SIMULTANEOUS NONVANISHING OF L-VALUES FOR MODULAR L-FUNCTIONS

YOUNGJU CHOIE, WINFRIED KOHNEN, AND YICHAO ZHANG

Abstract. A generalized Riemann hypothesis states that all zeros of the completed Hecke L-function $L^*(f,s)$ of a normalized Hecke eigenform f on the full modular group should lie on the vertical line $Re(s)=\frac{k}{2}$. It was shown in [5] that there exists a Hecke eigenform f of weight k such that $L^*(f,s)\neq 0$ for sufficiently large k and any point on the line segments $Im(s)=t_0, \frac{k-1}{2}<Re(s)<\frac{k}{2}-\epsilon, \frac{k}{2}+\epsilon<Re(s)<\frac{k+1}{2}$, for any given real number t_0 and a positive real number ϵ. This paper concerns the non-vanishing of the product $L^*(f,s)L^*(f,w)$ $(s,w\in\mathbb{C})$ on average.

1. Introduction

Let $L^*(f,s)\,(s\in\mathbb{C})$ be the complete Hecke L-function of a non-zero cuspidal Hecke eigenform f of integral weight k on $SL_2(\mathbb{Z})$. Although the generalized Riemann hypothesis, which states that all zeros should lie on the vertical line $Re(s)=\frac{k}{2}$, seems too far to prove in this stage, it is well known that zeros of $L^*(f,s)$ can occur only inside the critical strip $\frac{k-1}{2}<Re(s)<\frac{k}{2}$. However, Kohnen ([5]) showed that there exists a Hecke eigenform f of weight k such that $L^*(f,s)\neq 0$ for sufficiently large k and any point on the line segments $Im(s)=t_0, \frac{k-1}{2}<Re(s)<\frac{k}{2}-\epsilon, \frac{k}{2}+\epsilon<Re(s)<\frac{k+1}{2}$, for any given real number t_0 and a positive real number ϵ. This result and its method inspired various works on non-vanishing of L-values for different kinds of modular forms (see [2, 3, 6]).

This paper concerns the non-vanishing of the product $L^*(f,s)L^*(f,w)$ $(s,w\in\mathbb{C})$ on average. We shall prove that, given positive real numbers T and δ and for all k large enough the sum of the products $L^*(f,s)L^*(f,w)$ over the basis of Hecke eigenforms of weight k does not vanish on the region $Im(s),Im(w)\in[-T,T], \frac{k-1}{2}<Re(s),Re(w)<\frac{k+1}{2}, |Re(s)-\frac{k}{2}|>\delta, |Re(w)-\frac{k}{2}|>\delta$. For the proof we compute the Fourier coefficients of

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the double Eisenstein series. Since it is dual with respect to the Petersson scalar product of the values $L^*(f,s)L^*(f,w)$ by [1], we derive the result by estimating the first term of Fourier coefficients. This seems the first non-vanishing result for the product $L^*(f,s)L^*(f,w)$ inside the critical region.

2. **Notation**

- For complex numbers z and s with $z \neq 0$, fix the branch of $z^s = e^{s \log z}$ as $\log z = \log|z| + i\arg(z)$ and $-\pi < \arg(z) \leq \pi$.
- $\mathbb{H}$: the complex upper half plane.
- k : an even positive integer.
- Γ : the full modular group.
- Γ_∞ : the subgroup generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.
- $\mathcal{M}_n := \{\gamma \in M_{2\times 2}(\mathbb{Z}) : \det(\gamma) = n\}$.
- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \mathrm{Re}(s) > 1$.
- S_k the space of cusp forms of weight k on Γ with the Petersson scalar product $< \, , >$.
- $\mathcal{B}_k$ the basis of normalized Hecke eigenforms, i.e, eigenforms whose first Fourier coefficient equals 1, of S_k.

3. **Statement of Result**

For $f(\tau) = \sum_{n\geq 1} a_f(n) e^{2\pi i n\tau} (\tau \in \mathbb{H})$, let $L(f,s) := \sum_{n\geq 1} \frac{a_f(n)}{n^s} \, (Re(s) > \frac{k+1}{2})$ be the Hecke L-function associated to f. It is well-known that the complete $L^*(f,s)$ function

$$L^*(f,s) := (2\pi)^{-s}\Gamma(s)L(f,s)$$

has an analytic continuation and the functional equation

$$L^*(f,k-s) = (-1)^{\frac{k}{2}} L^*(f,s).$$

Theorem 3.1 (Main Theorem)**.** *For any fixed positive real numbers T,δ, let the region $\mathcal{R}_{T,\delta}$ of points $(s,w) \in \mathbb{C}^2$ such that*

- $Im(s), Im(w) \in [-T,T]$,
- $\frac{k-1}{2} < Re(s), Re(w) < \frac{k+1}{2}$,
- $|Re(s) - \frac{k}{2})| \geq \delta$ *and* $|Re(w) - \frac{k}{2})| \geq \delta$.

Then there exists a constant $C(T,\delta) > 0$ depending only on T and δ such that for $k > C(T,\delta)$, the following function

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f, f\rangle}$$

does not vanish at any pair $(s,w) \in \mathcal{R}_{T,\delta}$.

