GRADIENT HÖLDER REGULARITY FOR PARABOLIC NORMALIZED $p(x,t)$-LAPLACE EQUATION

YUZHOU FANG AND CHAO ZHANG*

Abstract. We consider the interior Hölder regularity of spatial gradient of viscosity solution to the parabolic normalized $p(x,t)$-Laplace equation

$$u_t = \left(\delta_{ij} + (p(x,t)-2)\frac{u_i u_j}{|Du|^2}\right) u_{ij}$$

with some suitable assumptions on $p(x,t)$, which arises naturally from a two-player zero-sum stochastic differential game with probabilities depending on space and time.

1. Introduction

Let $p(x,t) \in C^1_{\rm loc}(\mathbb{R}^{n+1})$ and $1 < p_- := \inf p(x,t) \le \sup p(x,t) =: p_+ < \infty$. In this work, we investigate the higher regularity properties of viscosity solutions to the following parabolic normalized $p(x,t)$-Laplacian

$$u_t(x,t) = \Delta^N_{p(x,t)} u(x,t), \tag{1.1}$$

where $\Delta^N_{p(x,t)}$ is the normalized $p(x,t)$-Laplace operator defined as

$$\Delta^N_{p(x,t)} u := \Delta u + (p(x,t)-2)\left\langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\right\rangle = \left(\delta_{ij} + (p(x,t)-2)\frac{u_i u_j}{|Du|^2}\right) u_{ij}.$$

Here the summation convention is utilized and the vector Du is the gradient with respect to the spatial variable x. In the rest of this paper, $D_{x,t}u = (\partial_t u, \partial_{x_1} u, \cdots, \partial_{x_n} u)^T$.

Over the last decade, equation (1.1) and related normalized equations in non-divergence form have received considerable attention, partly due to the stochastic zero-sum tug-of-war games defined by Peres-Schramm-Sheffield-Wilson [31], Peres-Sheffield [32] and Manfredi-Parviainen-Rossi [28]. When $p(x)$ is constant, Luiro-Parviainen-Saksman [27] proved the Harnack's inequality for the homogeneous normalized p-Laplace equation $-\Delta^N_p u = 0$. Ruosteenoja [35] studied the local Lipschitz continuity and Harnack's inequality for the inhomogeneous version $-\Delta^N_p u = f$. The first contribution on the $C^{1,\alpha}$ estimates for such equations is due to Jin-Silvestre [20], in which they established the local Hölder gradient estimates for the parabolic normalized p-Laplacian

$$\partial_t u = \Delta^N_p u. \tag{1.2}$$

This result was generalized to the inhomogeneous case by Attouchi-Parviainen in [2]. Additionally for the inhomogeneous elliptic analogue, Attouchi-Parviainen-Ruosteenoja in [3] verified that the solutions are locally of class $C^{1,\alpha}$, see also [9]. Later, Imbert-Jin-Silvestre [18]

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* Corresponding author.

proved the interior $C^{1,\alpha}$ regularity for a more general equation

$$\partial_t u = |Du|^\gamma \left(\delta_{ij} + (p-2)\frac{u_i u_j}{|Du|^2}\right) u_{ij}, \tag{1.3}$$

where $p \in (1, +\infty)$ and $\gamma > -1$. When $\gamma = 0$, it is nothing but (1.2); when $\gamma = p-2$, it is the usual parabolic p-Laplace equation

$$u_t = \operatorname{div}(|Du|^{p-2} Du). \tag{1.4}$$

It was well-known that viscosity solutions and weak solutions to (1.4) coincide (see [22]). Based on this equivalence and the $C^{1,\alpha}$ regularity of weak solutions to (1.4) in [13, 39], we know that the viscosity solutions are of class $C^{1,\alpha}$. For the inhomogeneous counterpart of (1.3)

$$\partial_t u - |Du|^\gamma \left(\delta_{ij} + (p-2)\frac{u_i u_j}{|Du|^2}\right) u_{ij} = f \tag{1.5}$$

with $-1 < \gamma < \infty$ and $1 < p < \infty$, the local higher regularity properties of solutions to (1.5) have been investigated in [1, 5], provided that f is bounded and continuous. One can find more related results in [6–8, 14, 16, 21, 24, 26, 29, 33, 34].

On the other hand, when it comes to the nonstandard growth case, Siltakoski [36] considered the normalized $p(x)$-Laplacian

$$\Delta^N_{p(x)} u := \left(\delta_{ij} + (p(x)-2)\frac{u_i u_j}{|Du|^2}\right) u_{ij} = 0 \tag{1.6}$$

and showed that the viscosity solution is locally $C^{1,\alpha}$ regular by means of the equivalence between viscosity solutions to (1.6) and weak solutions to strong $p(x)$-Laplace equation

$$\Delta^S_{p(x)} u = |Du|^{p(x)-2} \Delta^N_{p(x)} u.$$

And the local $C^{1,\alpha}$ regularity of weak solutions of strong $p(x)$-Laplace equation has been obtained by Zhang-Zhou [40]. For more results in the elliptic situation, see for instance [4, 10, 11, 19, 37] and references therein.

As interpreted in [15,30], parabolic equations of the type considered in (1.1) arise naturally from a two-player zero-sum stochastic differential game (SDG) with probabilities depending on space and time. It is defined in terms of an n-dimensional state process, and is driven by a $2n$-dimensional Brownian motion for $n \geq 2$. It is worth remarking that around "$2n$-dimensional Brownian motion" that the simplest versions use $(n+1)$-dimensional Brownian motion or even random walk. As far as we know, the present setting is less studied and it exhibits interesting features both from the tug-of-war games and from the mathematical viewpoint. In particular, Parviainen-Ruosteenoja [30] proved the Hölder and Harnack estimates for a more general game that was called $p(x,t)$-game without using the PDE techniques and showed that the value functions of the game converge to the unique viscosity solution of the Dirichlet problem to the normalized $p(x,t)$-parabolic equation

$$(n + p(x,t)) u_t(x,t) = \Delta^N_{p(x,t)} u(x,t).$$

In addition, Heino [15] formulated a stochastic differential game in continuous time and obtained that the viscosity solution to a terminal value problem involving the parabolic normalized $p(x,t)$-Laplace operator is unique under suitable assumptions. However, whether or not the spatial gradient Du of (1.1) is Hölder continuous was still unknown. In this paper we answer this question and prove the interior Hölder continuity for the spatial gradient of viscosity solutions to (1.1).

Let $Q_r := B_r \times (-r^2, 0] \subset \mathbb{R}^{n+1}$ be a parabolic cylinder, where B_r is a ball in $\mathbb{R}^n$ centered at the origin with the radius $r > 0$. Our main result is stated as follows.

