On the critical exponent of the one-dimensional Cucker-Smale model on a general graph

Seung-Yeal Ha

Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Republic of Korea
Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea
syha@snu.ac.kr

Zhuchun Li

Institute for Advanced Study in Mathematics and School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
lizhuchun@hit.edu.cn

Xiongtao Zhang*

Center for Mathematical Sciences, Huazhong University of Science and Technology, 1037 Luoyu Road, Wuhan, 430074, China
xtzhang@hust.edu.cn

We study a critical exponent of the flocking behavior to the one-dimensional Cucker-Smale(C-S) model with a regular inverse power law communication on a general network with a spanning tree. For this, we propose a new nonlinear functional which can control the velocity diameter and decays exponentially fast as time goes on. As an application of the time-evolution of the nonlinear functional, we show that the Cucker-Smale model on a line exhibits a unique critical exponent for unconditional flocking on a general network so that this improves an earlier result[22] on the all-to-all network. Our result also resolves the critical exponent conjecture posed in Cucker-Dong's work[12] for one-dimensional setting. Emergent behavior of the C-S model is independent of the special structure of the underlying network, as long as it contains a spanning tree.

Keywords: flocking; general topology; hypo-coercivity; critical exponent; exponential rate; the Cucker-Smale model.

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*Corresponding author.

1. Introduction

Collective behaviors in complex systems is ubiquitous in our nature, to name a few, e.g., aggregation of bacteria, flocking of birds, and swarming of fish in biological system, herding of volatilities in financial markets and formation of dominant opinions in social systems[2, 14–16, 25, 29, 33, 35–38] etc. In 2007, Cucker and Smale introduced a particle model in their famous paper.[13] The C-S model can be regarded as a dynamical system on a symmetric complete graph where all vertices(particles) are connected with symmetric weights. To fix the idea, we consider an ensemble of C-S particles on a directed and weighted network denoted by $\mathcal{G}$, and let (x_i, v_i) be the spatial position-velocity coordinate of the i-th particle in phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$. We denote the set of vertices, connectivity and weight by $\mathcal{V} := \{1, \cdots, N\}$, $\mathcal{E} = (\chi_{ij})$ and $\mathcal{W} = (w_{ij})$, respectively:

$$\mathcal{V} := \{1, \cdots, N\}, \quad w_{ij} := \frac{1}{(1 + |x_j - x_i|^2)^\beta}, \quad \beta \geq 0,$$

$$\chi_{ij} = \begin{cases} 1, & \text{if the } j\text{-th particle influences the } i\text{-th particle,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Under the setting (1.1), the dynamics of (x_i, v_i) is governed by the first-order dynamical system on a graph $\mathcal{G} = (\mathcal{N}, (\chi_{ij}), (w_{ij}))$:

$$\dot{x}_i = v_i, \quad t > 0, \quad i \in \mathcal{N},$$

$$\dot{v}_i = \kappa \sum_{k \in \mathcal{N}_i} \frac{1}{(1 + |x_j - x_i|^2)^\beta}(v_k - v_i), \tag{1.2}$$

where $\mathcal{N}_i := \{j \ : \ \chi_{ij} > 0\}$ is the set of neighbors of the i-th particle, or in other words, $\mathcal{N}_i$ consists of the agents which influence the particle i.

For the all-to-all case with $\mathcal{N}_i = \mathcal{N}$, system (1.2) has been extensively studied in previous literature in Ref. 7, 10, 13. In particular, Ha and Liu[22] introduced a nonlinear functional approach for (1.2) with a general ψ. The nonlinear functional approach has been successfully applied in the flocking analysis for the C-S model, for example, the collision avoidance,[3, 11] extra forces for special targets,[20, 34] white noise environment,[4, 21] multi-cluster flocking analysis[8, 9] and a variant with normalized weights,[27, 31] etc. This approach was also extended to the kinetic C-S model as a typical approach in its flocking analysis, e.g., Ref. 6, 7, 19. As a direct application of Ha-Liu's result, if ψ is non-integrable, i.e., $\beta \leq \frac{1}{2}$, then they showed that system (1.2) exhibits a mono-cluster flocking for any initial data and positive coupling strength $\kappa > 0$. In contrast, for an integrable ψ, i.e., $\beta > \frac{1}{2}$, system can exhibit multi-cluster flocking depending on the initial data and coupling strength κ (see Ref. 23 for a one-dimensional case). In literature, the exponent which distinguishes mono-cluster flocking and multi-cluster flocking is called "*critical exponent*" (see Ref. 12). On the other hand, for a general network topology (χ_{ij}) which is not all-to-all and non-symmetric, there are several technical difficulties due to the lack of

conservation of total momentum. It seems that the nonlinear functional approach in Ref. 22 cannot be applied for (1.2) as it is. Hence, it is also difficult to derive critical exponent unlike to the all-to-all case for the inverse power communication weight in (1.1).

Recently, the authors in Ref. 18, 30 considered a network topology with rooted leadership, and they found that unconditional mono-cluster flocking can be achieved for a small exponent β, which is less than $\frac{1}{2}$ and depends on the depth of the graph, see Ref. 30 for rooted leadership case and Ref. 18 for more general digraph. In fact, the approaches in Ref. 18, 30 rely on the decay estimate for the discrete-time iteration of relative velocities, and this type of approaches fails to provide the critical exponent for unconditional flocking since the decay is not strict at each iteration. Recall that the nonlinear functional approach in Ref. 22 crucially relies on the symmetry of the interaction topology. In Ref. 31, the nonlinear functional approach was extended to a variant of C-S model in which the interaction is not fully symmetric; however, the global structural symmetry was still imposed in the sense that the interaction should be all-to-all. In this paper, we will address the following questions:

- (Q1): Can we extend the nonlinear functional approach to (1.2) with a general network?

- (Q2): For a general network with weight $\psi(s) = \frac{1}{(1+s^2)^\beta}$, can the critical exponent for mono-cluster flocking be $\frac{1}{2}$? Moreover, can the unconditional flocking emerge at the critical exponent?

The purpose of this paper is to answer the above questions affirmatively in one-dimension. More precisely, we will study the following one-dimensional C-S system with a general communication weight ψ on a digraph containing a spanning tree (see Subsection 2.2)

$$\dot{x}_i = v_i, \quad t > 0, \quad i = 1, 2, \cdots, N,$$
$$\dot{v}_i = \kappa \sum_{k \in \mathcal{N}_i} \psi(x_k - x_i)(v_k - v_i), \qquad (1.3)$$

where the communication weight $\psi(r)$ satisfies parity, positivity, regularity and monotonicity conditions: there exists a positive constant $M > 0$ such that

$$\psi(-r) = \psi(r), \quad 0 \le \psi(r) \le M, \quad \forall\ r \in \mathbb{R},$$
$$\psi \text{ is monotonically decreasing (increasing) for } r \ge 0\ (r \le 0), \qquad (1.4)$$
$$\psi \text{ is not zero and an analytic function on } \mathbb{R}.$$

In this work, we develop a new nonlinear functional approach which do not need a symmetry of a network topology. Compared to the all-to-all case, there are two technical difficulties. First, the total momentum $\sum_{i=1}^{N} v_i$ may not be conserved due to the lack of symmetry in the network topology (χ_{ij}). Second, the decay rate of the velocity diameter may depend on the network structure. Therefore, we can

not expect a Gronwall type differential inequality with constant coefficient for the velocity diameter for all time. In fact, we will derive the decay rate through the hypo-coercivity type estimates which is highly non-trivial for nonlinear systems (see the related results[1,5] for linear systems). To overcome these two difficulties, we introduce two key ingredients. First, we introduce a node decomposition Lemma which provides a hierarchical structure and allows us to apply induction principle with respect to the graph (see Lemma 2.5). Second, we construct some nonlinear functionals Q^k and Y^k in terms of weighted mean position and mean velocity and show that these functionals are equivalent to the diameters (see (3.31) and (4.16)). The advantage is that, with the good properties of these quantities, we can still use the induction principle to yield the dissipation from hypo-coercivity estimates and derive a flocking estimate. Our main result can be summarized as follows.

Theorem 1.1. *Suppose that the network topology (χ_{ij}) contains a spanning tree, and let (x_i, v_i) be a solution to (1.3) with (1.4). Then, unconditional mono-cluster flocking emerges exponentially fast if and only if the communication weight function ψ is non-integrable. More precisely, there exist positive constants Λ and C such that*

$$\max_{1 \leq i,j \leq N} |v_j(t) - v_i(t)| \leq Ce^{-\Lambda t} \max_{1 \leq i,j \leq N} |v_j(0) - v_i(0)|, \quad t \geq 0.$$

Remark 1.1. For $\psi(s) = \frac{1}{(1+s^2)^\beta}$, the critical exponent for mono-cluster flocking is $\beta = \frac{1}{2}$, and mono-cluster flocking emerges for any $\beta \leq \frac{1}{2}$. Thus, this resolves the conjecture posed in Ref. 12. If $\psi(s)$ is short-ranged, i.e., integrable, then mono-cluster flocking may not emerge even for the all-to-all case, see Ref. 13 for details. Therefore, for a general network, the emergence of unconditional flocking occurs only when $\psi(s)$ is non-integrable.

The rest of the paper is organized as follows: in Section 2, we provide several concepts such as node and node decomposition to be used essentially in later sections. In Section 3, we review the first-order reduction of the second-order system (1.3), and then obtain a uniform upper bound for the relative distances in the long range interaction regime. In Section 4, we use this uniform bound to verify the exponential emergence of mono-cluster flocking. Finally, Section 5 is devoted to be a brief summary of our main results and remaining open problems to be explored in a future work.

2. Preliminaries

In this section, we present some basic concepts such as relative equilibria, mono-cluster flocking, spanning tree and node decomposition of a general network (1.3).

2.1. *Relative equilibria*

First, we note the Galilean invariance of the C-S model in the following lemma.

Lemma 2.1. *The C-S model* (1.3) *is Galilean invariant in the sense that for any solution* $\{(x_i, v_i)\}$ *to* (1.3) *and* $c \in \mathbb{R}^d$, $(x_i + tc, \ v_i + c)$ *is also a solution to* (1.3).

Proof. Since the right-hand side of (1.3) is given by the relative distances and velocities, i.e., $x_i - x_j$ and $v_i - v_j$, system (1.3) is clearly invariant under the Galilean transformation. $\qquad\square$

By direct inspection of system (1.3), all equilibrium solution (x_i, v_i) satisfies

$$v_i(t) = 0, \quad x_i(t) = x_i(0), \quad t \geq 0.$$

Thus, the traveling state with nonzero common velocity $v^\infty = c \neq 0$ is not an equilibrium for (1.2). Hence, we need to relax the concept of equilibrium as in the N-body system in celestial mechanics as follows.

Definition 2.1. We say (X^∞, V^∞) is a relative equilibrium for (1.2)-(1.4) if it can be represented by the following relations: there exist constant vectors $\tilde{X} := (\tilde{x}_1, \cdots, \tilde{x}_N)$ and $\tilde{V} = \tilde{v}(1, \ldots, 1)$ in $\mathbb{R}^N$ such that

$$X^\infty = \tilde{X} + t\tilde{V}, \qquad V^\infty = \tilde{V}, \quad t \in \mathbb{R}.$$

For simplicity, we set

$$\mathcal{X}(t) := \max_{1 \leq i,j \leq N} |x_j(t) - x_i(t)|, \quad \mathcal{V}(t) := \max_{1 \leq i,j \leq N} |v_j(t) - v_i(t)|.$$

Then, the concept of mono-cluster flocking (unconditional flocking) can be expressed in terms of $\mathcal{X}$ and $\mathcal{V}$.

Definition 2.2. System (1.3) exhibits a mono-cluster flocking if for any solution (x_i, v_i), the following two conditions hold.

$$\sup_{0 \leq t < \infty} \mathcal{X}(t) < \infty \quad \text{and} \quad \lim_{t \to \infty} \mathcal{V}(t) = 0.$$

2.2. *Spanning tree*

Next, we introduce some basic notions in digraph theory. Note that the network structure is registered by the neighbor set $\mathcal{N}_i$ which consists of all neighbors of the i-th particle, or in other words, $\mathcal{N}_i$ consists of the particles which influence particle i. For convenience, we associate the vertices $\{1, 2, \ldots, N\}$ of the C-S diagraph $\mathcal{G}$ with particles in system (1.3). Then, for a given set of $\{\mathcal{N}_i\}_{i=1}^N$ in system (1.3), we can use an associated digraph to model the interaction topology in the following definition.

Definition 2.3.

(1) The C-S digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated to (1.3) consists of a finite set $\mathcal{V} = \{1, 2, \ldots, N\}$ of vertices, a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of arcs with ordered pair $(j, i) \in \mathcal{E}$ if $j \in \mathcal{N}_i$.

(2) A path in $\mathcal{G}$ from i_1 to i_k is a sequence $i_1, i_2, \ldots, i_k$ such that

$$i_s \in \mathcal{N}_{i_{s+1}} \quad \text{for } 1 \leq s \leq k - 1.$$

If there exists a path from j to i, then vertex i is said to be reachable from vertex j.

(3) The C-S digraph $\mathcal{G}$ contains a *spanning tree* if we can find a vertex such that any other vertex of $\mathcal{G}$ is reachable from it.

Now, we introduce a lemma which provides a necessary condition for unconditional mono-cluster flocking in C-S network.

Lemma 2.2. *If the C-S digraph dose not contain a spanning tree, then mono-cluster flocking will not emerge for some initial configuration.*

We put the proof of Lemma 2.2 in the end of this chapter. Since we are interested in the unconditional mono-cluster flocking which means that mono-cluster flocking occurs for any initial data, we will always assume the existence of a spanning tree structure throughout the paper. Next, we will show a connectivity or transitivity of the spanning tree.