Corollary 3.2. *Let T and δ be positive real numbers. Then for $k > C(T,\delta)$ and any pair of complex numbers $(s,w) \in \mathcal{R}_{T,\delta}$, there exists a Hecke eigenform $f \in S_k$ such that $L^*(f,s)L^*(f,w) \neq 0$.*

On the region $\mathcal{R}_{T,\delta}$ in Theorem 3.1, s, w are away from the central lines $\mathrm{Re}(s), \mathrm{Re}(w) = \frac{k}{2}$. The points with $s = \frac{k}{2}$ or $w = \frac{k}{2}$ have to be removed, since the L-values $L^*(f, \frac{k}{2})$ are necessarily 0 for odd $\frac{k}{2}$. However we may try to enlarge the non-vanishing region by adding points with $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ equal to $\frac{k}{2}$. It turns out that this is closely related to a certain property of Riemann zeta function $\zeta(s)$. Since we do not know whether such property holds in general, we add it as an assumption.

Assumption. Fix any $\epsilon > 0$. Let $z_0 \in \mathbb{C}$ with $\mathrm{Re}(z_0) > 0$ and $\mathrm{Im}(z_0) \neq 0$, and $s_0 = it_0$ with $t_0 \neq 0$. Then there exists a neighborhood $W \subset \mathbb{C}^2$ of (s_0, z_0) and a positive integer N such that if $k \geq N$ and $(s,z) \in W$ then

$$\left| \left(\frac{4\pi}{k}\right)^s \frac{\zeta(1+z+s)}{\zeta(1+z-s)} \right| > 1+\epsilon \text{ or } < 1-\epsilon.$$

Note that when W is sufficiently small, the arguments in the zeta values stay in the right half plane $\mathrm{Re}(w) > 1$, which makes the zeta values non-vanishing. The Assumption is simply one that can make the proof of the following theorem to work. Roughly speaking, it says that the magnitude of the Riemann zeta function $|\zeta(s)|$ is not (locally) symmetric about a non-real horizontal line on the right-half plane $\mathrm{Re}(s) > 1$, so in particular it is likely not related to the Riemann Hypothesis.

Theorem 3.3. *For any fixed positive real numbers T, δ, let the region $\widetilde{\mathcal{R}_{T,\delta}}$ of points $(s,w) \in \mathbb{C}^2$ such that*

- $\delta \leq |Im(s)|, |Im(w)| \leq T$,
- $\frac{k-1}{2} < Re(s), Re(w) < \frac{k+1}{2}$,

- $|Re(s) - \frac{k}{2}| + |Re(w) - \frac{k}{2}| \geq \delta$,

Suppose the preceding assumption holds. Then there exists a constant $C(T,\delta) > 0$ depending only on T and δ such that for $k > C(T,\delta)$,

$$\sum_{f\in\mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f,f\rangle}$$

does not vanish at any pair $(s,w) \in \widetilde{\mathcal{R}_{T,\delta}}$.

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4. **Proof**

4.1. **Double Eisenstein Series and its Fourier expansion.** We recall the basics on double Eisenstein series (see [1]), and following the lines in [5], we compute its Fourier expansion.

For $s \in \mathbb{C}$ the double Eisenstein series is defined as

$$E_{s,k-s}(z,w) = \sum_{\gamma,\delta\in\Gamma_\infty\backslash\Gamma, c_{\gamma\delta^{-1}}>0} (c_{\gamma\delta^{-1}})^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}.$$

In [1] it is shown that $E_{s,k-s}(z,w)$ converges absolutely and uniformly on compact sets for $\mathcal{D}$, where

$$\mathcal{D} := \{2 < \mathrm{Re}(s) < k-2, \quad \mathrm{Re}(w) < \min\{\mathrm{Re}(s)-1, k-\mathrm{Re}(s)-1\}\}. \tag{4.1}$$

Define the completed double Eisenstein series

$$E^*_{s,k-s}(z,w) = \frac{e^{\pi is/2}\Gamma(s)\Gamma(k-s)\Gamma(k-w)\zeta(1-w+s)\zeta(1-w+k-s)}{2^{3-w}\pi^{k+1-w}\Gamma(k-1)} E_{s,k-s}(z,w).$$

Then the followings are proved:

Theorem 4.1. [1] *Let $k \geq 6$ be even. The series $E^*_{s,k-s}(z,w)$ has an analytic continuation to all $s,w \in \mathbb{C}$ and as a function of z is always in S_k. For any $f \in \mathcal{B}_k$, we have*

$$< E^*_{s,k-s}(z,w)\,, f >= L^*(f,s)L^*(f,w).$$

The followings are consequences of the above results given in [1]:

- The functional equations of $L^*(f,s)$ induces those of $E^*_{s,k-s}(z,w)$:

$$E^*_{s,k-s}(z,w) = E^*_{w,k-w}(z,s), E^*_{s,k-s}(z,w) = (-1)^{\frac{k}{2}} E^*_{s,k-s}(z,w).$$

-

$$\begin{aligned} E^*_{s,k-s}(z,w) &= \sum_{f\in\mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f,f\rangle} f(z) \\ &= \frac{e^{\pi is/2}\Gamma(s)\Gamma(k-s)\Gamma(k-w)}{2^{3-w}\pi^{k+1-w}\Gamma(k-1)} \sum_{n=1}^{\infty} n^{w-1} \sum_{\gamma\in\mathcal{M}_n} (\gamma z)^{-s} j(\gamma,z)^{-k} \end{aligned}$$

so that the right-hand side has analytic continuation to all of $(s,w)\in\mathbb{C}^2$.