Theorem 1.1. *Assume that u is a viscosity solution to (1.1) in Q_1. If $1 < p_- \le p_+ < \infty$ and $p(x,t) \in C^1(\overline{Q_1})$, then there exist two constants $\alpha \in (0,1)$ and C, both depending on n, p_-, p_+ and $\|D_{x,t}p\|_{L^\infty(Q_1)}$, such that*

$$\|Du\|_{C^\alpha(Q_{1/2})} \le C\|u\|_{L^\infty(Q_1)}$$

and

$$\sup_{Q_{1/2}} \frac{|u(x,t) - u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}} \le C\|u\|_{L^\infty(Q_1)}.$$

We would like to mention that our proof is much influenced by the ideas developed in [20]. To avoid the problem of vanishing gradient, we first approximate (1.1) with a regularized problem (3.1) below. Then we try to derive uniform *a priori* estimates regarding (3.1), so that we could pass to the limit through compactness argument eventually. Specifically, we verify that the oscillation of the spatial gradient decreases in a sequence of the shrinking parabolic cylinders. The iteration process is divided into two scenarios: either the gradient Du is close to a fixed vector e in a large portion of Q_{τ^k}, or it does not. We then have to combine these two alternatives to get the final result. In fact, by virtue of the similar structure between (1.1) and (1.2), we focus mainly on showing the improvement of oscillation for $|Du|$ (Lemmas 3.1 and 3.3 below) and demonstrate the higher Hölder regularity of solutions to the original equation (1.1) via approximation. It is worth pointing out that the comparison principle and stability of viscosity solutions play an important role in the proof of Theorem 3.12 (the approximation step). Although we basically follow the ideas in [20], there exist several noticeable differences and difficulties. First, to derive the improvement of oscillation of $|Du|$, there will be an additional term involving $Dp(x,t)$ when we differentiate the regularized equation (3.1) below with respect to the spatial variables x. Consequently, in comparison to the proof of [20, Lemma 4.1], we require more elaborate analysis and construct different auxiliary functions. Second, the comparison principle (Theorem 4.1) cannot directly follow from the well-known results owing to the variable coefficient $p(x,t)$. We have to make use of the information from the maximum principle of semicontinuous functions carefully, as well as the features such as the local Lipschitz continuity regarding the matrix square root. To the best of our knowledge, the comparison principle of (1.1) is new, which is also of independent interest.

This paper is organized as follows. In Section 2, we give the definition of viscosity solutions to (1.1) and state some known results that will be used later. Section 3 is devoted to showing firstly the Hölder gradient regularity of (1.1) under the assumption that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is small, then consummating the conclusion for all $p(x,t) \in C^1(\overline{Q_1})$. In Section 4, we prove the comparison principle and stability of viscosity solutions to (1.1), which are the indispensable ingredients for the proof of Theorem 3.12.

2. Preliminaries

Because equation (1.1) is not in divergence form, the concept of weak solutions with test functions under the integral sign is problematic. Thus, in this section we first recall the definition of viscosity solution to (1.1).

Definition 2.1 (viscosity solution)**.** *A lower (upper, resp.) semicontinuous function u in Q_1 is a viscosity supersolution (subsolution, resp.) to (1.1), if for any $\varphi \in C^2(Q_1)$, $u - \varphi$*

reaches a local minimum (maximum, resp.) at $(x_0,t_0)\in Q_1$*, then when* $D\varphi(x_0,t_0)\neq 0$*, it holds that*

$$\varphi_t \geq (\leq, resp.)\Delta\varphi + (p(x,t)-2)\left\langle D^2\varphi\frac{D\varphi}{|D\varphi|}, \frac{D\varphi}{|D\varphi|}\right\rangle$$

at (x_0,t_0)*; when* $D\varphi(x_0,t_0)=0$*, it holds that*

$$\varphi_t \geq (\leq, resp.)\Delta\varphi + (p(x,t)-2)\langle D^2\varphi q, q\rangle$$

at (x_0,t_0) *for some* $q\in\overline{B_1}(0)\subset\mathbb{R}^n$*. A function* u *is a viscosity solution to* (1.1) *if and only if it is both viscosity super- and subsolution.*

Next, we state some known results about solutions of linear uniformly parabolic equations, which will be used later. Consider the equation

$$u_t - a_{ij}(x,t)u_{ij} = 0 \quad \text{in } Q_1, \tag{2.1}$$

where the coefficient a_{ij} is uniformly parabolic, i.e., there exist two constants $0<\lambda\leq\Lambda<\infty$ such that

$$\lambda I \leq a_{ij}(x,t) \leq \Lambda I \quad \text{for all } (x,t)\in Q_1. \tag{2.2}$$

We begin with the following two lemmas (see [20]).

Lemma 2.2. *Let* $u\in C(\overline{Q_1})$ *be a solution to* (2.1) *satisfying* (2.2) *and* A *be a positive constant. If*

$$\mathrm{osc}_{B_1} u(\cdot,t) \leq A$$

for any $t\in[-1,0]$*, then we have*

$$\mathrm{osc}_{Q_1} u(x,t) \leq CA,$$

where $C>0$ *depends only on* n,Λ*.*

Lemma 2.3. *Let* η, u *be a positive constant and a smooth solution to* (2.1) *satisfying* (2.2) *respectively. Suppose* $|Du|\leq 1$ *in* Q_1 *and*

$$|\{(x,t)\in Q_1 : |Du-e|>\varepsilon_0\}| \leq \varepsilon_1$$

for some $e\in\mathbb{S}^{n-1}$ *(i.e.,* $|e|=1$*) and two positive constants* $\varepsilon_0,\varepsilon_1$*. Then there is a constant* $a\in\mathbb{R}$ *such that*

$$|u(x,t)-a-e\cdot x| \leq \eta$$

for any $(x,t)\in Q_{1/2}$*, provided that* $\varepsilon_0,\varepsilon_1$ *are small enough. Here* $\varepsilon_0,\varepsilon_1$ *depend on* n,λ,Λ *and* η*.*

Subsequently, we present an important conclusion about improvement of oscillation for solution to (2.1).

Lemma 2.4 ([20])**.** *Assume* $u\in C(Q_1)$ *is a nonnegative supersolution to* (2.1) *satisfying* (2.2)*. For any* $0<\mu<1$*, there is* $\tau\in(0,1)$ *depending only on* n,μ *and* $\gamma>0$ *depending on* n,μ,λ,Λ *such that if*

$$|\{(x,t)\in Q_1 : u\geq 1\}| > \mu|Q_1|,$$

then it holds that

$$u\geq\gamma \quad \text{in } Q_\tau.$$

We end this section by the following boundary estimates of solutions to (2.1) utilized in the proof of Theorem 3.12.

Lemma 2.5 ([20]). *Suppose that $u \in C(\overline{Q_1})$ is a solution to (2.1) satisfying (2.2) and that ρ is a modulus of continuity of boundary value $\varphi := u\,|_{\partial_p Q_1}$. Then there is another modulus of continuity ρ^* that depends on $n, \lambda, \Lambda, \rho, \|\varphi\|_{L^\infty(\partial_p Q_1)}$ such that*

$$|u(x,t) - u(y,s)| \le \rho^*(|x-y| \vee \sqrt{|t-s|})$$

for any $(x,t),(y,s) \in \overline{Q_1}$. Here $a \vee b$ denotes $\max\{a,b\}$.

3. Hölder regularity of spatial gradients

To avoid the lack of smoothness of viscosity solutions to (1.1), we first consider the following regularized equation

$$u_t = \left(\delta_{ij} + (p(x,t)-2)\frac{u_i u_j}{|Du|^2+\varepsilon^2}\right)u_{ij} \tag{3.1}$$

with $\varepsilon > 0$ in Q_1. We will focus on deriving the uniform estimates with respect to ε so that we can pass to the limit by letting $\varepsilon \to 0$ in the end. For later convenience, we denote

$$a_{ij}^\varepsilon := a_{ij}^\varepsilon(x,t,Du) = \delta_{ij} + (p(x,t)-2)\frac{u_i u_j}{|Du|^2+\varepsilon^2}$$

with u_i being the i-th component of Du.

Now we present the interior Lipschitz estimates independent of ε on the viscosity solutions of (3.1). This result is stated as follows.

