smooth quasi-projective variety. $\text{alb} : X \to \text{Alb}(X)$ is a morphism from X to a semi-abelian variety $\text{Alb}(X)$ such that for any morphism $f : X \to G$ to a semi-ableian variety G, there exists one unique morphism $g : \text{Alb}(X) \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\text{alb}} & \text{Alb}(X) \\ & \searrow^{f} & \downarrow{g} \\ & & G \end{array}$$

The Albanese map induces an isomorphism on the free part of H_1:

$$H_1(X, \mathbb{Z})/\mathrm{Torsion} \to H_1(\mathrm{Alb}(X), \mathbb{Z}).$$

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This gives us an isomorphism:

$$\mathrm{Char}(X) \cong \mathrm{Char}(\text{Alb}(X)).$$

The base change formula gives us that for any $\rho \in Char(X) \cong \mathrm{Char}(\mathsf{Alb}(X))$

$$H^i(X, \mathbb{C}_X \otimes L_\rho) \cong H^i(\mathsf{Alb}(X), (R\,\mathsf{alb}_* \mathbb{C}_X) \otimes L_\rho).$$

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If alb is proper, then **Decomposition Theorem (BBDG)** gives us that $R\,\mathsf{alb}_*\,\mathbb{C}_X$ is a direct sum of semi-simple perverse sheaves (with some shift).

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In general, alb is not proper. But one can still consider $R\,\mathsf{alb}_*\,\mathbb{C}_X$ as a complex of perverse sheaves.

Let G be a semi-abelian variety.
Then G is a complex algebraic group which is an extension

$$1 \to T \to G \to A \to 1,$$

where A is an abelian variety of dimension g and $T \cong (\mathbb{C}^*)^m$ is an algebraic affine torus of dimension m.
In particular, $\pi_1(G) \cong \mathbb{Z}^{m+2g}$ and $\dim G = m + g$.

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where A is an abelian variety of dimension g and $T \cong (\mathbb{C}^*)^m$ is an algebraic affine torus of dimension m.
In particular, $\pi_1(G) \cong \mathbb{Z}^{m+2g}$ and $\dim G = m + g$.
Hence $\mathrm{Char}(G) \cong (\mathbb{C}^*)^{m+2g}$.

Definition

Let $\mathcal{F}$ be a bounded complex of constructible sheaves with $\mathbb{C}$-coefficient on G, i.e., $\mathcal{F} \in D^b_c(G, \mathbb{C})$. The degree i cohomology jump loci of $\mathcal{F}$ are defined as:

$$\mathcal{V}^i(G, \mathcal{F}) := \{\rho \in \mathrm{Char}(G) \mid H^i(G, \mathcal{F} \otimes_{\mathbb{C}} L_\rho) \neq 0\}.$$

The category of perverse sheaves on G, denoted by $\mathrm{Perv}(G, \mathbb{C})$ is a sub-category of $D^b_c(G, \mathbb{C})$.

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So simple and semi-simple perverse sheaves are well-defined.
Every perverse sheaves has a finite length of composition series.

Affine torus case

Theorem (O. Gabber, F. Loeser 1996)

For any $\mathcal{F} \in D^b_c(T, \mathbb{C})$, $\mathcal{F}$ is a perverse sheaf on T, if and only if, for any $i > 0$,

$$\mathcal{V}^i(T, \mathcal{F}) = \emptyset,$$

and for any $i \geq 0$

$$\mathrm{codim} \mathcal{V}^{-i}(T, \mathcal{F}) \geq i \ .$$

Abelian variety case

Theorem (C. Schnell 2015)

For any $\mathcal{F} \in D_c^b(A, \mathbb{C})$, $\mathcal{F}$ is a perverse sheaf on A, if and only if, the following codimension lower bound holds:

$$\text{for any } i \geq 0, \ \operatorname{codim} \mathcal{V}^{\pm i}(A, \mathcal{F}) \geq |2i|.$$

Linearity theorem/ Structure Theorem

Definition

A closed irreducible subvariety V of $\mathrm{Char}(G)$ is called linear, if there exists a short exact sequence of semi-abelian varieties

$$1 \to G''(V) \to G \xrightarrow{q} G'(V) \to 1$$

and some $\rho \in \mathrm{Char}(G)$ such that

$$V := \rho \cdot \mathrm{Im}(q^{\#} : \mathrm{Char}(G'(V)) \to \mathrm{Char}(G)).$$

Let $T''(V)$ and $A''(V)$ denote the affine torus and, resp., the abelian variety part of $G''(V)$.

Theorem (N. Budur, B. Wang 2017)

For any $\mathcal{F} \in D^b_c(G, \mathbb{C})$, $\mathcal{V}^i(G, \mathcal{F})$ is a finite union of linear subvariety of $\mathrm{Char}(G)$.