Proposition 4.2. *(***Fourier expansion***)*

(1) $E^*_{s,k-s}(z,w)$ *has the Fourier expansion*

$$E^*_{s,k-s}(z,w) = \frac{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}{2^{2-w}\pi^{k+1-w}\Gamma(k-1)} \sum_{m=1}^{\infty} c_{s,w,k}(m) e^{2\pi imz},$$

where

$$\begin{aligned} &c_{s,w,k}(m) \\ =&\frac{(2\pi)^s}{\Gamma(s)} m^{s-1}\sigma_{w-s}(m)\zeta(k-s-w+1) \\ &+ (-1)^{\frac{k}{2}} \frac{(2\pi)^{k-s}}{\Gamma(k-s)} m^{k-s-1}\sigma_{w+s-k}(m)\zeta(s-w+1) \\ &+ (-1)^{\frac{k}{2}} \frac{(2\pi)^k m^{k-1}}{\Gamma(s)\Gamma(k-s)} \sum_{a,c>0,(a,c)=1} c^{s-k}a^{-s} \sum_{n\geq 1} n^{w-1} \sum_{r|m} r^{w-k} \\ &\times \left(e^{\pi is/2} e^{2\pi i \frac{m}{r}\frac{na'}{c}} {}_1f_1\left(s,k;-\frac{2\pi imn}{rac}\right) + e^{-\pi is/2} e^{-2\pi i \frac{m}{r}\frac{na'}{c}} {}_1f_1\left(s,k;\frac{2\pi imn}{rac}\right)\right). \end{aligned}$$

Here

$${}_1f_1(\alpha,\beta;z) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} {}_1F_1(\alpha,\beta;z),$$

with Kummer's degenerate hypergeometric function ${}_1F_1(\alpha,\beta;z)$ *and* $a' = a^{-1} \pmod{c}$.

(2) The series representation of $c_{s,w,k}(m)$ *is absolutely convergent on* $\mathcal{D}$ *in (4.1).*

Proof of Proposition 4.2: To compute the Fourier expansion of

$$\sum_{n=1}^{\infty} n^{w-1} \sum_{\gamma\in\mathcal{M}_n} (\gamma z)^{-s} j(\gamma, z)^{-k} = \sum_{m=1} c_{s,w,k}(m) e^{2\pi i m z}$$

we split it into four cases:

(**1**) Consider first the elements $\gamma \in \mathcal{M}_n$ with $c_\gamma = 0$. The contribution of such terms to the m-th Fourier coefficient is given by

$$\begin{aligned}
\mathbf{I} :=& \sum_{n\geq 1} n^{w-1} \sum_{ad=n} \sum_{b\in\mathbb{Z}} \int_{iC}^{iC+1} \left(\frac{az+b}{d}\right)^{-s} d^{-k} e^{-2\pi i m z} dz \\
=& 2\sum_{n\geq 1} n^{w-1} \sum_{ad=n, a>0} \sum_{b\in\mathbb{Z}} \int_{iC}^{iC+1} \left(\frac{az+b}{d}\right)^{-s} d^{-k} e^{-2\pi i m z} dz \\
=& 2\sum_{n\geq 1} n^{w-1} \sum_{ad=n, a>0} d^{s-k} \int_{iC}^{iC+1} \sum_{b\in\mathbb{Z}} (az+b)^{-s} e^{-2\pi i m z} dz,
\end{aligned}$$

where C is any fixed positive real number. Note that since we only work on $\mathcal{D}$ in this section, all interchanges of sums and/or integrals are justified by the absolute convergence. Applying Lipschitz's formula

$$\sum_{n\in\mathbb{Z}} (z+n)^{-s} = \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{2\pi i n\tau}, \quad \mathrm{Im}(\tau) > 0, \mathrm{Re}(s) > 1, \tag{4.2}$$

to the sum over b, we have

$$\begin{aligned}
\mathbf{I} &= 2\frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{w-1} \sum_{ad=n, a>0} d^{s-k} \int_{iC}^{iC+1} \sum_{r\geq 1} r^{s-1} e^{2\pi i r a z} e^{-2\pi i m z} dz \\
&= 2\frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{w-1} \sum_{ad=n, a|m} d^{s-k} (m/a)^{s-1}. \\
&= 2\frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{a|m} \sum_{d\geq 1} (ad)^{w-1} d^{s-k} (m/a)^{s-1}, \\
&= 2\frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} m^{s-1} \sigma_{w-s}(m) \zeta(k-s-w+1), \qquad (4.3)
\end{aligned}$$

where $\sigma_s(n) = \sum_{d|n} d^s$ is the divisor function.

(**2**) Similarly, we obtain the contribution **II** of the terms $\gamma \in \mathcal{M}_n$ with $a_\gamma = 0$ in the m-th Fourier coefficient:

$$\mathbf{II} = 2\frac{e^{\pi i(k-s)/2}(2\pi)^{k-s}}{\Gamma(k-s)} m^{k-s-1}\sigma_{w+s-k}(m)\zeta(s-w+1). \tag{4.4}$$

(**3**) Next we consider the contribution **III** of the terms $\gamma \in \mathcal{M}_n$ with $a_\gamma c_\gamma > 0$ in the m-th Fourier coefficient. The set of integral matrices with determinant n can be listed as follows:

$$\left\{ \begin{pmatrix} ar & nb_0/r + (t+r\ell)a \\ cr & nd_0/r + (t+r\ell)c \end{pmatrix} : r \mid n, \gcd(a,c) = 1, t \in \mathbb{Z}/r\mathbb{Z}, \ell \in \mathbb{Z} \right\}$$

where for each pair (a,c), b_0, d_0 are fixed so that $ad_0 - b_0 c = 1$. With this, we have