Let $l, k \in \mathbb{N}$ with $1 \leq l \leq k \leq N$, and let $C_{l,k} = (c_l, c_{l+1}, \ldots, c_k)$ be a vector in $\mathbb{R}^{k-l+1}$ such that

$$c_i \geq 0, \quad l \leq i \leq k, \quad \text{and} \quad \sum_{i=l}^{k} c_i = 1.$$

For an ensemble of N-particles with state $\{z_i := (x_i, v_i)\}_{i=1}^{N}$, we set $\mathcal{L}_l^k(C_{l,k})$ to be a convex combination of $\{z_i\}_{i=l}^{k}$ with the coefficient $C_{l,k}$:

$$\mathcal{L}_l^k(C_{l,k}) := \sum_{i=l}^{k} c_i z_i.$$

Note that each z_i is a convex combination of itself, in particular, $z_N = \mathcal{L}_N^N(1)$ and $z_1 = \mathcal{L}_1^1(1)$.

Definition 2.4 (Root and general root).

(1) We say z_k is a *root* if it is not affected by the rest particles; in other words, $j \notin \mathcal{N}_k$ for any $j \in \{1, 2, \ldots, N\} \backslash \{k\}$.

(2) We say $\mathcal{L}_l^k(C_{l,k})$ is a *general root* if $\mathcal{L}_l^k(C_{l,k})$ is not affected by the rest particles; in other words, for any $i \in \{l, l+1, \ldots, k\}$ and $j \in \{1, 2, \ldots, N\} \backslash \{l, l+1, \ldots, k\}$, we have $j \notin \mathcal{N}_i$.

Lemma 2.3. *The following assertions hold.*

> (1) *If the network contains a spanning tree, then there is at most one root.*
> (2) *If $\mathcal{L}_k^N(C_{k,N})$ is a general root, then $\mathcal{L}_1^l(C_{1,l})$ is not a general root for each $l \in \{1, 2, \ldots, k-1\}$.*

Proof. (1) Due to the existence of a spanning tree, it is impossible for two roots exist simultaneously. Otherwise, the two roots cannot affect each other through a directed path, which means each of them cannot be a root.

(2) The similar argument also implies that if $\mathcal{L}_k^N(C_{k,N})$ is a general root, then $\mathcal{L}_1^l(C_{1,l})$ is not a general root for any $1 \leq l \leq k-1$.

$\square$

2.3. *Node decomposition*

In this subsection, we will introduce the concept of *node*. Then we can introduce *node decomposition* to represent the whole graph $\mathcal{G}$ (or say vertex set $\mathcal{V}$) as a disjoint union of a sequence of nodes. The most important point is that the node decomposition shows a hierarchical structure which allows us to apply the induction principle. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{V}_1 \subset \mathcal{V}$, a subgraph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is the digraph with vertex set $\mathcal{V}_1$ and arc set $\mathcal{E}_1$ which consists of the arcs in $\mathcal{G}$ connecting members in $\mathcal{V}_1$. For simplicity, for a given digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ we will identify a subgraph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ with its vertex set $\mathcal{V}_1$. Let's introduce the definition of nodes below.

Definition 2.5 (Node). Let $\mathcal{G}$ be a digraph. A subset $\mathcal{G}_1$ of vertices is called a node if it satisfies the following condition: For any subset $\mathcal{G}_2$ of $\mathcal{G}_1$, $\mathcal{G}_2$ is affected by $\mathcal{G}_1 \backslash \mathcal{G}_2$. Moreover, if $\mathcal{G}_1$ is not affected by $\mathcal{G} \backslash \mathcal{G}_1$, we say $\mathcal{G}_1$ is a maximum node.

Intuitively, a node means that a set of particles can be viewed as a "large" particle. In the following, we will see that the concept of node can simplify the structure of the graph and help us to catch the attraction effect more clearly in a network.

Lemma 2.4. *Any graph $\mathcal{G}$ contains at least one maximum node.*

Proof. We will use induction principle to prove the lemma.
(Step 1) If $\mathcal{G}$ is a maximum node, we are done. If not, there exists a subset $\mathcal{G}_1$ such that $\mathcal{G}_1$ is not affected by $\mathcal{G} \backslash \mathcal{G}_1$.

(Step 2) If $\mathcal{G}_1$ is a node, $\mathcal{G}_1$ is obvious a maximum node. Otherwise, we can find $\mathcal{G}_2$ such that $\mathcal{G}_2 \subsetneqq \mathcal{G}_1$ and $\mathcal{G}_2$ is not affected by $\mathcal{G} \backslash \mathcal{G}_2$.

(Step 3) As $\mathcal{G}$ contains N particles, we can repeat Step 2 at most N times to find a strict decreasing sequence such that

$$\mathcal{G}_k \subsetneqq \mathcal{G}_{k-1} \cdots \subsetneqq \mathcal{G}_1 \subsetneqq \mathcal{G}, \quad 0 \leq k \leq N,$$

where $\mathcal{G}_k$ is a maximum node. This finishes the proof of the lemma $\qquad\square$

Lemma 2.5. *(Node decomposition) Suppose $\mathcal{G}$ to be any graph. Then we can decompose $\mathcal{G}$ to be a union as $\mathcal{G} = \bigcup\limits_{i=0}^{d} \Big(\bigcup\limits_{j=1}^{k_i} \mathcal{G}_i^j \Big)$ such that*

(1) *$\mathcal{G}_0^j$ are the maximum nodes of $\mathcal{G}$, where $1 \leq j \leq k_0$.*
(2) *For any p, q where $1 \leq p \leq d$ and $1 \leq q \leq k_p$, $\mathcal{G}_p^q$ are the maximum nodes of $\mathcal{G} \backslash \Big(\bigcup\limits_{i=0}^{p-1} \Big(\bigcup\limits_{j=1}^{k_i} \mathcal{G}_i^j \Big) \Big)$.*

Proof. This result is very intuitive. According to Lemma 2.4, $\mathcal{G}$ contains at least one maximum node. Therefore we can collect all maximum nodes and label them by $\mathcal{G}_0^j$ for $1 \leq j \leq k_0$, where k_0 is the number of maximum nodes of $\mathcal{G}$. Then we get rid of $\Big(\bigcup\limits_{j=1}^{k_0} \mathcal{G}_0^j \Big)$ and find all maximum nodes of the remain $\mathcal{G} \backslash \Big(\bigcup\limits_{j=1}^{k_0} \mathcal{G}_0^j \Big)$. Denote the maximum nodes of $\mathcal{G} \backslash \Big(\bigcup\limits_{j=1}^{k_0} \mathcal{G}_0^j \Big)$ by $\mathcal{G}_1^j$ for $1 \leq j \leq k_1$, provided there are k_1 maximum nodes. We can repeat this process and construct the maximum nodes $\mathcal{G}_p^q$ of $\mathcal{G} \backslash \Big(\bigcup\limits_{i=0}^{p-1} \Big(\bigcup\limits_{j=1}^{k_i} \mathcal{G}_i^j \Big) \Big)$ for $1 \leq q \leq k_p$. As $\mathcal{G}$ contains finite N particles, after d steps, we will obtain $\mathcal{G} = \bigcup\limits_{i=0}^{d} \Big(\bigcup\limits_{j=1}^{k_i} \mathcal{G}_i^j \Big)$. $\qquad\square$

Remark 2.1. Below, we give comments on important notation and properties to be used later.

(1) By definition of maximum node, we know $\mathcal{G}_p^q$ and $\mathcal{G}_p^{q'}$ do not affect each other for $1 \leq q \neq q' \leq k_p$. In fact, $\mathcal{G}_p^q$ will be only affected by $\mathcal{G}_0$ and $\mathcal{G}_i^j$, where $1 \leq i \leq p-1$. Therefore, without loss of generality, we may assume $k_i = 1$ for all $1 \leq i \leq d$ in the proof of the main theorems in the paper (see Lemma 3.6). Thus the decomposition can be expressed by

$$\mathcal{G} = \bigcup_{i=0}^{d} \mathcal{G}_i,$$

where $\mathcal{G}_p$ is a maximum node of $\mathcal{G} \backslash \bigcup_{i=0}^{p-1} \mathcal{G}_i$.

(2) Given a particle $z_i^{k+1} \in \mathcal{G}_{k+1}$, we denote the set of neighbors of z_i^{k+1} by $\bigcup\limits_{j=0}^{k+1} \mathcal{N}_i^{k+1}(j)$, where $\mathcal{N}_i^{k+1}(j)$ represents the neighbors of z_i^{k+1} in $\mathcal{G}_j$. The node decomposition and spanning tree in $\mathcal{G}$ guarantee that for any $0 \leq k \leq d-1$ there exists at least one particle in $\mathcal{G}_{k+1}$, say z_p^{k+1}, such that $\bigcup\limits_{j=0}^{k} \mathcal{N}_p^{k+1}(j) \neq \emptyset$.

Lemma 2.6. *A graph $\mathcal{G}$ contains a unique maximum node if and only if $\mathcal{G}$ contains a spanning tree.*

Proof. • (If part): Suppose $\mathcal{G}$ contains a spanning tree and two maximum nodes. Then the two nodes are not affected by each other which is a contradiction to the existence of spanning tree.

• (Only if part): Suppose $\mathcal{G}$ contains a unique maximum node $\mathcal{G}_0$. Then according to Lemma 2.5, we know all the maximum nodes $\mathcal{G}_1^j$ of the remain $\mathcal{G}\backslash\mathcal{G}_0$ should be reachable from $\mathcal{G}_0$. Otherwise, there are two maximum nodes of $\mathcal{G}$ which is a contradiction to the uniqueness assumption of the maximum node of $\mathcal{G}$. Then we can repeat this process to show that all the maximum nodes $\mathcal{G}_p^j$ of the remain $\mathcal{G}\backslash\Big(\bigcup\limits_{i=0}^{p-1}\Big(\bigcup\limits_{j=1}^{k_i}\mathcal{G}_i^j\Big)\Big)$ should be reachable from $\Big(\bigcup\limits_{i=0}^{p-1}\Big(\bigcup\limits_{j=1}^{k_i}\mathcal{G}_i^j\Big)\Big)$. In particular, this implies that all $\mathcal{G}_p^q$ are reachable from $\mathcal{G}_0$. Then combining the strong connectivity of each $\mathcal{G}_p^q$, we conclude that all the particles are reachable from any $z_i \in \mathcal{G}_0$, which implies the existence of the spanning tree. □

Now, with the definitions and notations above, we can apply the node decomposition in Lemma 2.5 to prove the Lemma 2.2 rigorously.

Proof of Lemma 2.2. According to Lemma 2.6, if the graph contains no spanning tree, we can find at least two maximum nodes $\mathcal{G}_0^1$ and $\mathcal{G}_0^2$ such that, $\mathcal{G}_0^1$ is not reachable from $\mathcal{G} \setminus \mathcal{G}_0^1$ and $\mathcal{G}_0^2$ is not reachable from $\mathcal{G} \setminus \mathcal{G}_0^2$. Then we set initial configuration as following

$$\begin{cases} x_i(0) = 0, & 1 \leq i \leq N, \\ v_i(0) = -1, & z_i = (x_i, v_i) \in \mathcal{G}_0^1, \\ v_i(0) = 1, & z_i = (x_i, v_i) \in \mathcal{G}_0^2, \\ v_i(0) = 0, & z_i = (x_i, v_i) \in \mathcal{G} \setminus \Big(\mathcal{G}_0^1 \cup \mathcal{G}_0^2\Big). \end{cases} \tag{2.1}$$

Now, as $\mathcal{G}_0^1$ is a maximum node, the dynamics of $z_i = (x_i, v_i)$ will not affected by $\mathcal{G} \setminus \mathcal{G}_0^1$. Moreover, as $z_i \in \mathcal{G}_0^1$ have the same initial velocity, we immediately have

$$(x_i(t), v_i(t)) = (x_i(0) + v_i(0)t, v_i(0)) = (-t, -1), \quad z_i \in \mathcal{G}_0^1.$$

Similarly we have for $z_i \in \mathcal{G}_0^2$ that

$$(x_i(t), v_i(t)) = (x_i(0) + v_i(0)t, v_i(0)) = (t, 1), \quad z_i \in \mathcal{G}_0^2.$$

Therefore, we conclude the non-existence of flocking emergence provided the initial data (2.1) and finish the proof.
□

3. Uniform boundedness of relative positions

In this section, we recall the first-order reformulation of the C-S model (1.3) on a real line, and we combine the node decomposition and induction principle to show that the relative positions are uniformly bounded for a digraph containing a spanning tree.

3.1. *A first-order reformulation*

In this subsection, we discuss the first-order system for position which can be derived from the second-order system (1.3).

Consider the following first-order system on the real line:

$$\begin{cases} \dot{x}_i = \nu_i + \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k - x_i), & t > 0, \\ x_i(0) = x_i^0, & i = 1, 2, \cdots, N, \end{cases} \tag{3.1}$$

where ν_i is the natural velocity of the i-th particle, and Ψ is the coupling function which is an anti-derivative of the communication function ψ:

$$\Psi(y) := \int_0^y \psi(s)ds, \quad y \in \mathbb{R}. \tag{3.2}$$

Then, it is easy to check the equivalence between the second-order system (1.3) and the first-order system (3.1) (see Ref. 23, 24). In fact, we have the following lemma.