Lemma 3.1. *Let u be a viscosity solution to (3.1) in Q_4 with $\varepsilon \in (0,1)$. Let $p(x,t)$ be uniformly Lipschitz continuous in the spatial variables, that is, there is a number $C_{\text{lip}} > 0$, independent of time variable, such that $|p(x,t) - p(y,t)| \le C_{\text{lip}}|x-y|$. Then there is a constant $C > 0$, which depends on $n, p_-, p_+, C_{\text{lip}}$ and $\|u\|_{L^\infty(Q_4)}$, such that*

$$|u(x,t) - u(y,t)| \le C|x-y|$$

for each $(x,t),(y,t) \in Q_3$ and $|x-y| < 1$.

Proof. As the proof of Lipschitz estimates in Section 2 in [18], this conclusion holds as well. It is enough to notice that the matrix

$$I + (p(x,t)-2)\frac{q \otimes q}{|q|^2+\varepsilon^2} \quad (q \in \mathbb{R}^n)$$

is uniformly elliptic. □

Remark 3.2. *If $p(x,t)$ is assumed to be of class $C^1(Q_4)$ in the above lemma, then the constant $C_{\text{lip}} > 0$ could be substituted with $\|D_{x,t}p(x,t)\|_{L^\infty(Q_4)}$.*

It follows from Lemma 3.1 that the spatial gradient Du is bounded (which is independent of ε). By normalization we may assume that $|Du| \le 1$ below. In the sequel, we are ready to prove the Hölder continuity of Du at the origin $(0,0)$, and conclude directly the local Hölder regularity for Du via standard translation arguments. To this end, we first verify that the solutions of equation (1.1) are locally of class $C^{1,\alpha}$ under the condition that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is small. Next, through doing a scaling work, we finally infer that the solutions are locally $C^{1,\alpha}$-regular under the assumption that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is finite, i.e., $|D_{x,t}p(x,t)| \le M$ in Q_1 (M is a large number).

3.1. **Hölder regularity of gradient for the case that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is small enough.** We shall prove that if the projection of Du onto the unit vector $e \in \mathbb{S}^{n-1}$ is away from 1 in a large part of Q_1, then the inner product $Du \cdot e$ has improved oscillation in a smaller cylinder.

Lemma 3.3. *Suppose that u is a smooth solution to (3.1) in Q_1. For every $0 < l < 1, \mu > 0$, if $p(x,t) \in C^1(\overline{Q_1})$ and $\|Dp\|_{L^\infty(Q_1)} \le \beta$, where β is a small enough constant depending on n, p_-, p_+, μ and l, then we can conclude that there are two positive constants τ and δ, the former depending only on n, μ and the latter depending on n, p_-, p_+, μ and l, such that for arbitrary $e \in \mathbb{S}^{n-1}$, if*

$$|\{(x,t) \in Q_1 : Du \cdot e \le l\}| > \mu |Q_1|,$$

we have

$$Du \cdot e < 1 - \delta \quad in\ Q_\tau.$$

Proof. Let

$$a^\varepsilon_{ij,m} := \frac{\partial a^\varepsilon_{ij}(x,t,Du)}{\partial u_m} = (p(x,t)-2)\left(\frac{\delta_{im}u_j + \delta_{jm}u_i}{|Du|^2+\varepsilon^2} - \frac{2u_iu_ju_m}{(|Du|^2+\varepsilon^2)^2}\right).$$

Differentiating equation (3.1) in x_k derives

$$(u_k)_t = a^\varepsilon_{ij}(u_k)_{ij} + a^\varepsilon_{ij,m}u_{ij}(u_k)_m + p_k\frac{u_iu_j}{|Du|^2+\varepsilon^2}u_{ij},$$

where $p_k := \frac{\partial p(x,t)}{\partial x_k}$. Define

$$w = (Du \cdot e - l + \rho|Du|^2)^+$$

with $\rho = \frac{l}{4}$. Here $(f)^+ := \max\{f, 0\}$. Then for the function $Du \cdot e - l$ we have

$$(Du\cdot e - l)_t = a^\varepsilon_{ij}(Du\cdot e - l)_{ij} + a^\varepsilon_{ij,m}u_{ij}(Du \cdot e - l)_m + Dp\cdot e\frac{u_iu_ju_{ij}}{|Du|^2+\varepsilon^2},$$

and for $|Du|^2$ derive

$$(|Du|^2)_t = a^\varepsilon_{ij}(|Du|^2)_{ij} + a^\varepsilon_{ij,m}u_{ij}(|Du|^2)_m + 2Dp\cdot Du\frac{u_iu_ju_{ij}}{|Du|^2+\varepsilon^2} - 2a^\varepsilon_{ij}u_{ki}u_{kj},$$

where Dp denotes the spatial gradient of $p(x,t)$.

Merging the previous two identities arrives at in the region $\Omega_+ := \{(x,t) \in Q_1 : w > 0\}$

$$w_t = a^\varepsilon_{ij}w_{ij} + a^\varepsilon_{ij,m}u_{ij}w_m + Dp\cdot(e + 2\rho Du)\frac{u_iu_ju_{ij}}{|Du|^2+\varepsilon^2} - 2\rho a^\varepsilon_{ij}u_{ki}u_{kj}.$$

Noting that $|Du| > \frac{l}{2}$ in Ω_+, we have

$$|a^\varepsilon_{ij,m}| \le \frac{4|p(x,t)-2|}{l} \le \frac{4}{l}\max\{|p_+-2|, |p_--2|\} =: \frac{4}{l}b. \tag{3.2}$$

By Cauchy-Schwarz inequality and (3.2), we obtain

$$\begin{aligned} w_t &\le a^\varepsilon_{ij}w_{ij} + \frac{4}{l}b|Dw|\sum_{i,j}^n|u_{ij}| + (1+2\rho)|Dp|\frac{|\langle D^2u\cdot Du, Du\rangle|}{|Du|^2+\varepsilon^2} - 2\rho a^\varepsilon_{ij}u_{ki}u_{kj} \\ &\le a^\varepsilon_{ij}w_{ij} + \varepsilon|D^2u|^2 + \frac{4n^2b^2}{\varepsilon l^2}|Dw|^2 + (1+2\rho)|Dp||D^2u| \\ &\quad - 2\rho\left(|D^2u|^2 + (p(x,t)-2)\frac{|D^2u\cdot Du|^2}{|Du|^2+\varepsilon^2}\right) \end{aligned}$$

$$\begin{aligned}&\le a^\varepsilon_{ij}w_{ij}+2\varepsilon|D^2u|^2+\frac{4n^2b^2}{\varepsilon l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\varepsilon}|Dp|^2\\&\quad-2\rho\left(|D^2u|^2+(p(x,t)-2)\frac{|D^2u\cdot Du|^2}{|Du|^2+\varepsilon^2}\right).\end{aligned}$$

Denote

$$\Omega_1:=\{p(x,t)\ge 2\}\cap\Omega_+\quad\text{and}\quad\Omega_2:=\{p(x,t)<2\}\cap\Omega_+.$$

In Ω_1, we get

$$w_t\le a^\varepsilon_{ij}w_{ij}+2\varepsilon|D^2u|^2+\frac{4n^2b^2}{\varepsilon l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\varepsilon}|Dp|^2-2\rho|D^2u|^2.$$

In Ω_2, we have

$$\begin{aligned}w_t&\le a^\varepsilon_{ij}w_{ij}+2\varepsilon|D^2u|^2+\frac{4n^2b^2}{\varepsilon l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\varepsilon}|Dp|^2+2\rho(1-p(x,t))|D^2u|^2\\&\le a^\varepsilon_{ij}w_{ij}+2\varepsilon|D^2u|^2+\frac{4n^2b^2}{\varepsilon l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\varepsilon}|Dp|^2+2\rho(1-p_-)|D^2u|^2.\end{aligned}$$