$$\begin{aligned}
\mathbf{III} &= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|n,t\in\mathbb{Z}/r\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \\
&\quad \times \int_{iC}^{iC+1} \left(\frac{ra(z+\ell) + nb_0/r + ta}{rc(z+\ell) + nd_0/r + tc} \right)^{-s} (rc(z+\ell) + nd_0/r + tc)^{-k} e^{-2\pi i m z} dz \\
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|n,t\in\mathbb{Z}/r\mathbb{Z}} \\
&\quad \times \int_{iC-\infty}^{iC+\infty} \left(\frac{raz + nb_0/r + ta}{rcz + nd_0/r + tc} \right)^{-s} (rcz + nd_0/r + tc)^{-k} e^{-2\pi i m z} dz \\
&\quad \text{(by the change of variable } z \to z - (nd_0 + trc)/(r^2 c)) \\
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|n,t\in\mathbb{Z}/r\mathbb{Z}} e^{2\pi i m \frac{nd_0 + trc}{r^2 c}} \\
&\quad \times \int_{iC-\infty}^{iC+\infty} \left(\frac{a}{c} - \frac{n}{c^2 r^2 z} \right)^{-s} (rcz)^{-k} e^{-2\pi i m z} dz. \\
&\quad \text{(since the summation over } t \text{ vanishes unless } r \mid m \text{)} \\
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|(n,m)} r e^{2\pi i \frac{m}{r}\frac{n}{r}\frac{a'}{c}} \\
&\quad \times \int_{iC-\infty}^{iC+\infty} \left(\frac{a}{c} - \frac{n}{c^2 r^2 z} \right)^{-s} (rcz)^{-k} e^{-2\pi i m z} dz \qquad (4.5) \\
&\quad \text{(with } a' \equiv a^{-1} \pmod{c} \text{)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|(n,m)} r e^{2\pi i \frac{m}{r}\frac{n}{r}\frac{a'}{c}} \\
&\quad \times \int_{iC-\infty}^{iC+\infty} z^s \left(\frac{a}{c}z - \frac{n}{c^2r^2}\right)^{-s} (rcz)^{-k} e^{-2\pi i m z} dz \\
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|(n,m)} r^{1-k} c^{-k} e^{2\pi i \frac{m}{r}\frac{n}{r}\frac{a'}{c}} \\
&\quad \times \int_{iC-\infty}^{iC+\infty} \left(\frac{a}{c}z - \frac{n}{c^2r^2}\right)^{-s} z^{s-k} e^{-2\pi i m z} dz. \\
&\quad \text{(the change of variable } z \mapsto \tfrac{c}{a} iz) \\
&= \sum_{n\geq 1} n^{w-1} \sum_{ac>0,(a,c)=1} \sum_{r|(n,m)} (-1)^{k/2} (c/a)^{s-k+1} r^{1-k} c^{-k} e^{2\pi i \frac{m}{r}\frac{n}{r}\frac{a'}{c}} \\
&\quad \times \frac{1}{i} \int_{C-i\infty}^{C+i\infty} \left(z + \frac{in}{c^2r^2}\right)^{-s} z^{s-k} e^{2\pi m c z/a} dz \\
&\quad \Big(\text{ using the following integral representation} \\
&\quad \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} (z+\alpha)^{-\mu}(z+\beta)^{-\nu} e^{pz} dz = \frac{p^{\mu+\nu-1}e^{-\beta p}}{\Gamma(\mu+\nu)}\, {}_1F_1(\mu, \mu+\nu; (\beta-\alpha)p), \\
&\quad \text{for all } p, \mu, \nu \in \mathbb{C} \text{ with } \mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0 \Big) \\
&= 2(-1)^{\frac{k}{2}} \frac{(2\pi)^k m^{k-1}}{\Gamma(k)} \sum_{a,c>0,(a,c)=1} c^{s-k} a^{-s} \sum_{n\geq 1} n^{w-1} \\
&\quad \times \sum_{r|m} r^{w-k} e^{2\pi i \frac{m}{r}\frac{na'}{c}}\, {}_1F_1\left(s, k; -\frac{2\pi i m n}{rac}\right). \qquad (4.6)
\end{aligned}
$$

(4) Finally the computation on terms with $ac < 0$ can be done similarly. One has to pay attention to the first equality in (4.5) and compute using

$$
\left(\frac{a}{c} - \frac{n}{c^2r^2z}\right)^{-s} = (-1)^{-s}\left(-\frac{a}{c} + \frac{n}{c^2r^2z}\right)^{-s} = e^{-\pi i s} z^{-s}\left(-\frac{a}{c}z + \frac{n}{c^2r^2}\right)^{-s}.
$$

Then by replacing (a, c) with $(-a, c)$, the above computation in the case of $ac > 0$ shows that

$$
\begin{aligned}
\mathbf{IV} =& 2(-1)^{\frac{k}{2}} \frac{(2\pi)^k m^{k-1}}{e^{\pi i s}\Gamma(k)} \sum_{a,c>0,(a,c)=1} c^{s-k} a^{-s} \sum_{n\geq 1} n^{w-1} \\
& \times \sum_{r|m} r^{w-k} e^{-2\pi i \frac{m}{r}\frac{na'}{c}}\, {}_1F_1\left(s, k; \frac{2\pi i m n}{rac}\right). \qquad (4.7)
\end{aligned}
$$

Combining the formulas (4.3), (4.4), (4.6) and (4.7) we conclude the result. □

Corollary 4.3. *The formula*

$$\begin{aligned}&\frac{2^{2-w}\pi^{k+1-w}\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\sum_{f\in\mathcal{H}_k}\frac{L^*(f,s)L^*(f,w)}{\langle f,f\rangle}\\ =&\frac{(2\pi)^s}{\Gamma(s)}\zeta(k-s-w+1)+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\zeta(s-w+1)\\ &+(-1)^{\frac{k}{2}}\frac{(2\pi)^k}{\Gamma(s)\Gamma(k-s)}\sum_{a,c>0,(a,c)=1}c^{s-k}a^{-s}\sum_{n\geq 1}n^{w-1}\\ &\times\left(e^{\pi is/2}e^{2\pi ina'/c}{}_1f_1\left(s,k;-\frac{2\pi in}{ac}\right)+e^{-\pi is/2}e^{-2\pi ina'/c}{}_1f_1\left(s,k;\frac{2\pi in}{ac}\right)\right)\end{aligned}$$

holds on $\mathcal{D}$ *and the double sum in the last term of the right-hand side is absolutely convergent on* $\mathcal{D}$*.*