Lemma 3.1. *For one-dimensional case, the second-order system* (1.3) *and the first-order system* (3.1) *are equivalent to each other.*

Proof. (i) (From the second-order to the first-order): Note that for one-dimensional case, we have

$$\psi(x_k - x_i)(v_k - v_i) = \frac{d}{dt} \int_0^{x_k - x_i} \psi(y)dy = \frac{d}{dt} \Psi(x_k - x_i). \tag{3.3}$$

Thus, the momentum equation in (1.3) and (3.3) yield

$$\frac{d}{dt} \Big[v_i - \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k - x_i) \Big] = 0.$$

We integrate the above relation to get

$$v_i(t) - \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k(t) - x_i(t)) = v_i^0 - \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k^0 - x_i^0) =: \nu_i(X^0, V^0),$$

or equivalently,

$$v_i(t) = \nu_i(X^0, V^0) + \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k(t) - x_i(t)). \tag{3.4}$$

Finally, we combine $(1.3)_1$ and (3.4) to get the first-order system for x_i:

$$\begin{cases} \dot{x}_i = \nu_i + \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k - x_i), \\ \nu_i = v_i^0 - \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k^0 - x_i^0), \\ x_i(0) = x_i^0, \quad t > 0, \quad i = 1, 2, \cdots, N. \end{cases} \tag{3.5}$$

(ii) (From the first-order to the second-order): We differentiate (3.1) with respect to t, and then obtain the second-order system (1.3) with initial data:

$$(x_i^0, v_i^0) := (x_i^0, \nu_i + \kappa \sum_{k \in \mathcal{N}_i} \Psi(x_k^0 - x_i^0)). \qquad \qquad \square$$

In next lemma, we provide some basic and important properties of the coupling function Ψ in (3.2) to be crucially used in later analysis.

Lemma 3.2. *Suppose that ψ satisfies the condition (1.4) and non-integrable. Then, the corresponding coupling function Ψ is odd, analytic, non-decreasing and unbounded:*

$$-\Psi(-s) = \Psi(s), \qquad \frac{d}{ds}\Psi(s) \geq 0, \qquad \lim_{s \to +\infty} \Psi(s) = +\infty.$$

Moreover, $\Psi(s)$ is concave for $s \geq 0$ and convex for $s \leq 0$.

Proof. The desired estimates are mainly due to the properties of ψ in (1.4). The facts that ψ is even, analytic and non-negativity imply that anti-deriviative Ψ is odd, analytic and non-decreasing. Moreover, $\Psi(s)$ is concave for $s \geq 0$ because ψ is non-increasing if $s \geq 0$. Similarly, we conclude Ψ is convex for $s \leq 0$. Finally, the unboundedness comes from the non-integrability of ψ. $\qquad \square$

3.2. *A priori estimates*

In this subsection we will derive some a priori estimates for the first-order system (3.1) leading to the uniform boundedness of relative positions for the second order system (1.3) in next subsection. For this purpose, we need to introduce an algorithm so that at each time we can derive a differential inequality for the functional $Q(t)$ which is equivalent to $\mathcal{X}(t)$. The algorithm consists of the following three steps:

Step 1: For any given time t, we reorder the particle indexes to make the particle position from minimum to maximum. More precisely, by relabelling the agents at time t, we set

$$x_1(t) \leq x_2(t) \cdots \leq x_N(t). \tag{3.6}$$

To introduce the following steps, we first introduce the processes of iterations for $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ and $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ as follows.

- $(\mathcal{A}_1)$: If $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ is not a general root, then we set

$$\bar{\mathcal{L}}_{k-1}^N(\bar{C}_{k-1,N}) = \frac{\bar{a}_{k-1}\bar{\mathcal{L}}_k^N(\bar{C}_{k,N}) + z_{k-1}}{\bar{a}_{k-1} + 1}.$$

- $(\mathcal{A}_2)$: If $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is not a general root, then we construct

$$\underline{\mathcal{L}}_1^{l+1}(C_{1,l+1}) = \frac{\underline{a}_{l+1}\underline{\mathcal{L}}_1^l(\underline{C}_{1,l}) + z_{l+1}}{\underline{a}_{l+1} + 1}.$$

Step 2: We start from z_N and follow the process $\mathcal{A}_1$ to construct $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ until either $k = 1$ or $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ is a general root at some $k > 1$. If the former happens, we denote $k^* = 1$. If the latter happens, we denote by k^* the first k producing a general root $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ in process $\mathcal{A}_1$.

Step 3: We start from z_1 and follow the process $\mathcal{A}_2$ until either $l = N$ or $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is a general root at some $l < N$. If the former happens, we denote $k_* = N$. If the latter happens, we denote by k_* the first l producing a general root $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ in process $\mathcal{A}_2$.

Remark 3.1.

(1) For convenience, the algorithm with Step 1 to Step 3 will be referred as *Algorithm $\mathcal{A}$*.

(2) At each iteration in $\mathcal{A}_1$, the right-hand side is a convex combination of $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ and z_{k-1}, hence it is a convex combination of subsets of $\{z_i\}_{i=k-1}^N$. Similarly, each $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is a convex combination of $\{z_i\}_{i=1}^l$. Here the coefficients $\bar{a}_{k-1} \geq 0$ and $\underline{a}_{l+1} \geq 0$ will be determined later (see Lemma 3.4).

(3) Note that $k^* \leq k_*$ because of the assumption of existence of spanning tree. In fact, according to Lemma 2.3, as $\bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N})$ is a general root, $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is not a general root for any $l < k^*$. This means that the iteration process $\mathcal{A}_2$ can continue until $l = k^* - 1$. Therefore, by Algorithm $\mathcal{A}$ we finally obtain two convex combination $\bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N})$ and $\underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*})$, which consist of two subgroups of agents with nonempty intersection.

In the next lemmas, we will show a monotone property of Ψ and construct an a priori estimate which will be used to prove the uniform bound of the relative distance of particles in the second order system (1.3), provided $\psi(s)$ is non-integrable.

Lemma 3.3. *Suppose that the network contains a spanning tree, and let (x_i) be a solution to the first-order system (3.1). Moreover we also assume the particles are*

well-ordered at time t as (3.6). *Then at time t, we have*

$$
\begin{cases}
\displaystyle\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i,\\ k\leq i}} \Psi(x_k - x_i) \leq \min_{k\in\cup_m^N \mathcal{N}_i} \Psi(x_k - x_N), & k^* \leq m \leq N,\\[3mm]
\displaystyle\sum_{i=1}^{m} \max_{\substack{k\in\mathcal{N}_i,\\ k\geq i}} \Psi(x_k - x_i) \geq \max_{k\in\cup_1^m \mathcal{N}_i} \Psi(x_k - x_1), & 1 \leq m \leq k_*,
\end{cases}
$$

where $k^*, k_* \in \{1, \ldots, N\}$ *are obtained in Algorithm* $\mathcal{A}$,

Proof. *We will prove the first relation, and the second one can be verified similarly. If* $k^* = N$, *i.e.,* N *is a root, then the desired result holds according to the monotonicity of* Ψ *and the well-ordering in* (x_i). *Next, we focus on the situation that* $k^* \leq N - 1$. *Note that Algorithm* $\mathcal{A}$ *means* $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ *is not a general root for any* k *with* $k^* + 1 \leq k \leq N$. *For given* $m \in [k^*, N]$, *we set*

$$
\bar{k} = \min_{k\in\cup_m^N \mathcal{N}_i} k.
$$

Then for $\bar{k}$ *we have*

$$
\Psi(x_{\bar{k}} - x_N) = \min_{k\in\cup_m^N \mathcal{N}_i} \Psi(x_k - x_N).
$$

Since $\bar{k} \in \cup_m^N \mathcal{N}_i$, *there exists* $m_0 \in [m, N]$ *such that* $\bar{k} \in \mathcal{N}_{m_0}$. *For* m_0, *as* $\bar{\mathcal{L}}_{m_0+1}^N(\bar{C}_{m_0+1,N})$ *is not a general root, there exist* $k_0 \leq m_0$ *and* $m_1 \in [m_0 + 1, N]$ *such that* $k_0 \in \mathcal{N}_{m_1}$. *For* m_1, *as* $\bar{\mathcal{L}}_{m_1+1}^N(\bar{C}_{m_1+1,N})$ *is not a general root, there exist* $k_1 \leq m_1$ *and* $m_2 \in [m_1 + 1, N]$ *such that* $k_1 \in \mathcal{N}_{m_2}$. *This process can be repeated until we find* $m_{j_0} = N$ *and* $k_{j_0-1} \leq m_{j_0-1}$ *such that* $k_{j_0-1} \in \mathcal{N}_N$. *Note that*

$$
\sum_{\substack{i=m}}^{N} \min_{\substack{k\in\mathcal{N}_i\\ k\leq i}} \Psi(x_k - x_i) \leq \Psi(x_{\bar{k}} - x_{m_0}) + \Psi(x_{k_0} - x_{m_1}) + \Psi(x_{k_1} - x_{m_2}) + \cdots + \Psi(x_{k_{j_0-1}} - x_N),
$$

where $k_i \leq m_i$ *for all* $0 \leq i \leq j_0$. *Then we can use the convexity of* Ψ *when* $s \leq 0$ *in Lemma 3.2 to obtain*

$$
\sum_{\substack{i=m}}^{N} \min_{\substack{k\in\mathcal{N}_i\\ k\leq i}} \Psi(x_k - x_i) \leq \Psi(x_{\bar{k}} - x_N).
$$

For the second assertion, we note that Lemma 2.3 tells that $\underline{\mathcal{L}}_1^k(\underline{C}_{1,k})$ *is not a general root for any* $k \in [1, m-1]$. *With the similar argument but applying the concavity of* $\Psi(s)$ *when* $s \geq 0$ *instead of convexity, we can prove*

$$
\sum_{i=1}^{m} \max_{\substack{k\in\mathcal{N}_i,\\ k\geq i}} \Psi(x_k - x_i) \geq \max_{k\in\cup_1^m \mathcal{N}_i} \Psi(x_k - x_1), \quad 1 \leq m \leq k_*.
$$

$\square$

According to Lemma 2.3, for any fixed m, either $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ is not a general root for $m \leq k \leq N$ or $\underline{\mathcal{L}}_1^k(\underline{C}_{1,k})$ is not a general root for $1 \leq k \leq m$. Moreover,

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Lemma 3.3 allows us to estimate the rate of change for the relative distance between $\bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N})$ and $\underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*})$, which is studied in the next lemma.

Lemma 3.4. *Suppose that the network contains a spanning tree, and let (x_i) be the solution to the first-order system* (3.1). *We can design suitable coefficients a_k's and a_l depending only on N, so that if we apply Algorithm $\mathcal{A}$ at each time t and set*

$$\bar{x}_k := \mathrm{Proj}_x \bar{\mathcal{L}}_k^N(\bar{C}_{k,N}), \quad \underline{x}_k := \mathrm{Proj}_x \underline{\mathcal{L}}_1^k(\underline{C}_{1,k}),$$

then we have:

(i) *there exist positive constants $C_1(\nu_i)$ and $C_2(N,\kappa)$ such that at each time t*

$$\frac{d}{dt}(\bar{x}_{k^*} - \underline{x}_{k_*}) \leq C_1 - C_2 \Psi(\mathcal{X}).$$

(ii) *at each time t,*

$$\frac{\mathcal{X}}{4} \leq \bar{x}_{k^*} - \underline{x}_{k_*} \leq \mathcal{X}.$$

Proof. (i) We will design suitable coefficients $\bar{a}_k$'s and $\underline{a}_l$'s inductively, and prove the desired relation in two steps. In the first step, we construct the differential inequality for $\bar{x}_{k^*}$. In the second step, we use the same method to construct similar differential inequality for $\underline{x}_{k_*}$ and finish the proof of (i).

• (Step 1): We apply the process $\mathcal{A}_1$ from x_N to x_{k^*} and construct

$$\bar{\mathcal{L}}_{k-1}^N(\bar{C}_{k-1,N}) \quad \text{with} \quad \bar{a}_N = 0, \quad \bar{a}_{k-1} = (N-k+2)(\bar{a}_k+1), \quad 2 \leq k \leq N. \quad (3.7)$$

By induction, we can derive

$$\bar{a}_{k-1} = \sum_{i=1}^{N-k+1} P(N-k+2, i), \quad 2 \leq k \leq N. \tag{3.8}$$

Then we consider the dynamic of the quantities $\bar{\mathcal{L}}_{k-1}^N(\bar{C}_{k-1,N})$ according to the first-order system. We set $\bar{x}_k := \mathrm{Proj}_x \bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ and have

$$\dot{x}_N = \nu_N + \kappa \sum_{k \in \mathcal{N}_N} \Psi(x_k - x_N) \leq \nu_N + \kappa \min_{k \in \mathcal{N}_N} \Psi(x_k - x_N).$$

The last inequality above comes from the negativity of $\Psi(x_k - x_N)$ due to the well-ordering assumption (3.6). Similarly, according to $\mathcal{A}_1$ and $\bar{a}_{N-1} = 2$ in (3.7), we

simply have for $\bar{\mathcal{L}}_{N-1}^N(\bar{C}_{N-1,N})$ that

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_{N-1} &= \frac{2\dot{x}_N + \dot{x}_{N-1}}{3} \\
&= \frac{2}{3}\kappa \sum_{k\in\mathcal{N}_N} \Psi(x_k - x_N) + \frac{1}{3}\kappa \sum_{k\in\mathcal{N}_{N-1}} \Psi(x_k - x_{N-1}) + \frac{2}{3}\nu_N + \frac{1}{3}\nu_{N-1} \\
&\leq \frac{2}{3}\kappa \min_{k\in\mathcal{N}_N} \Psi(x_k - x_N) + \frac{1}{3}\kappa\Big(\Psi(x_N - x_{N-1}) + \min_{\substack{k\in\mathcal{N}_{N-1} \\ k\leq N-1}} \Psi(x_k - x_{N-1})\Big) \\
&\quad + \frac{2}{3}\nu_N + \frac{1}{3}\nu_{N-1} \\
&\leq \frac{2}{3}\nu_N + \frac{1}{3}\nu_{N-1} + \frac{\kappa}{3}\Big(\min_{k\in\mathcal{N}_N} \Psi(x_k - x_N) + \min_{\substack{k\in\mathcal{N}_{N-1} \\ k\leq N-1}} \Psi(x_k - x_{N-1})\Big).
\end{aligned}
\tag{3.9}
$$