Case 1. If $p_-\ge 2$, then we obtain by choosing $\varepsilon=\rho$

$$\begin{aligned}w_t&\le a^\varepsilon_{ij}w_{ij}+\frac{4n^2b^2}{\rho l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\rho}|Dp|^2\\&\le a^\varepsilon_{ij}w_{ij}+\frac{4n^2b^2}{\rho l^2}|Dw|^2+\frac{(1+2\rho)^2}{4\rho}M^2\end{aligned}$$

in Ω_+, where $b=p_+-2$ and $M=\|Dp\|_{L^\infty(Q_1)}$. Let

$$\overline{c}=\frac{(1+2\rho)^2}{4\rho}M^2.$$

Thereby it satisfies in the viscosity sense that

$$w_t\le a^\varepsilon_{ij}w_{ij}+\frac{4n^2b^2}{\rho l^2}|Dw|^2+\overline{c}\quad\text{in } Q_1.$$

Set $\overline{w}=1-l+\rho+\overline{c}$ and $\nu=\frac{c_1}{\rho l^2}$, where c_1 will be chosen later. Define

$$U=\frac{1}{\nu}(1-e^{\nu(w-\overline{c}t-\overline{w})}).$$

Observe that

$$a^\varepsilon_{ij}w_{ij}+\nu a^\varepsilon_{ij}w_iw_j\ge a^\varepsilon_{ij}w_{ij}+\nu|Dw|^2.$$

Hence we can take $c_1=4n^2(p_+-2)^2$ such that

$$U_t\ge a^\varepsilon_{ij}U_{ij}\quad\text{in } Q_1$$

in the viscosity sense. Obviously, $U\ge 0$ in Q_1.

If $Du\cdot e\le l$, then it follows that

$$|\{(x,t)\in Q_1:U\ge\nu^{-1}(1-e^{\nu(l-1)})\}|>\mu|Q_1|.$$

Thus we can conclude from Lemma 2.4 that there exist two constants $\tau,\gamma_0>0$ such that

$$U\ge\nu^{-1}(1-e^{\nu(l-1)})\gamma_0\quad\text{in } Q_\tau,$$

where τ and γ_0 depend on μ,n and n,p_-,p_+,μ separately. Since $w\le\overline{w}+\overline{c}t$, we derive

$$U\le\overline{w}-w+\overline{c}t.$$

Therefore in Q_τ we get

$$Du\cdot e+\rho|Du|^2\le 1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\overline{c}+\overline{c}t.$$

By $|Du\cdot e|\le|Du|$, the above inequality becomes

$$Du\cdot e+\rho(Du\cdot e)^2\le 1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\overline{c}\quad\text{in } Q_\tau.$$

Furthermore,

$$Du\cdot e\le\frac{-1+\sqrt{1+4\rho(1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\overline{c})}}{2\rho}\quad\text{in } Q_\tau.$$

Here we need

$$\frac{-1+\sqrt{1+4\rho(1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\overline{c})}}{2\rho}<1.$$

Namely,

$$\begin{aligned}&\overline{c}<\nu^{-1}(1-e^{\nu(l-1)})\gamma_0\\ \Longleftrightarrow\ &\frac{(1+2\rho)^2}{4\rho}M^2<\nu^{-1}(1-e^{\nu(l-1)})\gamma_0\\ \Longleftrightarrow\ &M^2<\frac{4\rho\gamma_0}{\nu(1+2\rho)^2}(1-e^{\nu(l-1)}),\end{aligned}$$

where $\nu=\frac{4n^2}{\rho l^2}(p_+-2)^2$. In other words, when $M:=\|Dp\|_{L^\infty(Q_1)}$ is small enough depending on n,p_-,p_+,l and μ, we get

$$Du\cdot e\le 1-\delta\quad\text{in } Q_\tau,$$

where $\delta>0$ depends on n,p_-,p_+,l and μ.

Case 2. If $1<p_-<2$, we obtain

$$w_t\le a^\varepsilon_{ij}w_{ij}+\frac{4n^2b^2}{\rho l^2(p_--1)}|Dw|^2+\frac{(1+2\rho)^2}{4\rho(p_--1)}|Dp|^2\quad\text{in }\Omega_+,$$

where $b=\max\{|p_+-2|,|p_--2|\}$. Let

$$\widehat{c}=\frac{(1+2\rho)^2}{4\rho(p_--1)}M^2.$$

It follows that

$$w_t\le a^\varepsilon_{ij}w_{ij}+\frac{4b^2}{\rho l^2(p_--1)}|Dw|^2+\widehat{c}\quad\text{in } Q_1$$

in the viscosity sense.

Notice that

$$a^\varepsilon_{ij}w_{ij}+\nu a^\varepsilon_{ij}w_iw_j\ge a^\varepsilon_{ij}w_{ij}+\nu(p_--1)|Dw|^2$$

with $\nu=\frac{c_2}{\rho l^2(p_--1)}>0$, where c_2 is a constant determined later. Denote $\widehat{w}=1-l+\rho+\widehat{c}$ and $V=\frac{1}{\nu}(1-e^{\nu(w-\widehat{c}t-\widehat{w})})$. We take $c_2=\frac{4n^2b^2}{p_--1}$ such that

$$V_t\ge a^\varepsilon_{ij}V_{ij}\quad\text{in } Q_1$$

in the viscosity sense. Apparently, $V\ge 0$ in Q_1.

For $Du\cdot e\le l$, by the assumption we have

$$|\{(x,t)\in Q_1: V\ge\nu^{-1}(1-e^{\nu(l-1)})\}|>\mu|Q_1|.$$

Using again Lemma 2.4 deduces that there are two positive constants τ and γ_0, depending respectively on μ, n and n, p_-, p_+, μ, such that

$$V \geq \nu^{-1}(1-e^{\nu(l-1)})\gamma_0 \quad \text{in } Q_\tau.$$

We further obtain

$$Du\cdot e+\rho(Du\cdot e)^2 \leq 1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\widehat{c} \quad \text{in } Q_\tau.$$

Thus

$$Du\cdot e \leq \frac{-1+\sqrt{1+4\rho(1+\rho-\nu^{-1}(1-e^{\nu(l-1)})\gamma_0+\widehat{c})}}{2\rho} \quad \text{in } Q_\tau.$$

Analogous to Case 1, for $M=\|Dp\|_{L^\infty(Q_1)}$ sufficiently small and depending on n, p_-, p_+, l and μ, we arrive at

$$Du\cdot e \leq 1-\delta \quad \text{in } Q_\tau,$$

where $\delta>0$ depends on n, p_-, p_+, l and μ. We now complete the proof. □

Remark 3.4. *For the case that $p_- \geq 2$, we note that*

$$I \leq (p_- -1)I \leq (a^\varepsilon_{ij}(x,t,q))_{n\times n} \leq (p_+ -1)I$$

for all $\varepsilon\in(0,1), q\in\mathbb{R}^n$ and $(x,t)\in Q_1$, so the constant γ_0 appearing in Case 1 may not depend on p_-.

If Lemma 3.3 holds in all directions $e\in\mathbb{S}^{n-1}$, we then get the decay of oscillation of $|Du|$ in a smaller cylinder. This content is formulated by the following lemma.