Proof of Corollary 4.3: It follows easily from Proposition 4.2 and the formula (4.1). □

Remark 4.4. By (5-2) of [1], one may apply Kohnen's Fourier expansion for Cohen's kernel function ([5], Lemma 2) and obtain the same formula as in Corollary 4.3. However, the absolute convergence on $\mathcal{D}$ of the double sum in Corollary 4.3 is no longer clear. In our approach, the absolute convergence of $E^*_{s,k-s}(z,w)$ implies that of the double sum together with the integral for the hypergeometric function ${}_1F_1$, hence is the stronger than that of the double sum in Corollary 4.3.

4.2. **Analytic continuation.** The left-hand side of the identity in Corollary 4.3 is meromorphic on $\mathbb{C}^2$, while the right-hand side is only valid on $\mathcal{D}$. For later purpose, we shall analytically continue the right-hand side, and to this end, the following domains will be involved:

$$\mathcal{D}_1 := \{(s,w)\in\mathbb{C}^2 :\quad 2<\mathrm{Re}(s)<k-2,\quad \mathrm{Re}(w)<0\},$$
$$\mathcal{F} := \{(s,w)\in\mathbb{C}^2 :\quad 3/2<\mathrm{Re}(s),\mathrm{Re}(w)<k-2\}.$$

Proposition 4.5. *We have the following identity on* $\mathcal{F}$

$$\frac{2^{2-w}\pi^{k+1-w}\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\sum_{f\in\mathcal{H}_k}\frac{L^*(f,s)L^*(f,w)}{\langle f,f\rangle}$$

$$
\begin{aligned}
=&\frac{(2\pi)^s}{\Gamma(s)}\zeta(k-s-w+1)+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\zeta(s-w+1)\\
&+2(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w)\Gamma(s-w)}{\Gamma(s)\Gamma(k-w)}\cos(\pi(s-w)/2)\zeta(s-w)\\
&+2(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w)\Gamma(k-s-w)}{\Gamma(k-s)\Gamma(k-w)}\cos(\pi(s+w)/2)\zeta(k-s-w)\\
&+R(s,w),
\end{aligned}
$$

where $R(s,w)$ is holomorphic on $\mathcal{F}$ and bounded by (4.10).

Proof of Proposition 4.5: We only have to deal with the last term of the right-hand side of Corollary 4.3, namely

$$
\begin{aligned}
A(s,w):=(-1)^{\frac{k}{2}}&\frac{(2\pi)^k}{\Gamma(s)\Gamma(k-s)}\sum_{a,c>0,(a,c)=1}c^{s-k}a^{-s}\sum_{n\geq 1}n^{w-1}\\
&\times\left(e^{\pi is/2}e^{2\pi ina'/c}\,{}_1f_1\left(s,k;-\frac{2\pi in}{ac}\right)+e^{-\pi is/2}e^{-2\pi ina'/c}\,{}_1f_1\left(s,k;\frac{2\pi in}{ac}\right)\right).
\end{aligned}
$$

Step I Let $G_{a,c}(s,w):=G'_{a,c}(s,w)+G_{a,c}"(s,w)$, where

$$
G'_{a,c}(s,w):=\sum_{n\geq 1}n^{w-1}e^{\pi is/2}e^{2\pi ina'/c}\,{}_1f_1\left(s,k;-\frac{2\pi in}{ac}\right),
$$

$$
G''_{a,c}(s,w):=\sum_{n\geq 1}n^{w-1}e^{-\pi is/2}e^{-2\pi ina'/c}\,{}_1f_1\left(s,k;\frac{2\pi in}{ac}\right)
$$

As a subseries of an absolutely convergent series, the function $G_{a,c}(s,w)$ is holomorphic on $\mathcal{D}$, so in particular $G_{a,c}(s,w)$ is holomorphic on the smaller region $\mathcal{D}_1$. For $\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0$, it is known that ([5])

$$
{}_1f_1\left(\alpha,\beta;z\right)=\int_0^1 e^{zu}u^{\alpha-1}(1-u)^{\beta-\alpha-1}du.
$$

On $\mathcal{D}_1$, we have

$$
\begin{aligned}
G'_{a,c}(s,w)=&\sum_{n=1}^{\infty}n^{w-1}e^{2\pi ina'/c}e^{\pi is/2}\,{}_1f_1\left(s,k;\frac{-2\pi in}{ac}\right)\\
=&\sum_{n=1}^{\infty}n^{w-1}e^{2\pi ina'/c}e^{\pi is/2}\int_0^1 e^{\frac{-2\pi inu}{ac}}u^{s-1}(1-u)^{k-s-1}du
\end{aligned}
$$

$$=e^{\pi is/2}\int_0^1 u^{s-1}(1-u)^{k-s-1}\sum_{n=1}^{\infty} n^{w-1}e^{2\pi ina'/c}e^{\frac{-2\pi inu}{ac}}\,du,$$

where the interchange of summation and integration is justified because of absolute convergence on $\mathcal{D}_1$. We warn here that the above series with ${}_1f_1$ expanded is not necessarily absolutely convergence on $\mathcal{D}$, which is why we need the smaller $\mathcal{D}_1$. Now for $s \in \mathbb{C}$, $a > 0$, define

$$F(s,a) = \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s}, \quad \zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Therefore, originally on $\mathcal{D}_1$,

$$G'_{a,c}(s,w) = e^{\pi is/2}\int_0^1 u^{s-1}(1-u)^{k-s-1}F\left(1-w, \frac{a'}{c}-\frac{u}{ac}\right)du.$$