Next we use method of induction to deal with $\bar{\mathcal{L}}_m^N(\bar{C}_{m,N})$. In fact, suppose for $\bar{\mathcal{L}}_m^N(\bar{C}_{m,N})$, where $k^* + 1 \leq m \leq N-1$, we have

$$
\frac{d}{dt}\bar{x}_m \leq \frac{\nu_m}{\bar{a}_m + 1} + \sum_{i=m+1}^{N} \frac{\prod_{l=m}^{i-1} \bar{a}_l \nu_i}{\prod_{l=m}^{i}(\bar{a}_l + 1)} + \frac{\kappa}{\bar{a}_m + 1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k - x_i)\Big). \tag{3.10}
$$

Then (3.9) already shows (3.10) holds for $m = N-1$. Now by Algorithm $\mathcal{A}_1$ we obtain

$$
\frac{d}{dt}\bar{x}_{m-1} = \frac{\bar{a}_{m-1}\dot{\bar{x}}_m + \dot{x}_{m-1}}{\bar{a}_{m-1} + 1} = \frac{\bar{a}_{m-1}\dot{\bar{x}}_m}{\bar{a}_{m-1} + 1} + \frac{\dot{x}_{m-1}}{\bar{a}_{m-1} + 1} = \mathcal{I}_{11} + \mathcal{I}_{12}. \tag{3.11}
$$

$\diamond$ For the term $\mathcal{I}_{11}$, we can apply (3.7) and (3.10) to obtain

$$
\begin{aligned}
\mathcal{I}_{11} &\leq \frac{\bar{a}_{m-1}}{\bar{a}_{m-1} + 1}\Big(\frac{\nu_m}{\bar{a}_m + 1} + \sum_{i=m+1}^{N} \frac{\nu_i \prod_{l=m}^{i-1} \bar{a}_l}{\prod_{l=m}^{i}(\bar{a}_l + 1)}\Big) \\
&\quad + \frac{\bar{a}_{m-1}}{\bar{a}_{m-1} + 1}\Big(\frac{\kappa}{\bar{a}_m + 1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k - x_i)\Big)\Big) \\
&= \sum_{i=m}^{N} \frac{\nu_i \prod_{l=m-1}^{i-1} \bar{a}_l}{\prod_{l=m-1}^{i}(\bar{a}_l + 1)} + \frac{(N-m+2)\kappa}{\bar{a}_{m-1} + 1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k - x_i)\Big).
\end{aligned}
\tag{3.12}
$$

$\diamond$ For the term $\mathcal{I}_{12}$, we can use the equation (3.5) to have

$$
\mathcal{I}_{12} = \frac{1}{\bar{a}_{m-1} + 1}\Big(\nu_{m-1} + \kappa \sum_{k\in\mathcal{N}_{m-1}} \Psi(x_k - x_{m-1})\Big). \tag{3.13}
$$

Now, we combine (3.11), (3.12) and (3.13) to get

$$
\frac{d}{dt}\bar{x}_{m-1} \leq \underbrace{\frac{\nu_{m-1}}{\bar{a}_{m-1}+1} + \sum_{i=m}^{N} \frac{\nu_i \prod_{l=m-1}^{i-1} \bar{a}_l}{\prod_{l=m-1}^{i}(\bar{a}_l+1)}}_{\mathcal{I}_{21}}
$$
$$
+ \underbrace{\frac{(N-m+2)\kappa}{\bar{a}_{m-1}+1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k-x_i)\Big) + \frac{\kappa}{\bar{a}_{m-1}+1}\Big(\sum_{k\in\mathcal{N}_{m-1}} \Psi(x_k-x_{m-1})\Big)}_{\mathcal{I}_{22}}.
$$

$$(3.14)$$

$\diamond$ For the term $\mathcal{I}_{21}$, it already satifies the induction rule. Thus, we only need to focus on $\mathcal{I}_{22}$. Due to the well-ordering assumption (3.6), we have

$$
\mathcal{I}_{22} = \frac{\kappa}{\bar{a}_{m-1}+1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k-x_i) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k\leq m-1}} \Psi(x_k-x_{m-1})\Big)
$$
$$
+ \frac{\kappa}{\bar{a}_{m-1}+1}\Big((N-m+1)\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i, \\ k\leq i}} \Psi(x_k-x_i) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k>m-1}} \Psi(x_k-x_{m-1})\Big)
$$

$$(3.15)$$

According to Lemma 3.3 and the fact that $\bar{\mathcal{L}}_m^N(C_{m,N})$ is not a general root for all $k^*+1 \leq m \leq N$, we have

$$
\frac{\kappa}{\bar{a}_{m-1}+1}\Big((N-m+1)\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k-x_i) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k>m-1}} \Psi(x_k-x_{m-1})\Big)
$$
$$
\leq \frac{\kappa}{\bar{a}_{m-1}+1}\Big((N-m+1)\min_{k\in\cup_m^N\mathcal{N}_i} \Psi(x_k-x_N) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k>m-1}} \Psi(x_k-x_{m-1})\Big)
$$
$$
\leq \frac{\kappa}{\bar{a}_{m-1}+1}\Big((N-m+1)\Psi(x_{m-1}-x_N) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k>m-1}} \Psi(x_k-x_{m-1})\Big)
$$
$$
\leq 0.
$$

$$(3.16)$$

Therefore, we combine (3.14), (3.15) and (3.16) to get

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_{m-1} &\leq \frac{\nu_{m-1}}{\bar{a}_{m-1}+1} + \sum_{i=m}^{N} \frac{\nu_i \prod_{l=m-1}^{i-1}\bar{a}_l}{\prod_{l=m-1}^{i}(\bar{a}_l+1)} \\
&\quad + \frac{\kappa}{\bar{a}_{m-1}+1}\Big(\sum_{i=m}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k-x_i) + \sum_{\substack{k\in\mathcal{N}_{m-1} \\ k\leq m-1}} \Psi(x_k-x_{m-1}) \Big) \\
&\leq \frac{\nu_{m-1}}{\bar{a}_{m-1}+1} + \sum_{i=m}^{N} \frac{\nu_i \prod_{l=m-1}^{i-1}\bar{a}_l}{\prod_{l=m-1}^{i}(\bar{a}_l+1)} + \frac{\kappa}{\bar{a}_{m-1}+1}\Big(\sum_{i=m-1}^{N} \min_{\substack{k\in\mathcal{N}_i, \\ k\leq i}} \Psi(x_k-x_i) \Big).
\end{aligned}
$$

By induction hypothesis and Lemma 3.3, we can apply the connectivity to get

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_{k^*} &\leq \frac{\nu_{k^*}}{\bar{a}_{k^*}+1} + \sum_{i=k^*+1}^{N} \frac{\nu_i \prod_{l=k^*}^{i-1}\bar{a}_l}{\prod_{l=k^*}^{i}(\bar{a}_l+1)} + \frac{\kappa}{\bar{a}_{k^*}+1}\Big(\sum_{i=k^*}^{N} \min_{\substack{k\in\mathcal{N}_i \\ k\leq i}} \Psi(x_k-x_i) \Big) \\
&\leq \frac{\nu_{k^*}}{\bar{a}_{k^*}+1} + \sum_{i=k^*+1}^{N} \frac{\nu_i \prod_{l=k^*}^{i-1}\bar{a}_l}{\prod_{l=k^*}^{i}(\bar{a}_l+1)} + \frac{\kappa}{\bar{a}_{k^*}+1}\Big(\min_{k\in\cup_{k^*}^{N}\mathcal{N}_i} \Psi(x_k-x_N) \Big) \\
&= \frac{\nu_{k^*}}{\bar{a}_{k^*}+1} + \sum_{i=k^*+1}^{N} \frac{\nu_i \prod_{l=k^*}^{i-1}\bar{a}_l}{\prod_{l=k^*}^{i}(\bar{a}_l+1)} + \frac{\kappa}{\bar{a}_{k^*}+1}\Psi(x_{k^*}-x_N).
\end{aligned}
$$
(3.17)

Here the last equality is due to the case that $\bar{\mathcal{L}}_{k^*}^{N}(\bar{C}_{k^*,N})$ is a general root and $\bar{\mathcal{L}}_{k}^{N}(\bar{C}_{k,N})$ are not general root for all $k^* < k \leq N$.

• (Step 2): In order to estimate $\underline{\dot{x}}_{k_*}$, we recall Step 3 in Algorithm $\mathcal{A}$, and construct

$$
\underline{\mathcal{L}}_1^{k+1}(\underline{C}_{1,k+1}) \quad \text{with} \quad \underline{a}_1 = 0, \quad \underline{a}_{k+1} = (k+1)(\underline{a}_k+1), \quad 1 \leq k \leq N-1.
$$

Therefore, we have

$$
\underline{a}_{k+1} = \sum_{i=1}^{k} P(k+1,i), \quad 1 \leq k \leq N-1. \tag{3.18}
$$

According to Lemma 2.3, the process $\mathcal{A}_2$ can continue until k_*, thus we set

$$
\underline{x}_{k_*} := \mathrm{Proj}_x \underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*})
$$

and apply a similar method as before to derive

$$
\begin{aligned}
\frac{d}{dt}\underline{x}_{k_*} &\geq \frac{\nu_{k_*}}{\underline{a}_{k_*}+1} + \sum_{i=1}^{k_*-1} \frac{\nu_i \prod_{l=i+1}^{k_*}\underline{a}_l}{\prod_{l=i}^{k_*}(\underline{a}_l+1)} + \frac{\kappa}{\underline{a}_{k_*}+1}\Big(\sum_{i=1}^{k_*} \max_{\substack{k\in\mathcal{N}_i \\ k\geq i}} \Psi(x_k-x_i) \Big) \\
&\geq \frac{\nu_{k_*}}{\underline{a}_{k_*}+1} + \sum_{i=1}^{k_*-1} \frac{\nu_i \prod_{l=i+1}^{k_*}\underline{a}_l}{\prod_{l=i}^{k_*}(\underline{a}_l+1)} + \frac{\kappa}{\underline{a}_{k_*}+1}\Psi(x_{k^*}-x_1).
\end{aligned}
$$
(3.19)

Here, for the second inequality we used the fact that $\underline{\mathcal{L}}_1^k(\underline{C}_{1,k})$ is not a general root for any $k \in [1, k_* - 1]$. We combine (3.17) and (3.19) to get

$$
\begin{aligned}
\frac{d}{dt}&(\bar{x}_{k^*} - \underline{x}_{k_*}) \\
&\leq \frac{\nu_{k^*}}{\bar{a}_{k^*} + 1} + \sum_{i=k^*+1}^{N} \frac{\nu_i \prod_{l=k^*}^{i-1} \bar{a}_l}{\prod_{l=k^*}^{i}(\bar{a}_l + 1)} - \frac{\nu_{k_*}}{\underline{a}_{k_*} + 1} - \sum_{i=1}^{k_*-1} \frac{\nu_i \prod_{l=i+1}^{k_*} \underline{a}_l}{\prod_{l=i}^{k_*}(\underline{a}_l + 1)} \\
&\quad + \frac{\kappa}{\bar{a}_{k^*} + 1}\Psi(x_{k^*} - x_N) - \frac{\kappa}{\underline{a}_{k_*} + 1}\Psi(x_{k^*} - x_1) \\
&\leq \frac{\nu_{k^*}}{\bar{a}_{k^*} + 1} - \frac{\nu_{k_*}}{\underline{a}_{k_*} + 1} + \sum_{i=k^*+1}^{N} \frac{\nu_i \prod_{l=k^*}^{i-1} \bar{a}_l}{\prod_{l=k^*}^{i}(\bar{a}_l + 1)} - \sum_{i=1}^{k_*-1} \frac{\nu_i \prod_{l=i+1}^{k_*} \underline{a}_l}{\prod_{l=i}^{k_*}(\underline{a}_l + 1)} \\
&\quad + \min\left\{\frac{\kappa}{\bar{a}_{k^*} + 1}, \frac{\kappa}{\underline{a}_{k_*} + 1}\right\}\Psi(x_1 - x_N).
\end{aligned} \tag{3.20}
$$

The last inequality is due to the fact $k^* \leq k_*$ in Remark 3.1. Here, the constants $\bar{a}_{k^*}$ and $\underline{a}_{k_*}$ are given as in (3.8) and (3.18), i.e.,

$$
\bar{a}_{k^*} = \sum_{i=1}^{N-k^*} P(N - k^* + 1, i) \quad \text{and} \quad \underline{a}_{k_*} = \sum_{i=1}^{k_*-1} P(k_*, i).
$$

We can verify that the terms in (3.20) involving ν_i is the difference of a convex combination of $\nu_{k^*}, \nu_{k^*+1}, \ldots, \nu_N$ and a convex combination of $\nu_1, \nu_2, \ldots, \nu_{k_*}$ Therefore, it follows from (3.20) that

$$
\frac{d}{dt}(\bar{x}_{k^*} - \underline{x}_{k_*}) \leq |\nu_M - \nu_m| - \frac{\kappa}{\sum_{i=1}^{N-1} P(N, i) + 1}\Psi(\mathcal{X}).
$$

This finishes the proof of (i).