Lemma 3.5. *Let u be a smooth solution of (3.1) in Q_1. For any $0<l<1, \mu>0$, when $\|Dp\|_{L^\infty(Q_1)} \leq \beta$ with β being a sufficiently small constant depending on n, p_-, p_+, l, μ, there is τ (small) depending on n, μ, and $\delta>0$ depending on n, p_-, p_+, l and μ, such that for any nonnegative integer k, if*

$$|\{(x,t)\in Q_{\tau^i}: Du\cdot e \leq l(1-\delta)^i\}| > \mu|Q_{\tau^i}| \quad \textit{for all } e\in\mathbb{S}^{n-1}, \tag{3.3}$$

and $i=0,1,\cdots,k$, then

$$|Du| < (1-\delta)^{i+1} \quad \textit{in } Q_{\tau^{i+1}}$$

for all $i=0,1,\cdots,k$.

Proof. We prove this lemma by induction. For $k=0$, the conclusion holds obviously by Lemma 3.3. Suppose the conclusion is true for $i=0,1,\cdots,k-1$. We are going to verify it for $i=k$. Set

$$v(x,t) := \frac{1}{\tau^k(1-\delta)^k}u(\tau^k x, \tau^{2k}t).$$

Then v satisfies

$$v_t = \Delta v+(h_k(x,t)-2)\frac{v_i v_j}{|Dv|^2+\varepsilon^2(1-\delta)^{-2k}}v_{ij} \quad \text{in } Q_1,$$

where $h_k(x,t)=p(\tau^k x, \tau^{2k}t)$. We can see from the induction assumptions that $|Dv|<1$ in Q_1, and

$$|\{(x,t)\in Q_1: Dv\cdot e \leq l\}| > \mu|Q_1| \quad \text{for all } e\in\mathbb{S}^{n-1}.$$

Furthermore, we observe that

$$1<p_- \leq h_k(x,t) \leq p_+ < \infty$$

and

$$|Dh_k(x,t)| = |\tau^k Dp(y,s)| \leq \tau^k\|Dp\|_{L^\infty(Q_1)},$$

where $(y,s)=(\tau^k x,\tau^{2k}t)$ and $(x,t)\in Q_1$. Hence from Lemma 3.3 we get

$$Dv\cdot e\le 1-\delta \quad \text{in } Q_\tau$$

for all $e\in\mathbb{S}^{n-1}$. Namely, $|Dv|\le 1-\delta$ in Q_τ. Rescaling back, we arrive at

$$|Du|<(1-\delta)^{k+1} \quad \text{in } Q_{\tau^{k+1}}.$$

We finish the proof. ∎

Remark 3.6. *Noting that $0<\tau<1$, when $Dp(x,t)$ is bounded, we can see that*

$$|Dh_k(x,t)|\to 0 \quad \textit{uniformly in } Q_1,$$

by sending $k\to\infty$. That is to say, for k large enough, we could remove the restriction that $\|Dp\|_{L^\infty(Q_1)}$ is sufficiently small.

Remark 3.7. *To obtain the reduction of oscillation of $|Du|$, $\|Dp\|_{L^\infty(Q_1)}\le\beta$ (small enough) is required from Lemmas 3.3 and 3.5 above. Hence we could assume initially that $\sup_{(x,t)\in Q_1}|D_{x,t}p| =: \|D_{x,t}p\|_{L^\infty(Q_1)}\le\beta$. Naturally, $\sup_{(x,t)\in Q_1}|Dp| =: \|Dp\|_{L^\infty(Q_1)}\le \|D_{x,t}p\|_{L^\infty(Q_1)}\le\beta$. Consequently, these two lemmas still hold true, when we replace $\|Dp\|_{L^\infty(Q_1)}$ by $\|D_{x,t}p\|_{L^\infty(Q_1)}$ in Lemmas 3.3 and 3.5.*

If the previous iteration process could be carried out infinitely, we then easily infer the Hölder regularity for Du at the origin $(0,0)$. Nonetheless, unless $Du(0,0)=0$, the iteration will terminate at some step, that is, the condition (3.3) will fail to be fulfilled in some direction $e\in\mathbb{S}^{n-1}$. In this scenario, it follows from Lemmas 2.2 and 2.3 that the solution, u, is close to some linear function. Then we could employ the conclusion on the regularity of small perturbation solution in [38] to verify the Hölder continuity of Du.

Lemma 3.8. *Let u be a smooth solution to* (3.1) *in Q_1. For $\gamma=\frac{1}{2}$, there are two positive constants η (small) and C (large), both depending on n,p_-,p_+ and $\|D_{x,t}p\|_{L^\infty(Q_1)}$ such that if a linear function $L(x)$ with $\frac{1}{2}\le|DL|\le 2$ satisfies*

$$\|u(x,t)-L(x)\|_{L^\infty(Q_1)}\le\eta,$$

then

$$\|u(x,t)-L(x)\|_{C^{2,1/2}(Q_{1/2})}\le C.$$

Proof. We can reach this conclusion from Corollary 1.2 in [38], because $L(x)$ is a solution to (3.1) as well. ∎

Remark 3.9. *From Remark 3.7, we have known that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is small enough, so in Lemma 3.8 we may assume that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is smaller than some sufficiently large constant determined so that we can substitute $\|D_{x,t}p\|_{L^\infty(Q_1)}$ by that constant.*

In the following, we will give a uniformly *a priori* estimate for the solution to (3.1).

Theorem 3.10. *Let u be a smooth solution to* (3.1) *in Q_1. Suppose that $p(x,t)\in C^1(\overline{Q_1})$ and $\|D_{x,t}p\|_{L^\infty(Q_1)}\le\beta$, where β is a small constant depending only on n,p_-,p_+. Then there are two positive constants α and C, both of which depend on n,p_-,p_+, such that*

$$\|Du\|_{C^\alpha(Q_{1/2})}\le C(\|u\|_{L^\infty(Q_1)}+\varepsilon)$$

and

$$\sup_{Q_{1/2}}\frac{|u(x,t)-u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}}\le C(\|u\|_{L^\infty(Q_1)}+\varepsilon).$$

Proof. As the proof of Theorem 4.5 in [20], we can first deduce $Du \in C^\alpha(Q_{1/2})$ by combining Lemma 3.5 and Lemmas 2.3, 3.8. To this end, we choose η as the one in Lemma 3.8 with $\|D_{x,t}p\|_{L^\infty(Q_1)}$ replaced by some large constant fixed. And then we take $\varepsilon_0, \varepsilon_1 > 0$ such small constants that Lemma 2.3 holds. Next, we determine the constants l and μ to be $1 - \varepsilon_0^2/2$ and $\varepsilon_1/|Q_1|$ respectively.

Terminally, by $Du \in C^\alpha(Q_{1/2})$ and using Lemma 2.2, we show that u is $C^{\frac{1+\alpha}{2}}(Q_{1/2})$-regular in the t-variable. □

Lemma 3.11. *Let $g \in C(\partial_p Q_1)$. For $\varepsilon > 0$, there is a unique solution $u^\varepsilon \in C(\overline{Q_1}) \cap C^\infty(Q_1)$ of equation (3.1) satisfying $u^\varepsilon = g$ on $\partial_p Q_1$.*

For this lemma, we observe that equation (3.1) is uniformly parabolic and the coefficients $a^\varepsilon_{ij}(x,t,Du)$ are smooth with bounded derivatives for every $\varepsilon > 0$. So it can be concluded from the classical quasilinear equation theory (see Theorem 4.4 of [25], page 560) and the Schauder estimates.

Combining the previous conclusions, we now could establish an important intermediate result as follows.