Note that $0 < \frac{a'}{c} - \frac{u}{ac} < 1$, so by the identity between $F(s,a)$ and $\zeta(s,a)$ (see Formula 25.13.2 of [7]), we have

$$\begin{aligned}G'_{a,c}(s,w) =&(2\pi)^{-w}\Gamma(w)\int_0^1 u^{s-1}(1-u)^{k-s-1}\\ &\times\left(e^{\pi i(s+w)/2}\zeta\left(w, \frac{a'}{c}-\frac{u}{ac}\right)+e^{\pi i(s-w)/2}\zeta\left(w, 1-\frac{a'}{c}+\frac{u}{ac}\right)\right)du.\end{aligned}$$

Similarly,

$$G''_{a,c}(s,w) = e^{-\pi is/2}\int_0^1 u^{s-1}(1-u)^{k-s-1}F\left(1-w, -\frac{a'}{c}+\frac{u}{ac}\right)du$$

and

$$\begin{aligned}G''_{a,c}(s,w) =&(2\pi)^{-w}\Gamma(w)\int_0^1 u^{s-1}(1-u)^{k-s-1}\\ &\times\left(e^{-\pi i(s+w)/2}\zeta\left(w, \frac{a'}{c}-\frac{u}{ac}\right)+e^{\pi i(w-s)/2}\zeta\left(w, 1-\frac{a'}{c}+\frac{u}{ac}\right)\right)du.\end{aligned}$$

So, we have , on $\mathcal{D}_1$

$$\begin{aligned}(4.8)\quad &G_{a,c}(s,w)\\ &=2(2\pi)^{-w}\Gamma(w)\int_0^1 u^{s-1}(1-u)^{k-s-1}\\ &\times\left(\cos(\pi(s+w)/2)\zeta\left(w, \frac{a'}{c}-\frac{u}{ac}\right)+\cos(\pi(s-w)/2)\zeta\left(w, 1-\frac{a'}{c}+\frac{u}{ac}\right)\right)du.\end{aligned}$$

Note that the right-hand side of (4.8) is meromorphic on $\mathcal{D}$, forcing that (4.8) holds on $\mathcal{D}$ as well since $G_{a,c}(s,w)$ is holomorphic on $\mathcal{D}$.

Replacing the expression $G_{a,c}(s,w)$ in $A(s,w)$ with the right-hand side of (4.8), the following equality

(4.9)
$$A(s,w) = 2(-1)^{\frac{k}{2}} \frac{(2\pi)^{k-w}\Gamma(w)}{\Gamma(s)\Gamma(k-s)} \sum_{(a,c)=1,a,c>0} c^{-k+s}a^{-s} \int_0^1 u^{s-1}(1-u)^{k-s-1}$$
$$\times \left(\cos(\pi(s+w)/2)\zeta\left(w, \frac{a'}{c} - \frac{u}{ac}\right) + \cos(\pi(s-w)/2)\zeta\left(w, 1 - \frac{a'}{c} + \frac{u}{ac}\right)\right) du$$

holds on $\mathcal{D}$.

Step II Next, we prove that the series on the right-hand side of (4.9) is absolutely convergent on $\mathcal{F}$ after removing the desired main terms. Since $\mathcal{D}\cap\mathcal{F} \neq \emptyset$, we then see that the equality (4.9) holds on $\mathcal{F}$.

(**1**) First consider the terms with $c > 1$. Note that for $\mathrm{Re}(w) > \frac{3}{2}$ and each pair a, c, we always have $1 - \frac{a'}{c} \geq \frac{1}{c}$, so

$$\left|\zeta\left(w, 1 - \frac{a'}{c} + \frac{u}{ac}\right)\right| \leq \left|\left(1 - \frac{a'}{c} + \frac{u}{ac}\right)^{-w}\right| + \zeta(\mathrm{Re}(w))$$
$$\leq \left(\frac{1}{c} + \frac{u}{ac}\right)^{-\mathrm{Re}(w)} + \zeta(3/2) \leq c^{\mathrm{Re}(w)} + \zeta(3/2) \ll c^{\mathrm{Re}(w)}.$$

Then the sum over such pairs (a,c) on the second ζ-term is bounded absolutely by

$$e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)} \sum_{c=2}^{\infty}\sum_{m=1}^{\infty} \sum_{a \mod c}{}' c^{-k+\mathrm{Re}(s)+\mathrm{Re}(w)}(a+cm)^{-\mathrm{Re}(s)}$$
$$\leq e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)} \sum_{c=1}^{\infty}\sum_{m=1}^{\infty} c\cdot c^{-k+\mathrm{Re}(s)+\mathrm{Re}(w)}(cm)^{-\mathrm{Re}(s)}$$
$$= e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)}\zeta(k-1-\mathrm{Re}(w))\zeta(\mathrm{Re}(s)).$$

The same bound can be obtained for $a > 1$ in the first ζ-term , while the $a = 1$ part contributes

$$\frac{\Gamma(s)\Gamma(k-s-w)}{\Gamma(k-w)} \cos(\pi(s+w)/2)\left(\zeta(k-s-w) - 1\right).$$