(ii) From the convex combination structure of $\bar{x}_{k^*}(t)$ and $\underline{x}_{k_*}(t)$, we immediately have

$$
\bar{x}_{k^*}(t) - \underline{x}_{k_*}(t) \leq \mathcal{X}(t).
$$

We now prove the left part of the desired relation. In fact, as we assume the order of position, we can continue (3.7) to construct $\bar{\mathcal{L}}_1^N(\bar{C}_{1,N})$. Similarly, we construct $\underline{\mathcal{L}}_1^N(\underline{C}_{1,N})$. Clearly, one has

$$
\text{Proj}_x \bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N}) \geq \text{Proj}_x \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}), \qquad \text{Proj}_x \underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*}) \leq \text{Proj}_x \underline{\mathcal{L}}_1^N(\underline{C}_{1,N}).
$$

Therefore we have

$$
\begin{aligned}
\bar{x}_{k^*}(t) &- \underline{x}_{k_*}(t) \\
&= \text{Proj}_x \bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N}) - \text{Proj}_x \underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*}) \geq \text{Proj}_x \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}) - \text{Proj}_x \underline{\mathcal{L}}_1^N(\underline{C}_{1,N}).
\end{aligned} \tag{3.21}
$$

According to (3.8) and (3.18), we have

$$
\bar{a}_{N-i} = \underline{a}_{1+i}.
$$

Therefore, we immediately obtain a symmetric property:

$$\frac{\prod_{l=i+1}^{N} \underline{a}_l}{\prod_{l=i}^{N} (\underline{a}_l + 1)} = \frac{\prod_{l=1}^{N-i} \bar{a}_l}{\prod_{l=1}^{N-i+1} (\bar{a}_l + 1)}.$$

With above relation, we can apply (3.7) to have

$$\text{Proj}_x \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}) - \text{Proj}_x \underline{\mathcal{L}}_1^N(\underline{C}_{1,N})$$

$$= \frac{x_1}{\bar{a}_1 + 1} + \sum_{i=2}^{N} \frac{x_i \prod_{l=1}^{i-1} \bar{a}_l}{\prod_{l=1}^{i} (\bar{a}_l + 1)} - \frac{x_N}{\underline{a}_N + 1} - \sum_{i=1}^{N-1} \frac{x_i \prod_{l=i+1}^{N} \underline{a}_l}{\prod_{l=i}^{N} (\underline{a}_l + 1)}$$

$$= \frac{x_1}{\bar{a}_1 + 1} + \sum_{i=2}^{N} \frac{x_i \prod_{l=1}^{i-1} \bar{a}_l}{\prod_{l=1}^{i} (\bar{a}_l + 1)} - \frac{x_N}{\bar{a}_1 + 1} - \sum_{i=1}^{N-1} \frac{x_i \prod_{l=1}^{N-i} \bar{a}_l}{\prod_{l=1}^{N-i+1} (\bar{a}_l + 1)} \qquad (3.22)$$

$$= \frac{1}{\bar{a}_1 + 1} \sum_{i=1}^{N} x_i \Big(\frac{N!}{(N-i+1)!} - \frac{N!}{i!} \Big).$$

With the symmetric structure, we can rewrite (3.22) as

$$\text{Proj}_x \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}) - \text{Proj}_x \underline{\mathcal{L}}_1^N(\underline{C}_{1,N})$$

$$= \frac{N! - 1}{\bar{a}_1 + 1}(x_N - x_1) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor - 1} \frac{(x_{N-k} - x_{k+1})}{\bar{a}_1 + 1} \Big(\frac{N!}{(k+1)!} - \frac{N!}{(N-k)!} \Big) \qquad (3.23)$$

$$\geq \frac{N! - 1}{\bar{a}_1 + 1}(x_N - x_1).$$

According to (3.8), we have

$$\bar{a}_1 = \sum_{i=1}^{N-1} P(N, i).$$

We combine the above formula of $\bar{a}_1$ and (3.23) to find

$$\text{Proj}_x \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}) - \text{Proj}_x \underline{\mathcal{L}}_1^N(\underline{C}_{1,N})$$

$$\geq \frac{N!}{2(\sum_{i=1}^{N-1} P(N,i) + 1)}(x_N - x_1) = \frac{\mathcal{X}}{2(\sum_{i=1}^{N} \frac{1}{i!})} \geq \frac{\mathcal{X}}{2(\sum_{i=0}^{N-1} \frac{1}{2^i})} \geq \frac{\mathcal{X}}{4}. \qquad (3.24)$$

Finally, we combine (3.21) and (3.24) to finish the proof of (ii). $\square$

3.3.   *Uniform boundedness of relative positions*

In next lemma, we show that the particles in a maximum node $\mathcal{G}_0$ will aggregate and thus the maximum node $\mathcal{G}_0$ can be really viewed as one particle.

Lemma 3.5. *Suppose that the graph $\mathcal{G}$ contains a spanning tree, and let (x_i) be a solution to the system (3.1). Moreover, we denote the unique maximum node by $\mathcal{G}_0$. If Ψ is unbounded, then the diameter of spatial variable $\mathcal{X}_0(t)$ of $\mathcal{G}_0$ is uniformly bounded i.e. there exists a positive constant M_0 such that*

$$\sup_{0 \leq t < \infty} \max_{i,j \in \mathcal{G}_0} |x_j(t) - x_i(t)| \leq 4M_0 < \infty.$$

Proof. According to Lemma 2.4, the maximum node $\mathcal{G}_0$ is not affected by $\mathcal{G}\backslash\mathcal{G}_0$. Therefore, we can get rid of the other particles and we only consider the subgraph $\mathcal{G}_0$. For simplicity, we say that $\mathcal{G}_0$ consists of N_0 particles labeled $\{1, 2, \ldots, N_0\}$. Moreover, according to Lemma 3.2, Ψ is analytic and thus the system (3.1) has a unique analytical solution. Therefore, for any two particles, they either collide finite times or always stay together in any finite time period $[0, T]$. Thus we fix the time period $[0, T]$ and consider these two cases separately.

• (Case 1): No pair of particles stay together through $[0, T]$. In this situation, only finite many collisions happen in $[0, T]$. Therefore, we can construct the set

$$\mathcal{T} = \{t \mid x_i(t) = x_j(t), \quad \text{for some } i \neq j\},$$

which has finite cardinality. Without loss of generality, we can arrange $\mathcal{T} \cup \{0, T\}$ as

$$0 = t_1 < t_2 \cdots < t_M = T.$$

In the following, we will prove the uniform boundedness of $\mathcal{X}_0(t)$ in three steps. In the first two steps, we will construct a quantity $Q^0(t)$ and show the equivalence between $Q^0(t)$ and $\mathcal{X}_0(t)$. Then in the third step we will show that $Q^0(t)$ is uniformly bounded which implies the uniform boundedness of $\mathcal{X}_0(t)$.

$\diamond$ (Step 1): It is obvious that, in each interval (t_i, t_{i+1}) where $1 \leq i \leq M - 1$, no particles collide and the order of x_i's are fixed. This enables us to apply algorithm $\mathcal{A}$ to the maximum node $\mathcal{G}_0$. In fact, we can construct $\bar{\mathcal{L}}_{k^*}^{N_0}(\bar{C}_{k^*, N_0})$ and $\underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1, k_*})$ in the same way for all $t \in J_i := [t_i, t_{i+1})$. Therefore, following Lemma 3.4, we can construct $\bar{x}_{k^*}$ and $\underline{x}_{k_*}$ so that their form does not depend on $t \in J_i$. Then we denote

$$Q_i^0(t) := \bar{x}_{k^*}(t) - \underline{x}_{k_*}(t), \quad t \in J_i.$$

By Lemma 3.4, we have

$$\frac{d}{dt} Q_i^0(t) \leq |\nu_M - \nu_m| - \frac{\kappa}{\sum_{i=1}^{N_0 - 1} P(N_0, i) + 1} \Psi(\mathcal{X}_0(t)), \quad t \in J_i.$$

$\diamond$ (Step 2): As $\mathcal{G}_0$ is a node, we immediately have $k^* = 1$ and $k_* = N_0$ for any J_i. In other words, for $i \neq j$, the terms Q_i^0 and Q_j^0 are convex combinations of all the members in $\mathcal{G}_0$. Therefore, we have

$$\lim_{t \to t_{i+1}} Q_i^0(t) = Q_{i+1}^0(t_{i+1}), \quad 1 \leq i \leq M.$$

Then, if we define $Q^0(t) = Q_i^0(t)$ where $t \in J_i$, we can conclude that $Q^0(t)$ is a well defined Lipschitz function and it has the following properties

$$\begin{cases} \dfrac{\mathcal{X}_0(t)}{4} \leq Q^0(t) \leq \mathcal{X}_0(t), \quad t \in J_i, \\ \dfrac{d}{dt} Q^0(t) \leq |\nu_M - \nu_m| - \dfrac{\kappa}{\sum_{i=1}^{N_0 - 1} P(N_0, i) + 1} \Psi(\mathcal{X}_0(t)) \quad a.e. \ t \in [0, T]. \end{cases} \tag{3.25}$$

$\diamond$ (Step 3): In this part, we prove the uniform boundedness of $\mathcal{X}_0$. As Ψ is unbounded and monotone increasing, we can find a positive constant M_0 such that

$$M_0 := \max\left\{ Q^0(0), \; \max\left\{ s \;\Big|\; \Psi(s) \leq \frac{|\nu_M - \nu_m|\left(\sum_{i=1}^{N_0-1} P(N_0, i) + 1\right)}{\kappa} \right\} \right\}. \tag{3.26}$$

Next we claim

$$Q^0(t) \leq M_0 \quad \text{for all } t \in [0, T].$$

Proof of the claim: Suppose not, then we have some $\bar{t} \in [0, T]$ such that

$$Q^0(\bar{t}) > M_0.$$

Consider the set:

$$\mathcal{M}_0 := \{t \mid t < \bar{t}, \; Q^0(t) \leq M_0\}.$$

It is obvious that $0 \in \mathcal{M}_0$, therefore $\mathcal{M}_0$ is not empty and we denote $t^* := \sup \mathcal{M}_0$. Moreover, it is easy to show that $t^* < \bar{t}$. According to the construction of M_0 in (3.26), we have

$$Q^0(t^*) = M_0 \quad \text{and} \quad |\nu_M - \nu_m| - \frac{\kappa}{\sum_{i=1}^{N_0-1} P(N_0, i) + 1} \Psi(Q^0(t)) \leq 0, \quad t^* \leq t \leq \bar{t}. \tag{3.27}$$

Now we apply $(3.25)_2$ to have

$$\begin{aligned} 0 < Q^0(\bar{t}) - M_0 &= Q^0(\bar{t}) - Q^0(t^*) \\ &\leq \int_{t^*}^{\bar{t}} |\nu_M - \nu_m| - \frac{\kappa}{\sum_{i=1}^{N_0-1} P(N_0, i) + 1} \Psi(Q^0(t)) dt \leq 0, \end{aligned} \tag{3.28}$$

which is a contradictory. Therefore, we finish the proof of claim and show that

$$Q^0(t) \leq M_0 \quad \text{for all } t \in [0, T].$$

Finally, it follows from $(3.25)_1$ that

$$\mathcal{X}_0(t) \leq 4Q^0(t) \leq 4M_0, \quad t \in [0, T].$$

Since T is arbitrary chosen, and M_0 in (3.26) is independent of T, we conclude

$$\mathcal{X}_0(t) \leq 4M_0, \quad t \geq 0.$$

$\bullet$ (Case 2): If there are some x_i and x_j stay together in all period $[0, T]$, then we can view them as one particle and thus the total number of particles that we need to study is reduced. Therefore, for this even simpler situation, we can apply previous method in the same way to obtain

$$\mathcal{X}_0(t) \leq 4M_0, \quad t \geq 0. \qquad \qquad \square$$

In next lemma, we will further show the uniform boundedness of the spatial diameter of the whole graph $\mathcal{G}$. However, as $\mathcal{G}_k$'s are not nodes in $\mathcal{G}$ for $k \geq 1$, we have to reproduce $\bar{a}_i^k$ and $\underline{a}_j^k$ to construct the coefficients of convex combination.

Lemma 3.6. *Suppose that the network $\mathcal{G}$ contains a spanning tree, and let (x_i) be a solution to the system* (3.1). *If Ψ is unbounded, then the diameter of spatial variable $\mathcal{X}(t)$ is uniformly bounded: there exists a positive constant $\bar{M}$ such that*

$$\sup_{0 \leq t < \infty} \max_{1 \leq i,j \leq N} |x_j(t) - x_i(t)| \leq \bar{M} < \infty.$$

Proof. According to Remark 2.1, we apply node decomposition to represent $\mathcal{G}$ as

$$\mathcal{G} = \bigcup_{k=0}^{d} \mathcal{G}_k, \quad |\mathcal{G}_k| = N_k.$$

Now we define a sequence of quantities $Q^k(t)$ as following. First, we denote the particles in $\mathcal{G}_k$ by $z_i^k = (x_i^k, v_i^k)$ with $1 \leq i \leq N_k$. Then we can assume x_i^k are well ordered as below

$$x_1^k \leq x_2^k \cdots \leq x_{N_k}^k.$$

According to Lemma 2.5 and Remark 2.1, $\mathcal{G}_k$ is the maximum node in $\mathcal{G} \backslash \bigcup_{i=0}^{k-1} \mathcal{G}_i$. Therefore, we can define $\bar{\mathcal{L}}_1^{N_k}(\bar{C}_{1,N_k})$ and $\underline{\mathcal{L}}_1^{N_k}(\underline{C}_{1,N_k})$ for each $\mathcal{G}_k$ similar to Lemma 3.4. In fact, we can set $\bar{a}_i^k$ and $\underline{a}_i^k$ as below, where $1 \leq i \leq N_k$:

$$\begin{cases} \bar{a}_{N_k}^k = 0, \quad \bar{a}_{i-1}^k = (N_k - i + 2 + g_{k-1})(\bar{a}_i^k + 1), \quad g_{k-1} := \sum_{j=1}^{k-1} N_j, \quad 2 \leq i \leq N_k, \\ \underline{a}_1^k = 0, \quad \underline{a}_{i+1}^k = (i + 1 + g_{k-1})(\underline{a}_i^k + 1), \quad 1 \leq i \leq N_k - 1. \end{cases} \tag{3.29}$$

By induction argument, we can derive

$$\begin{cases} \bar{a}_{i-1}^k = \sum_{j=1}^{N_k-i+1} P(N_k - i + 2 + g_{k-1}, j + g_{k-1}), \quad 2 \leq i \leq N_k, \\ \underline{a}_{i+1}^k = \sum_{j=1}^{i} P(i + 1 + g_{k-1}, j + g_{k-1}), \quad 1 \leq i \leq N_k - 1. \end{cases} \tag{3.30}$$

Next, we set

$$\bar{x}_k := \mathrm{Proj}_x \bar{\mathcal{L}}_1^{N_k}(\bar{C}_{1,N_k}) \quad \text{and} \quad \underline{x}_k := \mathrm{Proj}_x \underline{\mathcal{L}}_1^{N_k}(\underline{C}_{1,N_k}).$$

Then we can define

$$Q^k(t) := \max_{0 \leq i \leq k} \{\bar{x}_i\} - \min_{0 \leq i \leq k} \{\underline{x}_i\}. \tag{3.31}$$

It is obvious that $Q^k(t)$ is Lipschitz continuous and Lemma 3.5 provides positive constants C_1^0, C_2^0 and C_3^0 such that

$$\frac{d}{dt}Q^0(t) \leq C_1^0 - C_2^0\Psi(Q^0), \quad \mathcal{X}^0(t) \leq C_3^0.$$

Note that the construction of $\bar{x}_k$ and $\underline{x}_k$ do not involve the particles in $\mathcal{G}_i$ for $i \neq k$. However, for $k \geq 1$, we cannot directly apply the previous method that we used for Q^0. This is because the particles in $\mathcal{G}_i$ with $i < k$ perform as a source and thus we cannot miss the information from $\mathcal{G}_i$ with $i < k$. Therefore, we will study Q^k which contains all informations of $\mathcal{G}_i$ with $i \leq k$. In the following, we will construct the induction process in two steps to finish the proof.