Theorem 3.12. *Let u be a viscosity solution of (1.1) in Q_1. Assume that $p(x,t) \in C^1(\overline{Q_1})$ and $\|D_{x,t}p\|_{L^\infty(Q_1)} \le \beta$ with β being a small enough constant that depends on n, p_-, p_+. Then there are two positive constants $\alpha \in (0,1)$ and C, both depending on n, p_- and p_+, such that*

$$\|Du\|_{C^\alpha(Q_{1/2})} \le C\|u\|_{L^\infty(Q_1)}$$

and

$$\sup_{Q_{1/2}} \frac{|u(x,t) - u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}} \le C\|u\|_{L^\infty(Q_1)}.$$

Proof. Without loss of generality, we can suppose $u \in C(\overline{Q_1})$. It follows from Lemma 3.11 that there is a unique viscosity solution $u^\varepsilon \in C(\overline{Q_1}) \cap C^\infty(Q_1)$ to (3.1) such that $u^\varepsilon = u$ on $\partial_p Q_1$. Based on the proof of Theorem 1.1 in [20], we note that it suffices to show that u^ε converges to u uniformly in $\overline{Q_1}$ as $\varepsilon \to 0$ (up to a subsequence). To this end, we shall make use of comparison principle and stability property for viscosity solution to (1.1), which are two counterparts to Theorems 2.9 and 2.10 in [20]. Fortunately, these two conclusions hold true, whose proof will be presented in Section 4. □

3.2. **Hölder regularity of gradient for the case that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is finite.** In this subsection, we shall establish the Hölder estimates for the gradients of solutions to (1.1), under the condition that $|D_{x,t}p(x,t)|$ possesses a more general bound.

Set

$$\widetilde{u}(x,t) := u(\epsilon x, \epsilon^2 t), \quad \widetilde{p}(x,t) := p(\epsilon x, \epsilon^2 t)$$

with $0 < \epsilon < 1$. By a scaling argument for equation (1.1), it follows that $\widetilde{u}$ satisfies (in the viscosity sense) that

$$\widetilde{u}_t = \left(\delta_{ij} + (\widetilde{p}(x,t) - 2)\frac{\widetilde{u}_i \widetilde{u}_j}{|D\widetilde{u}|^2}\right)\widetilde{u}_{ij} \quad \text{in } Q_{\epsilon^{-1}}. \tag{3.4}$$

When $|D_{x,t}p| \le M$ in Q_1 with M being a large quantity, by the following equalities

$$D\widetilde{p}(x,t) = \epsilon Dp(y,s),$$
$$\partial_t \widetilde{p}(x,t) = \epsilon^2 \partial_s p(y,s),$$

where $(y,s) := (\epsilon x, \epsilon^2 t)$, then we have

$$\|D_{x,t}\widetilde{p}\|_{L^\infty(Q_{\epsilon^{-1}})} \le \epsilon \|D_{x,t}p\|_{L^\infty(Q_1)} \le \epsilon M < \beta$$

by choosing ϵ so small that

$$\epsilon \le \frac{\beta}{M+1}.$$

According to Theorem 3.12, we know that ϵ will only depend on n, p_-, p_+ and $\|D_{x,t}p\|_{L^\infty(Q_1)}$. Observe that the structure of (3.4) is similar to that of (1.1). Therefore this permits us to employ the previous results in subsection 3.1 to show the local $C^{1,\alpha}$-regularity of the solution $\widetilde{u}$ to (3.4). In turn, by rescaling back, we can deduce that the solution u to (1.1) is of class $C^{1,\alpha}_{\text{loc}}$ provided that $\|D_{x,t}p\|_{L^\infty(Q_1)}$ is finite. As a consequence, we reach the conclusion that if function $p(x,t) \in C^1(\overline{Q_1})$, then the viscosity solution to (1.1) is locally $C^{1,\alpha}$-regular.

As has been stated above, we now complete the proof of Theorem 1.1.

4. Comparison principle and stability for viscosity solution

In this section, we are going to prove the comparison principle and stability properties for viscosity solutions of (1.1). We shall make use of Ishii-Lions' method to show the comparison principle and we remark that the comparison principle is new. Some ideas of the proof are inspired by that of comparison principle in [23], in which the $p(x)$-Laplace equation was considered. Here we investigate these two properties in a more general domain. Let Ω be a bounded domain of $\mathbb{R}^n$. We denote a general parabolic cylinder by $\Omega_T := \Omega \times [0,T)$, and $\partial_p\Omega_T$ denotes its parabolic boundary.

Theorem 4.1 (comparison principle)**.** *Suppose that the function $p(x,t)$ in equation* (1.1) *is Lipschitz continuous in Ω_T. Let u be a viscosity subsolution and v be a continuous viscosity supersolution to* (1.1)*. If $u \le v$ on $\partial_p\Omega_T$, then we can conclude*

$$u \le v \quad \text{in } \Omega_T. \tag{4.1}$$

Proof. For convenience, we can assume v is a strict supersolution, i.e.,

$$v_t - \left(\Delta v + (p(x,t)-2)\left\langle D^2 v\frac{Dv}{|Dv|}, \frac{Dv}{|Dv|}\right\rangle\right) > 0$$

in the viscosity sense by considering $w := v + \frac{\varepsilon}{T-t}$ instead, and $w \to \infty$ as $t \to T$. Indeed, we suppose $\varphi \in C^2(\Omega_T)$ such that $w - \varphi$ has a local minimum at $(x_0,t_0) \in \Omega_T$, then so does $v - \phi$ by letting $\phi(x,t) := \varphi(x,t) - \frac{\varepsilon}{T-t}$. Notice that

$$\begin{aligned} D\phi(x_0,t_0) &= D\varphi(x_0,t_0), \\ \partial_t\phi(x_0,t_0) &= \partial_t\varphi(x_0,t_0) - \frac{\varepsilon}{(T-t_0)^2}, \end{aligned}$$

and

$$D^2\phi(x_0,t_0) = D^2\varphi(x_0,t_0).$$

Because v is a viscosity supersolution, we obtain

$$\begin{aligned} 0 &\le \partial_t\phi(x_0,t_0) - \left(\text{tr}D^2\phi(x_0,t_0) + (p(x_0,t_0)-2)\left\langle D^2\phi(x_0,t_0)\frac{\phi(x_0,t_0)}{|\phi(x_0,t_0)|}, \frac{\phi(x_0,t_0)}{|\phi(x_0,t_0)|}\right\rangle\right) \\ &= \partial_t\varphi(x_0,t_0) - \frac{\varepsilon}{(T-t_0)^2} \\ &\quad - \left(\text{tr}D^2\varphi(x_0,t_0) + (p(x_0,t_0)-2)\left\langle D^2\varphi(x_0,t_0)\frac{\varphi(x_0,t_0)}{|\varphi(x_0,t_0)|}, \frac{\varphi(x_0,t_0)}{|\varphi(x_0,t_0)|}\right\rangle\right), \end{aligned}$$

when $D\varphi(x_0,t_0)\neq 0$. Here we denote by $\mathrm{tr}M$ the trace of matrix M. Furthermore,

$$\begin{aligned}0&<\frac{\varepsilon}{(T-t_0)^2}\\&\leq \partial_t\varphi(x_0,t_0)-\left(\mathrm{tr}D^2\varphi(x_0,t_0)+(p(x_0,t_0)-2)\left\langle D^2\varphi(x_0,t_0)\frac{\varphi(x_0,t_0)}{|\varphi(x_0,t_0)|},\frac{\varphi(x_0,t_0)}{|\varphi(x_0,t_0)|}\right\rangle\right).\end{aligned}$$

When $D\varphi(x_0,t_0)=0$, for some $|\eta|\leq 1$ (i.e., $\eta\in\overline{B_1(0)}$) we get

$$0\leq \partial_t\phi(x_0,t_0)-(\mathrm{tr}D^2\phi(x_0,t_0)+(p(x_0,t_0)-2)\langle D^2\phi(x_0,t_0)\cdot\eta,\eta\rangle).$$

Namely,

$$0<\frac{\varepsilon}{(T-t_0)^2}\leq \partial_t\varphi(x_0,t_0)-(\mathrm{tr}D^2\varphi(x_0,t_0)+(p(x_0,t_0)-2)\langle D^2\varphi(x_0,t_0)\cdot\eta,\eta\rangle).$$

In conclusion, we have verified that $w:=v+\frac{\varepsilon}{T-t}$ is a strict supersolution.