(**2**) Next consider the terms with $c = 1$. Separating the first term in the Hurwitz zeta functions, we have the following four terms

$$\begin{aligned}&\sum_{a=1}^{\infty} a^{-s} \int_0^1 u^{s-1}(1-u)^{k-s-1}\cos(\pi(s+w)/2)(1-u/a)^{-w}du\\+&\sum_{a=1}^{\infty} a^{-s} \int_0^1 u^{s-1}(1-u)^{k-s-1}\cos(\pi(s-w)/2)(u/a)^{-w}du\\+&\sum_{a=1}^{\infty} a^{-s} \int_0^1 u^{s-1}(1-u)^{k-s-1}\cos(\pi(s+w)/2)\zeta\left(w,2-\frac{u}{a}\right)du\\+&\sum_{a=1}^{\infty} a^{-s} \int_0^1 u^{s-1}(1-u)^{k-s-1}\cos(\pi(s-w)/2)\zeta\left(w,1+\frac{u}{a}\right)du.\end{aligned}$$

The third and the fourth term are absolutely bounded by

$$e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)}\zeta(\mathrm{Re}(s))\zeta(\mathrm{Re}(w)),$$

hence giving holomorphic functions on $\mathcal{F}$. The second term gives

$$\frac{\Gamma(s-w)\Gamma(k-s)}{\Gamma(k-w)}\cos(\pi(s-w)/2)\zeta(s-w),$$

which is meromorphic everywhere. Finally, we employ the elementary inequality for the first term

$$|(1-u/a)^{-w}| \le (1-u)^{-\mathrm{Re}(w)/a}, \quad a \in \mathbb{Z}_{>0}, \mathrm{Re}(w) > 0, u \in (0,1).$$

The $a = 1$ term gives

$$\frac{\Gamma(s)\Gamma(k-s-w)}{\Gamma(k-w)}\cos(\pi(s+w)/2),$$

where the rest in the first term gives a series absolutely convergent on $\mathcal{F}$, where it is bounded by

$$e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)}\zeta(\mathrm{Re}(s)).$$

(**3**) Putting everything together, we have on $\mathcal{F}$,

$$\begin{aligned}&A(s,w)\\=&2(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-w}\Gamma(w)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\&\times\left(\Gamma(s-w)\Gamma(k-s)\cos(\pi(s-w)/2)\zeta(s-w)+\Gamma(s)\Gamma(k-s-w)\cos(\pi(s+w)/2)\right)\end{aligned}$$

$$+ R(s,w),$$

where $R(s,w)$ is holomorphic on $\mathcal{F}$ and is bounded by

$$2e^{\pi(|\mathrm{Im}(s)|+|\mathrm{Im}(w)|)}\frac{(2\pi)^{k-\mathrm{Re}(w)}|\Gamma(w)|\zeta(\mathrm{Re}(s))}{|\Gamma(s)\Gamma(k-s)|}\left(\zeta(k-1-\mathrm{Re}(w))+\zeta(\mathrm{Re}(w))+1\right). \tag{4.10}$$

This concludes the result. □

For fixed positive real numbers δ, T, we consider the following smaller region

$$\mathcal{F}_1 := \left\{(s,w)\in\mathbb{C}^2 : \frac{k-1}{2} < \mathrm{Re}(s),\mathrm{Re}(w) < \frac{k+1}{2}, \mathrm{Re}(s)+\mathrm{Re}(w) < k-\delta, \mathrm{Im}(s),\mathrm{Im}(w)\in[-T,T]\right\}.$$

Lemma 4.6. *We have the following identity on* $\mathcal{F}_1$

$$\begin{aligned}
&\frac{2^2\pi^{k+1}\Gamma(k-1)}{\Gamma(s)\Gamma(w)\Gamma(k-s)\Gamma(k-w)}\sum_{f\in\mathcal{H}_k}\frac{L^*(f,s)L^*(f,w)}{\langle f,f\rangle}\\
=&\frac{(2\pi)^{s+w}}{\Gamma(s)\Gamma(w)}\zeta(k-s-w+1)+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k-s+w}}{\Gamma(w)\Gamma(k-s)}\zeta(s-w+1)\\
&+(-1)^{\frac{k}{2}}\frac{(2\pi)^{k+s-w}}{\Gamma(s)\Gamma(k-w)}\zeta(w-s+1)+\frac{(2\pi)^{2k-s-w}}{\Gamma(k-s)\Gamma(k-w)}\zeta(s+w-k+1)\\
&+R(s,w),
\end{aligned}$$

where $R(s,w)$ (different from that in Proposition 4.5) is holomorphic on $\mathcal{F}_1$ and bounded by

$$\frac{(2\pi)^k}{|\Gamma(s)\Gamma(k-s)|}$$

up to a constant depending only on T and δ.

Proof of Lemma 4.6: It follows from Proposition 4.5 by simplifying the corresponding terms of the right-hand side therein. Explicitly, multiply both sides by $(2\pi)^w/\Gamma(w)$ and apply the functional equation of Riemann zeta function

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$$

and then we have the desired terms. The bound for $R(s,w)$ follows directly from (4.10). □

4.3. **Proof of Theorem 3.1.** Now we can prove the main theorem. Fix positive real numbers T and δ and consider

$$\mathcal{R}'_{T,\delta} = \{(s,w) \in \mathcal{F}_1 \colon \mathrm{Re}(s), \mathrm{Re}(w) \leq k/2 - \delta\}.$$

We prove instead that on $\mathcal{R}'_{T,\delta}$ the right-hand side of Lemma 4.6 is non-vanishing. To this end, we show that the first term is the main term, that is, we prove that when multiplied by $\Gamma(s)\Gamma(w)(2\pi)^{-s-w}$ the sum of the rest terms has limit 0.