• (Step 1): In this step, we prove that for $0 \leq k \leq d-1$, if there exist positive constants C_1^k, C_2^k and C_4^k such that

$$\frac{d}{dt}Q^k(t) \leq C_1^k - C_2^k\Psi(Q^k), \quad \mathcal{D}_k \leq C_4^k,$$

where $\mathcal{D}_k := \left(\max\limits_{0 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\} - \min\limits_{0 \leq i \leq k} \min\limits_{1 \leq j \leq N_i} \{x_j^i\} \right)$. Then, we can find positive constants C_1^{k+1} and C_2^{k+1} such that

$$\frac{d}{dt}Q^{k+1}(t) \leq C_1^{k+1} - C_2^{k+1}\Psi(Q^{k+1}). \tag{3.32}$$

Please see Section Appendix A.1 for details.

• (Step 2): In this step, we prove that if

$$\frac{d}{dt}Q^{k+1}(t) \leq C_1^{k+1} - C_2^{k+1}\Psi(Q^{k+1}), \quad 0 \leq k \leq d-1,$$

then there exist positive constants C_3^{k+1} and C_4^{k+1} such that

$$\mathcal{X}^{k+1} \leq C_3^{k+1}, \quad \mathcal{D}_{k+1} \leq C_4^{k+1}.$$

Please see Section A.2 for details.

• (Step 3): Now, we are ready to finish the proof of the lemma. According to Lemma 3.5, we have

$$\frac{d}{dt}Q^0(t) \leq C_1^0 - C_2^0\Psi(Q^0) \quad \text{and} \quad \mathcal{D}_0 = \mathcal{X}^0 \leq C_3^0 = C_4^0.$$

Then, with the analysis in (Step 1) and (Step 2), we can apply induction principle to conclude that there exists a positive constant $\bar{M}$ such that

$$\sup_{0 \leq t < \infty} \max_{1 \leq i,j \leq N} |x_j(t) - x_i(t)| = \sup_{0 \leq t < \infty} \mathcal{D}_d \leq C_4^d = \bar{M} < \infty. \qquad \square$$

According to the equivalence between the first-order system (3.1) and second-order system (1.3) from Lemma 3.1, we can apply Lemma 3.6 to obtain the corresponding result for second order system (1.3) as follows.

Corollary 3.1. *Suppose that (x_i, v_i) is a solution to the system (1.3) and the network contains a spanning tree. If ψ is non-integrable and satisfies (1.4), then the diameter of spatial variable $\mathcal{X}(t)$ is uniform bounded: there exists a positive constant M_1 such that*

$$\sup_{0 \leq t < \infty} \max_{1 \leq i,j \leq N} |x_j(t) - x_i(t)| \leq M_1 < \infty.$$

Proof. For any fixed initial data (x_i^0, v_i^0), we apply Lemma 3.1 to construct the equivalent first-order system as (3.5). Then we apply Lemma 3.6 to obtain the uniform bound of the diameter of relative distance. □

4. Unconditional flocking

In this section, we show the unconditional flocking for the second-order system (1.3) with long ranged interactions. The method is quite similar to the previous section. In fact, we will construct a reduction algorithm for velocity, and then construct a quantity which is exponentially decaying. Finally we use this quantity to control the velocity diameter and show the unconditional flocking emergence.

4.1. *Algorithm and a priori estimate*

In this subsection, we will first introduce algorithm $\mathcal{B}$ and then construct a basic a priori estimate for velocity diameter which is very important for flocking of $\mathcal{G}_0$. The following algorithm consisting of Step 1-Step 3, will be referred as Algorithm $\mathcal{B}$.

Step 1: For any given time t, we reorder particle velocities from minimum to maximum. More precisely, by relabelling the agents at time t, we let

$$v_1(t) \leq v_2(t) \cdots \leq v_N(t).$$

To introduce the following steps, we first introduce the processes of iterations for $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ and $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ as follows.

$(\mathcal{B}_1)$: If $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ is not a general root, then we set

$$\bar{\mathcal{L}}_{k-1}^N(\bar{C}_{k-1,N}) = \frac{\bar{b}_{k-1}\bar{\mathcal{L}}_k^N(\bar{C}_{k,N}) + z_{k-1}}{\bar{b}_{k-1} + 1}.$$

$(\mathcal{B}_2)$: If $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is not a general root, then we construct

$$\underline{\mathcal{L}}_1^{l+1}(\underline{C}_{1,l+1}) = \frac{\underline{b}_{l+1}\underline{\mathcal{L}}_1^l(\underline{C}_{1,l}) + z_{l+1}}{\underline{b}_{l+1} + 1}.$$

Step 2: We start from z_N and follow the process $\mathcal{B}_1$ to construct $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ until either $k = 1$ or $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ is a general root at some $k > 1$. If the former happens, then we denote $k^* = 1$. If the later happens, we denote by k^* the first k producing a general root $\bar{\mathcal{L}}_k^N(\bar{C}_{k,N})$ in the process $\mathcal{B}_1$.

Step 3: We start from z_1 and follow the process $\mathcal{B}_2$ until either $l = N$ or $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ is a general root at some $l < N$. If the former happens, then we denote $k_* = N$. If the later happens, we denote by k_* the first l producing a general root $\underline{\mathcal{L}}_1^l(\underline{C}_{1,l})$ in process $\mathcal{B}_2$.

Algorithm $\mathcal{B}$ is very similar to Algorithm $\mathcal{A}$ except that $\mathcal{B}$ is designed according to the velocities. By Algorithm $\mathcal{B}$ with suitable coefficients $\{\bar{b}_k\}$ and $\{\underline{b}_l\}$, we will finally obtain two convex combination $\bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N})$ and $\underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*})$. Now we denote

$$\bar{v}_k := \mathrm{Proj}_v \bar{\mathcal{L}}_k^N(\bar{C}_{k,N}), \quad \underline{v}_k := \mathrm{Proj}_v \underline{\mathcal{L}}_1^k(\underline{C}_{1,k}).$$

As in Section 3, the quantity $\bar{v}_{k^*} - \underline{v}_{k_*}$ is non-negative but may not be a continuous function due to the possible change of order of velocity. But we can use the analyticity of v_i to imply that both $\bar{v}_{k^*}$ and $\underline{v}_{k_*}$ are piecewise smooth in any time interval $[0, T]$. Therefore, the time derivative of $\bar{v}_{k^*}$ and $\underline{v}_{k_*}$ are well defined a.e. $t \in [0, T]$ except for finite many t_k. The next lemma provides an a priori estimate for $\bar{v}_{k^*} - \underline{v}_{k_*}$ in any interval such that both $\bar{v}_{k^*}$ and $\underline{v}_{k_*}$ are smooth.

Lemma 4.1. *(Hypo-coercivity) Suppose that there is a spanning tree in the network topology for system* (1.3) *and ψ is non-integrable which satisfies* (1.4)*. Then, on any interval J which preserves the order of $\{v_i\}$, we can design suitable coefficients $\bar{b}_k$'s and $\underline{b}_l$'s depending only on N such that*

$$\frac{d}{dt}\left(\bar{v}_{k^*} - \underline{v}_{k_*}\right) \leq -B\psi_m \mathcal{V}(t), \qquad \frac{2\psi_0 - \psi_m}{2\psi_0}\mathcal{V}(t) \leq \left(\bar{v}_{k^*} - \underline{v}_{k_*}\right) \leq \mathcal{V}(t).$$

Here B is a positive constant, $\psi_0 := \psi(0)$ and $\psi_m := \psi(M_1)$ with M_1 being obtained in Lemma 3.6 or Corollary 3.1. Moreover, the constant B depends only on N and the initial data.

Proof. In order to construct proper $\bar{v}_{k^*}$ and $\underline{v}_{k_*}$, we first introduce the inductive process to produce $\bar{b}_k$ and $\underline{b}_l$:

$$\begin{cases} \bar{b}_N = 0, \quad \bar{b}_{k-1} = \dfrac{\psi_0}{\psi_m}(N - k + 2)(\bar{b}_k + 1), \quad 2 \leq k \leq N, \\[2mm] \underline{b}_1 = 0, \quad \underline{b}_{l+1} = \dfrac{\psi_0}{\psi_m}(l + 1)(\underline{b}_l + 1), \quad 1 \leq l \leq N - 1. \end{cases} \tag{4.1}$$

By direct calculation, we have

$$
\begin{cases}
\bar{b}_{k-1} = \displaystyle\sum_{i=1}^{N-k+1} \left(\frac{\psi_0}{\psi_m}\right)^{N-k+2-i} P(N-k+2, i), & 2 \le k \le N, \\[3mm]
\underline{b}_{l+1} = \displaystyle\sum_{i=1}^{l} \left(\frac{\psi_0}{\psi_m}\right)^{l+1-i} P(l+1, i), & 1 \le l \le N-1.
\end{cases}
\tag{4.2}
$$

Now we are going to prove lemma based on the setting (4.1) and (4.2). If $N = 2$, it is clear and thus we assume $N \ge 3$. Moreover, according to Lemma 3.6 and Corollary 3.1, the relative positions among particles are uniformly bounded by M_1. Therefore we have

$$
\psi_m \le \psi(x_i(t) - x_j(t)) \le \psi_0, \quad \forall\, 1 \le i, j \le N, \quad \forall\, t \ge 0.
$$

Similar to Lemma 3.4, we can split the proof in two steps.

• (Step 1): In this step, we will show the equivalence between $\mathcal{V}$ and the quantity $\left(\bar{v}_{k^*} - \underline{v}_{k_*}\right)$ and finish the proof of the first part:

$$
\frac{2\psi_0 - \psi_m}{2\psi_0} \mathcal{V}(t) \le \left(\bar{v}_{k^*} - \underline{v}_{k_*}\right) \le \mathcal{V}(t).
\tag{4.3}
$$

In fact, by the property of convex combination, we immediately have

$$
\left(\bar{v}_{k^*} - \underline{v}_{k_*}\right) \le \mathcal{V}(t).
$$

Next, with the same argument as in Lemma 3.4 (ii), we apply the construction (4.1) and (4.2) to have

$$
\begin{aligned}
\bar{v}_{k^*}&(t) - \underline{v}_{k_*}(t) \\
&= \mathrm{Proj}_v \bar{\mathcal{L}}_{k^*}^N(\bar{C}_{k^*,N}) - \mathrm{Proj}_v \underline{\mathcal{L}}_1^{k_*}(\underline{C}_{1,k_*}) \ge \mathrm{Proj}_v \bar{\mathcal{L}}_1^N(\bar{C}_{1,N}) - \mathrm{Proj}_v \underline{\mathcal{L}}_1^N(\underline{C}_{1,N}) \\
&\ge \Big(\frac{\prod_{l=1}^{N-1} \bar{b}_l}{\prod_{l=1}^{N}(\bar{b}_l + 1)} - \frac{1}{\bar{b}_1 + 1}\Big)(v_N - v_1) = \Big(\frac{\left(\frac{\psi_0}{\psi_m}\right)^{N-1} N! - 1}{\bar{b}_1 + 1}\Big)(v_N - v_1).
\end{aligned}
\tag{4.4}
$$

The last equality is due to the relation:

$$
\bar{b}_{k-1} = \frac{\psi_0}{\psi_m}(N - k + 2)(\bar{b}_k + 1).
$$

Now we apply (4.2) and (4.4) to obtain

$$
\begin{aligned}
\bar{v}_{k^*}(t) - \underline{v}_{k_*}(t) &\ge \Big(\frac{\left(\frac{\psi_0}{\psi_m}\right)^{N-1} N!}{2(\bar{b}_1 + 1)}\Big)(v_N - v_1) = \frac{\mathcal{V}(t)}{2 \sum_{i=1}^{N-1} \left(\frac{\psi_m}{\psi_0}\right)^{i-1} \frac{1}{i!}} \\
&\ge \frac{\mathcal{V}(t)}{2 \sum_{i=0}^{N-2} \left(\frac{\psi_m}{2\psi_0}\right)^{i}} \ge \frac{2\psi_0 - \psi_m}{4\psi_0} \mathcal{V}(t).
\end{aligned}
\tag{4.5}
$$