This theorem built on a long series of partial results due to Green-Lazarsfeld, Arapura, Simpson, Dimca-Papadima, etc.

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Theorem (N. Budur, B. Wang 2015)

Let X be a smooth quasi-projective variety. Then $\mathcal{V}^i(X)$ is a finite union of torsion translated linear subvariety of $\mathrm{Char}(X)$.

Semi-abelian variety case

Definition

Let V be a linear irreducible close sub-variety of $\mathrm{Char}(G)$. We define the *semi-abelian codimension* of V by

$$\mathrm{codim}_{sa} V = \dim G''(V)$$

and its *abelian codimension* by

$$\mathrm{codim}_{a} V = \dim A''(V).$$

Theorem (–, Maxim, Wang 2018)

For any $\mathcal{F} \in D^b_c(G, \mathbb{C})$, $\mathcal{F}$ is a perverse sheaf on G, if and only if, for any $i \geq 0$,

$$\operatorname{codim}_a \mathcal{V}^i(G, \mathcal{F}) \geq i,$$

and

$$\operatorname{codim}_{sa} \mathcal{V}^{-i}(G, \mathcal{F}) \geq i \ .$$

Proof

$\Longrightarrow$ can be done by induction on $\dim T$.

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$\Longleftarrow$ can be done by showing that the codimension lower bound is sharp and induction on the length of perverse sheaves.

Corollary

For any perverse sheaf $\mathcal{P} \in \mathrm{Perv}(G, \mathbb{C})$, the cohomology jump loci of $\mathcal{P}$ satisfy the following properties:

(i) *Propagation property:*

$$\mathcal{V}^{-m-g}(G, \mathcal{P}) \subseteq \cdots \subseteq \mathcal{V}^0(G, \mathcal{P}) \supseteq \cdots \supseteq \mathcal{V}^g(G, \mathcal{P}).$$

(ii) *Generic vanishing: for any $i \neq 0$ and generic $\rho \in \mathrm{Char}(G)$,*

$$H^i(G, \mathcal{P} \bigotimes L_\rho) = 0.$$

Hence $\chi(G, \mathcal{P}) \geq 0$.

If $\chi(G, \mathcal{P}) \neq 0$, then $\mathcal{V}^0(G, \mathcal{P}) = \mathrm{Char}(G)$.

Theorem (-, Maxim, Wang 2018)

If $\mathcal{P} \in \mathrm{Perv}(G, \mathbb{C})$ *is a simple perverse sheaf on* G *with* $\chi(G, \mathcal{P}) = 0$*, then* $\mathcal{V}^0(G, \mathcal{P})$ *is irreducible. Moreover, there exists a rank 1 local system* L_ρ *on* G*, a positive dimensional semi-abelian subvariety* G'' *of* G*, and a perverse sheaf* $\mathcal{P}'$ *on* $G' = G/G''$ *with* $\chi(G', \mathcal{P}') \neq 0$ *such that*

$$\mathcal{P} \cong L_\rho \otimes q^*\mathcal{P}'[\dim G'']$$

holds for the quotient map $q : G \to G/G''$*. In particular,* $\mathcal{V}^0(G, \mathcal{P}) = \mathcal{V}^i(G, \mathcal{P})$ *for any* $i \in [-\mathrm{codim}_{sa}\mathcal{V}^0(G, \mathcal{P}), \mathrm{codim}_a\mathcal{V}^0(G, \mathcal{P})]$*.*

Corollary

Assume that $\mathcal{P} \in \mathrm{Perv}(G, \mathbb{C})$ *is a simple perverse sheaf on* G*. Then* $\mathcal{V}^0(G, \mathcal{P})$ *is an isolated point, if and only if,* $\mathcal{P}$ *is a rank 1 local system.*

Corollary

Let X be a n-dimensional smooth complex quasi-projective variety with $\mathrm{alb} : X \to \mathrm{Alb}(X)$ *being proper. If* $\bigcup_{i=0}^{2n} \mathcal{V}^i(X)$ *has an isolated point, then* alb *is surjective.*

Proof.

By decomposition theorem,

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Then there exists at least one simple perverse sheaf $\mathcal{P}_j$ such that $\mathcal{V}^0(\mathcal{P}_j)$ is exactly this isolated point.

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Then there exists at least one simple perverse sheaf $\mathcal{P}_j$ such that $\mathcal{V}^0(\mathcal{P}_j)$ is exactly this isolated point. So $\mathcal{P}_j$ is indeed a rank 1 local system on $\mathrm{Alb}(X)$, hence alb is surjective. □

Corollary

Let X be a n-dimensional smooth complex quasi-projective variety with $\mathrm{alb} : X \to \mathrm{Alb}(X)$ being proper. If X is homotopy equivalent to a real torus, then X is isomorphic to a semi-abelian variety.

Thank you !