As indicated in the proof of Lemma 4.6, the fourth term therein can be put into the remaining term $R(s,w)$, while

$$\left|\Gamma(s)\Gamma(w)(2\pi)^{-s-w} R(s,w)\right| \ll_{T,\delta} \left|\frac{(2\pi)^{k-s-w}\Gamma(w)}{\Gamma(k-s)}\right|.$$

By (**??**),

$$\frac{(2\pi)^{k-s-w}\Gamma(w)}{\Gamma(k-s)} \sim (4\pi/k)^{k-s-w}, \quad k \to \infty.$$

Since $\mathrm{Re}(s) + \mathrm{Re}(w) < k - \delta$, it follows that the fourth term and the remaining term give limit 0 as expected.

So we only have to show that

$$\frac{(2\pi)^{k-2s}\Gamma(s)}{\Gamma(k-s)}\zeta(s-w+1) + \frac{(2\pi)^{k-2w}\Gamma(w)}{\Gamma(k-w)}\zeta(w-s+1)$$

approaches 0 as $k \to \infty$. Since the sum is holomorphic, the singularities of the two terms on $s = w$ cancel. Rewriting it as

$$\begin{aligned}(4.11)\quad & \frac{(2\pi)^{k-2s}\Gamma(s)}{\Gamma(k-s)}\left(\zeta(s-w+1) - \frac{1}{s-w}\right) + \frac{(2\pi)^{k-2w}\Gamma(w)}{\Gamma(k-w)}\left(\zeta(w-s+1) - \frac{1}{w-s}\right) \\ & + \frac{(2\pi)^{k-2s}\Gamma(s)}{\Gamma(k-s)}\frac{1}{s-w} + \frac{(2\pi)^{k-2w}\Gamma(w)}{\Gamma(k-w)}\frac{1}{w-s}.\end{aligned}$$

The first two terms of (4.11) approach 0 by (**??**) and the same argument as above, so it suffices to show

$$\frac{(2\pi)^{k-2s}\Gamma(s)}{\Gamma(k-s)}\frac{1}{s-w} + \frac{(2\pi)^{k-2w}\Gamma(w)}{\Gamma(k-w)}\frac{1}{w-s} \to 0, \quad k \to \infty.$$

It is clear that such a function is absolutely bounded by $\sup_z |g'(z)|$ where

$$g(z) = \frac{(2\pi)^{k-2z}\Gamma(z)}{\Gamma(k-z)}$$

and z ranges over $\{z\colon -1/2 \le \mathrm{Re}(z) - k/2 \le -\delta, |\mathrm{Im}(z)| \le T\}$. It is easy to obtain its derivative

$$g'(z) = \frac{(2\pi)^{k-2z}\Gamma(z)}{\Gamma(k-z)} \left(-2\log(2\pi) + \psi(z) - \psi(k-z)\right),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Since $\psi(z) \sim \log z$ (see (5.11.2) of [7]), we have $\sup_z |g'(z)| \to 0$ as desired.

The above argument shows that on $\mathcal{R}'_{T,\delta}$, if k is large enough,

$$\sum_{f \in \mathcal{H}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f, f\rangle} \neq 0.$$

Finally, by applying the functional equations of $E^*_{s,k-s}(z,w)$ we obtain the theorem. □

4.4. **Proof of Theorem 3.3.** By Theorem 3.1 and the symmetries, we only have to show that for each $(s_0, w_0) = (1/2 + it_0, w_0)$ with $t_0 \neq 0, \mathrm{Re}(w_0) \le 1/2 - \delta$ and $\mathrm{Im}(w_0) \neq 0$, there exists a neighborhood on which the right-hand side of the equation of Lemma 4.6 (after shifting by $\frac{k-1}{2}$) is non-vanishing when k is large. Following the proof of Theorem 3.1, we see that the remainder on the right-hand side of the equation of Lemma 4.6 is dominated by the first two terms as $k \to \infty$. Therefore, we need to prove that on some neighborhood of (s_0, w_0), the quantity

$$1 + (-1)^{\frac{k}{2}}(2\pi)^{1-2s}\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(\frac{k+1}{2}-s)}\frac{\zeta(1+s-w)}{\zeta(2-s-w)} + (-1)^{\frac{k}{2}}(2\pi)^{1-2w}\frac{\Gamma(w+\frac{k-1}{2})}{\Gamma(\frac{k+1}{2}-w)}\frac{\zeta(1+w-w)}{\zeta(2-s-w)}$$
$$+ (2\pi)^{2-2s-2w}\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(\frac{k+1}{2}-s)}\frac{\Gamma(w+\frac{k-1}{2})}{\Gamma(\frac{k+1}{2}-w)}\frac{\zeta(s+w)}{\zeta(2-s-w)}$$

is non-vanishing when k is large.

The third and the fourth term approach 0 when k approaches ∞, as we have seen in the proof of Theorem 3.1. Now

$$\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(\frac{k+1}{2}-s)} = \left(\frac{k}{2}\right)^{1-2s}\left(1 + O(k^{-1})\right), k \to \infty,$$

so together with the Assumption, it implies the existence of a desired neighborhood, on which the sum of first two terms stays away from 0. □

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Department of Mathematics, Pohang University of Science and Technology, Pohang, 790–784, Korea

Email address: yjc@postech.ac.kr

Department of Mathematics, University of Heidelberg, Germany

Email address: winfried@mathi.uni-heidelberg.de

Department of Mathematics and Institute of Advanced Studies of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R.China

Email address: yichao.zhang@hit.edu.cn