• (Step 2): In this step, we will study on the interval where both $\bar{v}_{k^*}$ and $\underline{v}_{k_*}$ are smooth. Then we will construct the differential inequality for the quantity $(\bar{v}_{k^*} - \underline{v}_{k_*})$:

$$\frac{d}{dt}\Big(\bar{v}_{k^*} - \underline{v}_{k_*}\Big) \leq -B\mathcal{V}(t).$$

We will use similar method as in Lemma 3.4 to construct the inequality by induction process. In fact, we claim that

$$\frac{d}{dt}\bar{v}_l \leq \frac{\kappa\psi_m}{\bar{b}_l + 1}\Big(\min_{k\in\cup_{i=l}^N\mathcal{N}_i}\{v_k\} - v_N\Big), \quad k^* \leq l \leq N. \tag{4.6}$$

We will prove the claim by induction. If $l = N$, then we immediately have

$$\frac{d}{dt}v_N(t) = \kappa\sum_{i\in\mathcal{N}_N}\psi(x_i - x_N)(v_i - v_N) \leq \kappa\psi_m\Big(\min_{k\in\mathcal{N}_N}\{v_k\} - v_N\Big).$$

Therefore, we assume that (4.6) holds for $\bar{v}_N, \ldots, \bar{v}_l$ and consider the term $\bar{v}_{l-1}$, for $k^* + 1 \leq l \leq N$. Similar to (3.11), we have

$$\frac{d}{dt}\bar{v}_{l-1} = \frac{\bar{b}_{l-1}\dot{\bar{v}}_l + \dot{v}_{l-1}}{\bar{b}_{l-1} + 1} = \frac{\bar{b}_{l-1}\dot{\bar{v}}_l}{\bar{b}_{l-1} + 1} + \frac{\dot{v}_{l-1}}{\bar{b}_{l-1} + 1} = \mathcal{I}_{31} + \mathcal{I}_{32}. \tag{4.7}$$

Now for the term $\mathcal{I}_{31}$, we can apply (4.1) and (4.6) to obtain

$$\mathcal{I}_{31} \leq \frac{\bar{b}_{l-1}}{\bar{b}_{l-1} + 1}\Big(\frac{\kappa}{\bar{b}_l + 1}\psi_m\Big(\min_{k\in\cup_{i=l}^N\mathcal{N}_i}\{v_k\} - v_N\Big)\Big) = \frac{(N - l + 2)\kappa\psi_0}{\bar{b}_{l-1} + 1}\Big(\min_{k\in\cup_{i=l}^N\mathcal{N}_i}\{v_k\} - v_N\Big). \tag{4.8}$$

For the term $\mathcal{I}_{32}$, we can use the equation (1.3) to have

$$\begin{aligned}
\mathcal{I}_{32} &= \frac{\kappa}{\bar{b}_{l-1} + 1}\sum_{k\in\mathcal{N}_{l-1}}\psi(x_k - x_{l-1})(v_k - v_{l-1}) \\
&\leq \frac{\kappa}{\bar{b}_{l-1} + 1}\Big(\sum_{\substack{k\in\mathcal{N}_{l-1}\\k\leq l-1}}\psi_m(v_k - v_{l-1}) + \sum_{\substack{k\in\mathcal{N}_{l-1}\\k\geq l-1}}\psi_0(v_k - v_{l-1})\Big) \\
&\leq \frac{\kappa}{\bar{b}_{l-1} + 1}\Big(\psi_m\Big[\min_{\substack{k\in\mathcal{N}_{l-1}\\k\leq l-1}}\{v_k\} - v_{l-1}\Big] + (N - l + 1)\psi_0(v_N - v_{l-1})\Big).
\end{aligned} \tag{4.9}$$

We combine (4.7), (4.8) and (4.9) to obtain

$$\begin{aligned}
\frac{d}{dt}\bar{v}_{l-1} &\leq \frac{(N - l + 1)\kappa\psi_0}{\bar{b}_{l-1} + 1}\Big(\min_{k\in\cup_{i=l}^N\mathcal{N}_i}\{v_k\} - v_N + v_N - v_{l-1}\Big) \\
&\quad + \frac{\kappa\psi_0}{\bar{b}_{l-1} + 1}\Big(\min_{k\in\cup_{i=l}^N\mathcal{N}_i}\{v_k\} - v_N\Big) + \frac{\kappa\psi_m}{\bar{b}_{l-1} + 1}\Big(\min_{\substack{k\in\mathcal{N}_{l-1}\\k\leq l-1}}\{v_k\} - v_{l-1}\Big) \quad (4.10) \\
&= \mathcal{I}_{41} + \mathcal{I}_{42}.
\end{aligned}$$

For the term $\mathcal{I}_{41}$, note that $\mathcal{L}_l^N(C_{l,N})$ is not a general root for each $k^* + 1 \leq l \leq N$. Therefore, we have

$$\min_{k \in \cup_{i=l}^N \mathcal{N}_i} \{v_k\} \leq v_{l-1}, \quad \text{i.e.,} \quad \mathcal{I}_{41} \leq 0. \tag{4.11}$$

With the same argument, we have

$$\begin{aligned}
\mathcal{I}_{42} &= \frac{\kappa \psi_0}{\overline{b}_{l-1} + 1} \Big(\min_{k \in \cup_{i=l}^N \mathcal{N}_i} \{v_k\} - v_N \Big) + \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\min_{\substack{k \in \mathcal{N}_{l-1} \\ k \leq l-1}} \{v_k\} - v_{l-1} \Big) \\
&\leq \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\min_{k \in \cup_{i=l}^N \mathcal{N}_i} \{v_k\} - v_N + \min_{\substack{k \in \mathcal{N}_{l-1} \\ k \leq l-1}} \{v_k\} - v_{l-1} \Big) \\
&= \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\min \Big\{ \min_{k \in \cup_{i=l}^N \mathcal{N}_i} \{v_k\}, \min_{\substack{k \in \mathcal{N}_{l-1} \\ k \leq l-1}} \{v_k\} \Big\} - v_N \Big) \\
&\quad + \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\max \Big\{ \min_{k \in \cup_{i=l}^N \mathcal{N}_i} \{v_k\}, \min_{\substack{k \in \mathcal{N}_{l-1} \\ k \leq l-1}} \{v_k\} \Big\} - v_{l-1} \Big) \\
&\leq \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\min_{k \in \cup_{i=l-1}^N \mathcal{N}_i} \{v_k\} - v_N \Big).
\end{aligned} \tag{4.12}$$

Finally, we combine (4.10), (4.11) and (4.12) to get

$$\frac{d}{dt} \overline{v}_{l-1} \leq \frac{\kappa \psi_m}{\overline{b}_{l-1} + 1} \Big(\min_{k \in \cup_{i=l-1}^N \mathcal{N}_i} \{v_k\} - v_N \Big).$$

This completes the proof of the claim (4.6). Therefore, we set $l = k^* + 1$, and we immediately obtain

$$\frac{d}{dt} \overline{v}_{k^*} \leq \frac{\kappa \psi_m}{\overline{b}_{k^*} + 1} \Big(\min_{k \in \cup_{i=k^*}^N \mathcal{N}_i} \{v_k\} - v_N \Big) = \frac{\kappa \psi_m}{\overline{b}_{k^*} + 1} (v_{k^*} - v_N). \tag{4.13}$$

Similarly, we can derive the differential inequality for $\underline{v}_{k_*}$ and obtain that

$$\frac{d}{dt} \underline{v}_{k_*} \geq \frac{\kappa \psi_m}{\underline{b}_{k_*} + 1} (v_{k_*} - v_1). \tag{4.14}$$

Now, due to the existence of spanning tree, we have $k_* \geq k^*$ and thus $v_{k_*} \geq v_{k^*}$. Therefore we combine (4.13), (4.14) to obtain the following estimates:

$$\frac{d}{dt} (\overline{v}_{k^*} - \underline{v}_{k_*}) \leq - \min \Big\{ \frac{\kappa}{\overline{b}_{k^*} + 1}, \frac{\kappa}{\underline{b}_{k_*} + 1} \Big\} \psi_m \mathcal{V}(t).$$

Now, according to the rules (4.1) and (4.2), we find that

$$\overline{b}_{k^*} = \sum_{i=1}^{N-k^*} \Big(\frac{\psi_0}{\psi_m} \Big)^{N-k^*+1-i} P(N-k^*+1, i), \qquad \underline{b}_{k_*} = \sum_{i=1}^{k_*-1} \Big(\frac{\psi_0}{\psi_m} \Big)^{k_*-i} P(k_*, i),$$

both of which are not greater than $\sum_{i=1}^{N-1} \Big(\frac{\psi_0}{\psi_m} \Big)^{N-i} P(N, i)$. Therefore, we combine the analysis in Step 1 and Step 2 to obtain

$$\frac{2\psi_0 - \psi_m}{4\psi_0} \mathcal{V}(t) \leq \Big(\overline{v}_{k^*} - \underline{v}_{k_*} \Big) \leq \mathcal{V}(t), \qquad \frac{d}{dt} (\overline{v}_{k^*} - \underline{v}_{k_*}) \leq -B \psi_m \mathcal{V}(t),$$

where B is a positive constant defined by

$$B = \frac{\kappa}{\sum_{i=1}^{N-1} \left(\frac{\psi_0}{\psi_m}\right)^{N-i} P(N,i) + 1}.$$ □

4.2. *Unconditional flocking*

In this subsection, we prove the emergence of unconditional flocking (Theorem 1.1). According to Lemma 2.5 and Remark 2.1, we have the node decomposition $\mathcal{G} = \bigcup_{i=0}^{d} \mathcal{G}_i$, where $\mathcal{G}_p$ is a maximum node of $\mathcal{G} \setminus \bigcup_{i=0}^{p-1} \mathcal{G}_i$. We denote particles in $\mathcal{G}_i$ by z_j^i where $1 \leq j \leq N_i$ and denote the velocity diameter of $\mathcal{G}_i$ by $\mathcal{V}^i$. Then, we will use induction principle based on the node decomposition to prove Theorem 1.1. The following lemma performs as the initial step of the induction.

Lemma 4.2. *Under the condition in Theorem 1.1, unconditional flocking emerges in the maximum node $\mathcal{G}_0$. More precisely, we can find positive constants C_6^0 and C_7^0 such that*

$$\mathcal{V}^0(t) \leq C_6^0 e^{-C_7^0 t} \mathcal{V}^0(0).$$

Proof. Consider a finite time interval $[0,T]$. We follow the argument in Lemma 3.5 to construct a time sequence $0 = t_1 < t_2 < \cdots < t_M = T$ such that the order of velocity v_i is preserved in each $J_i = [t_i, t_{i+1}]$. We assume that $\mathcal{G}_0$ consists of N_0 particles which can be labeled as $\{1, 2, \ldots, N_0\}$. Then we can apply the same method in Lemma 3.5 to define

$$Y^0(t) := (\bar{v}_{k^*}(t) - \underline{v}_{k_*}(t)), \quad t \in J_i.$$

With a similar argument as in Lemma 3.5, since $\mathcal{G}_0$ is a maximum node, we conclude

$$k_* = N_0 \quad \text{and} \quad k^* = 1.$$

Therefore, $Y^0(t)$ is Lipschitz continuous in $[0,T]$, and we follow Lemma 4.1 to obtain

$$\frac{d}{dt} Y^0(t) \leq -B\psi_m \mathcal{V}^0(t), \quad \frac{2\psi_0 - \psi_m}{4\psi_0} \mathcal{V}^0(t) \leq Y^0(t) \leq \mathcal{V}^0(t), \quad a.e. \ 0 \leq t \leq T. \tag{4.15}$$

Here $\mathcal{V}^0 := \max\{|v_i^0 - v_j^0|\}$. We combine the two inequalities in (4.14) to find

$$\frac{d}{dt} Y^0(t) \leq -B\psi_m Y^0(t), \quad a.e. \ 0 \leq t \leq T.$$

Therefore, we have the decay estimate of the quantity $Y(t)$:

$$Y^0(t) \leq e^{-B\psi_m t} Y^0(0), \quad 0 \leq t \leq T.$$

As T was arbitrary chosen and all the constants are independent of T in above formula, we immediately obtain the exponential decay of $Y^0(t)$:

$$Y^0(t) \leq e^{-B\psi_m t} Y^0(0), \quad t \geq 0.$$

Then, we apply the inequality $Y^0(t) \leq \mathcal{V}^0(t) \leq \frac{4\psi_0}{2\psi_0 - \psi_m} Y^0(t)$ to obtain that

$$\mathcal{V}^0(t) \leq \frac{4\psi_0}{2\psi_0 - \psi_m} e^{-B\psi_m t} \mathcal{V}^0(0), \quad t \geq 0.$$

We set

$$C_6^0 = \frac{4\psi_0}{2\psi_0 - \psi_m} \quad \text{and} \quad C_7^0 = -B\psi_m$$

to finish the proof. □

Now we are ready to prove Theorem 1.1 by induction principle. The proof is very similar to Lemma 3.6 and thus we may omit some details.