To demonstrate this conclusion, we argue by contradiction. Suppose (4.1) is not valid. Then it holds that for some $(\widehat{x},\widehat{t})\in\Omega\times(0,T)$, we have

$$\theta:=u(\widehat{x},\widehat{t})-v(\widehat{x},\widehat{t})=\sup_{\Omega_T}(u-v)>0.$$

Set

$$\Psi_j(x,y,t,s)=u(x,t)-v(y,s)-\Phi_j(x,y,t,s),$$

where $\Phi_j(x,y,t,s)=\frac{j}{q}|x-y|^q+\frac{j}{2}(t-s)^2$ with $q>2$.

Without loss of generality, in what follows, we take a special value of q, i.e., $q=4$. Let (x_j,y_j,t_j,s_j) be the maximum point of Ψ_j in $\overline{\Omega}\times\overline{\Omega}\times[0,T)\times[0,T)$. We can prove that $(x_j,y_j,t_j,s_j)\in\Omega\times\Omega\times(0,T)\times(0,T)$ and $(x_j,y_j,t_j,s_j)\to(\widehat{x},\widehat{x},\widehat{t},\widehat{t})$ as $j\to\infty$ by the Lemma 7.2 in [12].

Case 1. If $x_j=y_j$, then

$$\begin{aligned}0&=D_x\Phi_j(x_j,y_j,t_j,s_j)=D_y\Phi_j(x_j,y_j,t_j,s_j),\\0&=D_x^2\Phi_j(x_j,y_j,t_j,s_j)=D_y^2\Phi_j(x_j,y_j,t_j,s_j).\end{aligned}$$

Observe that

$$u(x_j,t_j)-v(y_j,s_j)-\Phi_j(x_j,y_j,t_j,s_j)\geq u(x_j,t_j)-v(y,s)-\Phi_j(x_j,y,t_j,s).$$

Denote

$$\Theta(y,s):=-\Phi_j(x_j,y,t_j,s)+\Phi_j(x_j,y_j,t_j,s_j)+v(y_j,s_j).$$

Obviously, $v(y,s)-\Theta(y,s)$ reaches the local minimum at (y_j,s_j). Due to v a strict supersolution, we arrive at

$$\begin{aligned}0&<\partial_s\Theta(y_j,s_j)-(\mathrm{tr}D^2\Theta(y_j,s_j)+(p(y_j,s_j)-2)\langle D^2\Theta(y_j,s_j)\cdot\eta,\eta\rangle)\\&=j(t_j-s_j)\end{aligned}$$

for some $|\eta|\leq 1$. Analogously, letting

$$\beta(x,t):=\Phi_j(x,y_j,t,s_j)-\Phi_j(x_j,y_j,t_j,s_j)+u(x_j,t_j),$$

we can obtain

$$0\geq\partial_t\beta(x_j,t_j)=j(t_j-s_j).$$

From the previous two inequalities, we get

$$0<j(t_j-s_j)-j(t_j-s_j)=0.$$

This is a contradiction.

Case 2. If $x_j \neq y_j$, we have the following results.

By theorem of sums (see [12]), for every $\mu > 0$, there are $X_j, Y_j \in \mathcal{S}^n$ ($\mathcal{S}^n$ denotes the set of all symmetric $n \times n$ matrices) such that

$$(\partial_t \Phi_j, D_x \Phi_j, X_j) \in \overline{\mathcal{P}}^{2,+} u(x_j, t_j), \quad (-\partial_s \Phi_j, -D_y \Phi_j, Y_j) \in \overline{\mathcal{P}}^{2,-} v(y_j, s_j)$$

and

$$\begin{pmatrix} X_j & \\ & -Y_j \end{pmatrix} \leq D^2 \Phi_j + \frac{1}{\mu} (D^2 \Phi_j)^2,$$

where all the derivatives are computed at (x_j, y_j, t_j, s_j) and

$$D^2 \Phi_j = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

with $B := j|x_j - y_j|^2 I + 2j(x_j - y_j) \otimes (x_j - y_j)$. Furthermore, taking $\mu = j$ gets

$$\begin{aligned} \begin{pmatrix} X_j & \\ & -Y_j \end{pmatrix} \leq {} & j(|x_j - y_j|^2 + 2|x_j - y_j|^4) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ & + 2j(1 + 8|x_j - y_j|^2) \begin{pmatrix} G & -G \\ -G & G \end{pmatrix}, \end{aligned} \tag{4.2}$$

where $G := (x_j - y_j) \otimes (x_j - y_j)$. Note that (4.2) implies for any $\xi, \zeta \in \mathbb{R}^n$

$$\langle X_j \xi, \xi \rangle - \langle Y_j \zeta, \zeta \rangle \leq (3j|x_j - y_j|^2 + 18j|x_j - y_j|^4)|\xi - \zeta|^2. \tag{4.3}$$

By virtue of the equivalent definition of viscosity solution emphasized by terminology of semi jets, we obtain

$$-\partial_s \Phi_j - \left(\operatorname{tr} Y_j + (p(y_j, s_j) - 2) \left\langle Y_j \frac{-D_y \Phi_j}{|D_y \Phi_j|}, \frac{-D_y \Phi_j}{|D_y \Phi_j|} \right\rangle \right) > 0 \tag{4.4}$$

and

$$\partial_t \Phi_j - \left(\operatorname{tr} X_j + (p(x_j, t_j) - 2) \left\langle X_j \frac{D_x \Phi_j}{|D_x \Phi_j|}, \frac{D_x \Phi_j}{|D_x \Phi_j|} \right\rangle \right) \leq 0. \tag{4.5}$$

Here we observe that

$$\partial_t \Phi_j = j(t_j - s_j) = -\partial_s \Phi_j$$

and

$$\eta_j := D_x \Phi_j = -D_y \Phi_j = j|x_j - y_j|^2 (x_j - y_j).$$

η_j is nonzero, which is crucial. Denote

$$A(x, t, \eta) := I + (p(x, t) - 2) \frac{\eta}{|\eta|} \otimes \frac{\eta}{|\eta|},$$

which is positive definite so that it possesses matrix square root denoted by $A^{\frac{1}{2}}(x, t, \eta)$. We denote the k-th column of $A^{\frac{1}{2}}(x, t, \eta)$ as $A_k^{\frac{1}{2}}(x, t, \eta)$. Subtracting (4.5) from (4.4), we derive

$$\begin{aligned} 0 & < \operatorname{tr}(A(x_j, t_j, \eta_j) X_j) - \operatorname{tr}(A(y_j, s_j, \eta_j) Y_j) \\ & = \sum_{k=1}^n X_j A_k^{\frac{1}{2}}(x_j, t_j, \eta_j) \cdot A_k^{\frac{1}{2}}(x_j, t_j, \eta_j) - \sum_{k=1}^n Y_j A_k^{\frac{1}{2}}(y_j, s_j, \eta_j) \cdot A_k^{\frac{1}{2}}(y_j, s_j, \eta_j) \\ & \leq Cj|x_j - y_j|^2 \|A^{\frac{1}{2}}(x_j, t_j, \eta_j) - A^{\frac{1}{2}}(y_j, s_j, \eta_j)\|_2^2 \\ & \leq \frac{Cj|x_j - y_j|^2}{(\lambda_{\min}(A^{\frac{1}{2}}(x_j, t_j, \eta_j)) + \lambda_{\min}(A^{\frac{1}{2}}(y_j, s_j, \eta_j)))^2} \|A(x_j, t_j, \eta_j) - A(y_j, s_j, \eta_j)\|_2^2, \end{aligned} \tag{4.6}$$

where the penultimate inequality is obtained by (4.3) and the last inequality is derived from the local Lipschitz continuity of $A \mapsto A^{\frac{1}{2}}$ (see [17], page 410). Here $\lambda_{\min}(M)$ denotes the smallest eigenvalue of a symmetric $n\times n$ matrix M.

Now we estimate

$$\begin{aligned}\|A(x_j,t_j,\eta_j)-A(y_j,s_j,\eta_j)\|_2^2 &= \left\|(p(x_j,t_j)-p(y_j,s_j))\frac{\eta_j}{|\eta_j|}\otimes\frac{\eta_j}{|\eta_j|}\right\|_2^2\\ &= |(p(x_j,t_j)-p(y_j,s_j))|^2\\ &\le C(|x_j-y_j|^2+|t_j-s_j|^2),\end{aligned}$$

where in the last inequality we employ the condition that $p(x,t)$ is Lipschitz continuous, i.e., $|p(x,t)-p(y,s)|\le C|(x-y,t-s)|$. Moreover,

$$\lambda_{\min}(A^{\frac{1}{2}}(x,t,\eta)) = (\lambda_{\min}(A(x,t,\eta))^{\frac{1}{2}} = \min\{1,\sqrt{p_--1}\}.$$

Hence (4.6) turns into

$$\begin{aligned}0 &< \frac{Cj|x_j-y_j|^2}{4\min\{1,p_--1\}}(|x_j-y_j|^2+|t_j-s_j|^2)\\ &= Cj|x_j-y_j|^4+Cj|t_j-s_j|^2|x_j-y_j|^2.\end{aligned}$$

On the other hand, we note that

$$\begin{aligned}u(x_j,t_j)-v(x_j,t_j) &\le \max_{\overline{\Omega}\times[0,T)}\{u(x,t)-v(x,t)\}\\ &\le u(x_j,t_j)-v(y_j,s_j)-\frac{j}{4}|x_j-y_j|^4-\frac{j}{2}(t_j-s_j)^2.\end{aligned}$$

So we further get

$$\begin{aligned}\frac{j}{4}|x_j-y_j|^4+\frac{j}{2}(t_j-s_j)^2 &\le v(x_j,t_j)-v(y_j,s_j)\\ &\to v(\widehat{x},\widehat{t})-v(\widehat{x},\widehat{t})=0,\end{aligned}$$

by sending $j\to\infty$, where we have assumed v is continuous in Ω_T.

Consequently, we reach a contradiction that

$$0<Cj|x_j-y_j|^4+Cj|t_j-s_j|^2|x_j-y_j|^2\to 0$$

as $j\to\infty$, observing that both x_j and y_j converge to the point $\widehat{x}$. □

We now end this section by stability properties of viscosity solutions.

Theorem 4.2 (stability). *Let $\{u_i\}$ be a sequence of viscosity solutions to* (3.1) *in Ω_T with $\varepsilon_i\ge 0$ such that $\varepsilon_i\to 0$, and $u_i\to u$ locally uniformly in Ω_T. Then u is a viscosity solution to* (1.1) *in Ω_T.*

Proof. We only show that u is a viscosity supersolution of (1.1). The proof of u being a subsolution is similar to that. Suppose $\varphi\in C^2(\Omega_T)$ such that $u-\varphi$ attains a local minimum at $(x_0,t_0)\in\Omega_T$. We know, from u_i converging to u locally uniformly, that there is a sequence $\{(x_i,t_i)\}\subset\Omega_T$ with $(x_i,t_i)\to(x_0,t_0)$ as $i\to\infty$, such that

$$u_i-\varphi \quad \text{has a local minimum at } (x_i,t_i).$$

If $D\varphi(x_0,t_0)\ne 0$, then by u_i viscosity supersolution, we obtain

$$\partial_t\varphi(x_i,t_i)\ge \mathrm{tr}D^2\varphi(x_i,t_i)+(p(x_i,t_i)-2)$$

$$\cdot \left\langle D^2\varphi(x_i,t_i)\frac{D\varphi(x_i,t_i)}{(|D\varphi(x_i,t_i)|^2+\varepsilon_i^2)^{\frac{1}{2}}}, \frac{D\varphi(x_i,t_i)}{(|D\varphi(x_i,t_i)|^2+\varepsilon_i^2)^{\frac{1}{2}}}\right\rangle.$$

Letting $i \to \infty$, the above inequality becomes

$$\partial_t\varphi(x_0,t_0) \geq F(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)),$$

where $F(x,t,\eta,X) := \mathrm{tr}X + (p(x,t)-2)\langle X\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\rangle$.

If $D\varphi(x_0,t_0) = 0$, we divide the proof into two cases. When $D\varphi(x_i,t_i) \neq 0$ for i large enough, it follows that

$$\begin{aligned}\partial_t\varphi(x_i,t_i) \geq{}& \mathrm{tr}D^2\varphi(x_i,t_i) + (p(x_i,t_i)-2)\\ &\cdot \left\langle D^2\varphi(x_i,t_i)\frac{D\varphi(x_i,t_i)}{(|D\varphi(x_i,t_i)|^2+\varepsilon_i^2)^{\frac{1}{2}}}, \frac{D\varphi(x_i,t_i)}{(|D\varphi(x_i,t_i)|^2+\varepsilon_i^2)^{\frac{1}{2}}}\right\rangle.\end{aligned}$$

For some vector $\xi \in \mathbb{R}^n$ with $|\xi| \leq 1$, we deduce by sending $i \to \infty$

$$\partial_t\varphi(x_0,t_0) \geq \mathrm{tr}D^2\varphi(x_0,t_0) + (p(x_0,t_0)-2)\langle D^2\varphi(x_0,t_0)\xi,\xi\rangle.$$

When $D\varphi(x_i,t_i) \equiv 0$ for i sufficiently large, by the definition of supersolution, we have

$$\partial_t\varphi(x_i,t_i) \geq \mathrm{tr}D^2\varphi(x_i,t_i) + (p(x_i,t_i)-2)\langle D^2\varphi(x_i,t_i)\xi_i,\xi_i\rangle,$$

where $\xi_i \in \mathbb{R}^n$ satisfies $|\xi_i| \leq 1$. Thus it follows that for some vector $|\xi| \leq 1$

$$\partial_t\varphi(x_0,t_0) \geq \mathrm{tr}D^2\varphi(x_0,t_0) + (p(x_0,t_0)-2)\langle D^2\varphi(x_0,t_0)\xi,\xi\rangle,$$

as $i \to \infty$. Therefore, we prove that u is a viscosity supersolution. □

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Yuzhou Fang
School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
E-mail address: 18b912036@hit.edu.cn

Chao Zhang
School of Mathematics and Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, China
E-mail address: czhangmath@hit.edu.cn