Proof of Theorem 1.1: According to Remark 2.1, we apply node decomposition to represent $\mathcal{G}$ as

$$\mathcal{G} = \bigcup_{k=0}^{d} \mathcal{G}_k, \quad |\mathcal{G}_k| = N_k.$$

Now we define a sequence of quantities $Q^k(t)$ as follows. First we assume v_i^k are well ordered as below

$$v_1^k \leq v_2^k \cdots \leq v_{N_k}^k.$$

According to Lemma 2.5 and Remark 2.1, $\mathcal{G}_k$ is the maximum node in $\mathcal{G} \backslash \bigcup_{i=0}^{k-1} \mathcal{G}_i$. Therefore, we can define $\bar{\mathcal{L}}_1^{N_k}(\bar{C}_{1,N_k})$ and $\underline{\mathcal{L}}_1^{N_k}(\underline{C}_{1,N_k})$ for each $\mathcal{G}_k$ similar to Lemma 3.7 and Lemma 3.9. In fact, we can set $\bar{b}_i^k$ and $\underline{b}_i^k$ as below, where $1 \leq i \leq N_k$:

$$\begin{cases} \bar{b}_{N_k}^k = 0, \quad \bar{b}_{i-1}^k = \frac{\psi_0}{\psi_m}(N_k - i + 2 + g_{k-1})(\bar{b}_i^k + 1), \quad g_{k-1} := \sum_{j=1}^{k-1} N_j, \quad 2 \leq i \leq N_k, \\ \\ \underline{b}_1^k = 0, \quad \underline{b}_{i+1}^k = \frac{\psi_0}{\psi_m}(i + 1 + g_{k-1})(\underline{b}_i^k + 1), \quad 1 \leq i \leq N_k - 1. \end{cases}$$

By induction, we can derive that

$$\begin{cases} \bar{b}_{i-1}^k = \sum_{j=1}^{N_k - i + 1} \left(\frac{\psi_0}{\psi_m}\right)^{N_k - i + 2 - j} P(N_k - i + 2 + g_{k-1}, j + g_{k-1}), \quad 2 \leq i \leq N_k, \\ \\ \underline{b}_{i+1}^k = \sum_{j=1}^{i} \left(\frac{\psi_0}{\psi_m}\right)^{i+1-j} P(i + 1 + g_{k-1}, j + g_{k-1}), \quad 1 \leq i \leq N_k - 1. \end{cases}$$

Next, we set

$$\bar{v}_k := \text{Proj}_v \bar{\mathcal{L}}_1^{N_k}(\bar{C}_{1,N_k}) \quad \text{and} \quad \underline{v}_k := \text{Proj}_v \underline{\mathcal{L}}_1^{N_k}(\underline{C}_{1,N_k}),$$

and we can define

$$Y^k(t) := \max_{0 \leq i \leq k} \{\bar{v}_i\} - \min_{0 \leq i \leq k} \{\underline{v}_i\}. \tag{4.16}$$

Then, it is clear to see that $Y^k(t)$ is Lipschitz continuous.

• (Step 1): In this step, we prove that, for $0 \leq k \leq d-1$, if there exist positive constants C_5^k, C_6^k, C_7^k and C_8^k such that

$$\frac{d}{dt}Y^k(t) \leq -C_5^k Y^k + C_8^k \mathcal{R}_{k-1}(t), \quad \mathcal{R}_k(t) \leq C_6^k e^{-C_7^k t} \mathcal{R}_k(0),$$

where $\mathcal{R}_{-1} := 0$ and $\mathcal{R}_k := \left(\max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i} \{v_j^i\} - \min_{0 \leq i \leq k} \min_{1 \leq j \leq N_i} \{v_j^i\} \right)$, then we can find positive constants C_5^{k+1} and C_8^{k+1} such that

$$\frac{d}{dt}Y^{k+1}(t) \leq -C_5^{k+1} Y^{k+1} + C_8^{k+1} \mathcal{R}_k(t). \tag{4.17}$$

Please see Section B.1 for details.

• (Step 2): In this step, we prove that if

$$\frac{d}{dt}Y^{k+1}(t) \leq -C_5^{k+1} Y^{k+1} + C_8^{k+1} \mathcal{R}_k(t), \quad \mathcal{R}_k(t) \leq C_6^k e^{-C_7^k t} \mathcal{R}_k(0), \quad 0 \leq k \leq d-1, \tag{4.18}$$

then there exist positive constants C_6^{k+1} and C_7^{k+1} such that

$$\mathcal{R}_{k+1}(t) \leq C_6^{k+1} e^{-C_7^{k+1} t} \mathcal{R}_{k+1}(0).$$

Please see Section B.2 for details.

• (Step 3): Now, we are ready to finish the proof of lemma. According to Lemma 4.2, we have

$$\frac{d}{dt}Y^0(t) \leq -C_5^0 Y^0 \qquad \text{and} \qquad \mathcal{R}_0(t) = \mathcal{V}^0(t) \leq C_6^0 e^{-C_7^0 t} \mathcal{V}^0(0) = C_6^0 e^{-C_7^0 t} \mathcal{R}_0(0).$$

Then, with the analysis in (Step 1) and (Step 2), we can apply induction principle to conclude that there exist positive constants B and C such that

$$\mathcal{V}(t) = \mathcal{R}_d(t) \leq C_6^d e^{-C_7^d t} \mathcal{R}_d(0) = C e^{-Bt} \mathcal{V}(0), \quad t \geq 0.$$

□

Remark 4.1. In Ref. 18, the author derived an estimate depending on the depth of the graph. In our result, all the information of the depth is contained in the exponent. In fact, the exponential decay rate is far from optimal, we can obtain similar decay rate depending on depth as in Ref. 18.

5. Conclusion

In this paper, we have studied the critical exponent of the Cucker-Smale model on a line with an algebraically decaying communication weight, and obtained the sufficient and necessary condition for unconditional flocking (mono-cluster flocking).

For this, we introduced a node decomposition argument to construct new nonlinear functionals which are equivalent to diameters. In fact, these quantities can be bounded by the diameter and, on the other hand, bound the half diameter. Thus we can yield the decay rate of the diameter by the estimates of the more relaxed quantities. This idea is reminiscent of the Harnack inequality in Laplace's equation, and the result is highly independent of graph structures. In fact, we only need to assume that the network topology contains a spanning tree. There are several issues that we did not cover in this paper. For example, our methodology relies on the first-order reduction for position variable which is valid for one-dimensional setting at present, although a node decomposition still works for multi-dimensional setting. Thus, it would be interesting whether our result on the critical exponent for unconditional flocking can be extended for the multi-dimensional setting or not.

Appendix A. Proof of Lemma 3.6

In this section, we provide all the details of Step 1 and Step 2 in the proof of Lemma 3.6, respectively.

A.1. *Proof of Step 1*

Since $\bar{x}_i$ and $\underline{x}_i$ are analytic for all i, $\bar{x}_i$ and $\bar{x}_j$ will always stay together or just collide for finite times in $[0, T]$. As the discussion in Lemma 3.5, without loss of generality, we can assume $\bar{x}_i$ and $\bar{x}_j$ do not always stay together for $[0, T]$. Therefore, the order of $\bar{x}_i$ will only exchange finite times in $[0, T]$, so does $\underline{x}_i$. So we can set

$$[0, T] := \bigcup_{i=1}^{q} J_i,$$

where $J_i = [t_{i-1}, t_i]$ and $0 = t_0 < t_1 \cdots < t_q = T$, such that the order of both $\{\bar{x}_j\}_{j=0}^{k+1}$ and $\{\underline{x}_j\}_{j=0}^{k+1}$ are preserved in each J_i. Now we pick out any J_p, where $1 \leq p \leq q$, and consider four cases depending on the relative position between $\bigcup_{i=0}^{k} \mathcal{G}_i$ and $\mathcal{G}_{k+1}$.

- (Case 1): Consider the case

$$\max_{0 \leq i \leq k+1} \{\bar{x}_i\} = \max_{0 \leq i \leq k} \{\bar{x}_i\}, \quad \min_{0 \leq i \leq k+1} \{\underline{x}_i\} = \min_{0 \leq i \leq k} \{\underline{x}_i\} \quad \text{on } J_p.$$

In this case, we immediately have $Q^{k+1}(t) = Q^k(t)$ and

$$\frac{d}{dt} Q^{k+1}(t) = \frac{d}{dt} Q^k(t) \leq C_1^k - C_2^k \Psi(Q^k) = C_1^k - C_2^k \Psi(Q^{k+1}), \quad t_{p-1} < t < t_p.$$

Therefore, we simply let $C_1^{k+1} = C_1^k$ and $C_2^{k+1} = C_2^k$ to obtain (3.32).

- (Case 2): Consider the case

$$\max_{0 \leq i \leq k+1} \{\bar{x}_i\} = \bar{x}_{k+1}, \quad \min_{0 \leq i \leq k+1} \{\underline{x}_i\} = \underline{x}_{k+1} \quad \text{on } J_p.$$

In this case, we assume

$$x_1^{k+1} \leq x_2^{k+1} \leq \cdots \leq x_{N_{k+1}}^{k+1}, \quad \bar{x}_j^{k+1} := \operatorname{Proj}_x \bar{\mathcal{L}}_j^{N_{k+1}}(\bar{C}_{j, N_{k+1}}), \quad \underline{x}_j^{k+1} := \operatorname{Proj}_x \underline{\mathcal{L}}_1^j(\underline{C}_{1,j}).$$

It is obvious that

$$\bar{x}_1^{k+1} = \bar{x}_{k+1} \quad \text{and} \quad \underline{x}_{N_{k+1}}^{k+1} = \underline{x}_{k+1}.$$

Similar to the formula (3.10), we claim that: for $1 \leq m \leq N_{k+1}$ we have

$$
\begin{aligned}
\frac{d}{dt} \bar{x}_m^{k+1} &\leq \frac{\nu_m^{k+1}}{\bar{a}_m^{k+1} + 1} + \sum_{i=m+1}^{N_{k+1}} \frac{\prod_{l=m}^{i-1} \bar{a}_l^{k+1} \nu_i^{k+1}}{\prod_{l=m}^{i} (\bar{a}_l^{k+1} + 1)} + \kappa g_k \Psi(\mathcal{D}_k) \\
&\quad + \frac{\kappa}{\bar{a}_m^{k+1} + 1} \Big(\sum_{i=m}^{N_{k+1}} \min_{\substack{j \in \mathcal{N}_i^{k+1}(k+1) \\ j \leq i}} \Psi(x_j^{k+1} - x_i^{k+1}) \Big).
\end{aligned}
\tag{A.1}
$$

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Proof of claim (A.1): We will prove the claim by induction.

$\diamond$ (Step 1): As an initial step, we verify that (A.1) holds for $m = N_{k+1}$. In fact, we have the differential equation of $\bar{x}^{k+1}_{N_{k+1}}$ as below:

$$
\begin{aligned}
\frac{d}{dt}\bar{x}^{k+1}_{N_{k+1}} &= \frac{d}{dt}x^{k+1}_{N_{k+1}} \\
&= \nu^{k+1}_{N_{k+1}} + \kappa \sum_{j\in\mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \Psi(x^{k+1}_j - x^{k+1}_{N_{k+1}}) + \kappa \sum_{l=0}^{k} \sum_{j\in\mathcal{N}^{k+1}_{N_{k+1}}(l)} \Psi(x^l_j - x^{k+1}_{N_{k+1}}).
\end{aligned}
$$
(A.2)

Since $x^{k+1}_{N_{k+1}}$ is assumed to be the largest one among the particles in the $(k+1)$th node, we have

$$
\Psi(x^{k+1}_j - x^{k+1}_{N_{k+1}}) \le 0 \quad \text{for all } j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1).
$$

Therefore, we obtain from Lemma 3.3 that

$$
\sum_{j\in\mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \Psi(x^{k+1}_j - x^{k+1}_{N_{k+1}}) \le \min_{j\in\mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \Psi(x^{k+1}_j - x^{k+1}_{N_{k+1}}).
$$
(A.3)

On the other hand, as $\max_{0\le i\le k+1}\{\bar{x}_i\} = \bar{x}_{k+1}$, we immediately have

$$
x^{k+1}_{N_{k+1}} \ge \min_{0\le i\le k} \min_{1\le j\le N_i}\{x^i_j\}.
$$

Otherwise, the convex combination $\bar{x}_{k+1}$ will be strictly smaller than all $\bar{x}_i$ for $0 \le i \le k$. Therefore, we have

$$
\Psi(x^l_j - x^{k+1}_{N_{k+1}}) \le \Psi(x^l_j - \min_{0\le i\le k}\min_{1\le j\le N_i}\{x^i_j\}) \le \Psi(\mathcal{D}_k).
$$
(A.4)

We combine (A.2), (A.3) and (A.4) to get

$$
\frac{d}{dt}\bar{x}^{k+1}_{N_{k+1}} \le \nu^{k+1}_{N_{k+1}} + \kappa \min_{j\in\mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \Psi(x^{k+1}_j - x^{k+1}_{N_{k+1}}) + \kappa g_k \Psi(\mathcal{D}_k).
$$

Therefore, (A.1) holds for $\bar{x}^{k+1}_{N_{k+1}}$.

$\diamond$ (Step 2): Now we suppose that (A.1) holds for m with $2 \le m \le N_{k+1}$. Next, we verify (A.1) for $m-1$. The calculation is almost the same as Lemma 3.4. In fact,

we use the same argument in Lemma 3.4 to derive

$$
\begin{aligned}
\frac{d}{dt}&\Big(\frac{\bar{a}_{m-1}^{k+1}\bar{x}_m^{k+1}}{\bar{a}_{m-1}^{k+1}+1}+\frac{x_{m-1}^{k+1}}{\bar{a}_{m-1}^{k+1}+1}\Big) \\
&\leq \frac{\nu_{m-1}^{k+1}}{\bar{a}_{m-1}^{k+1}+1}+\sum_{i=m}^{N_{k+1}}\frac{\prod_{l=m-1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=m-1}^{i}(\bar{a}_l^{k+1}+1)}+\frac{\bar{a}_{m-1}^{k+1}}{\bar{a}_{m-1}^{k+1}+1}\kappa g_k\Psi(\mathcal{D}_k) \\
&\quad +\frac{\kappa(N_{k+1}-m+2+g_k)}{\bar{a}_{m-1}^{k+1}+1}\Big(\sum_{i=m}^{N_{k+1}}\min_{\substack{j\in\mathcal{N}_i^{k+1}(k+1)\\ j\leq i}}\Psi(x_j^{k+1}-x_i^{k+1})\Big) \\
&\quad +\frac{\kappa}{\bar{a}_{m-1}^{k+1}+1}\Big(\underbrace{\sum_{j\in\mathcal{N}_{m-1}^{k+1}(k+1)}\Psi(x_j^{k+1}-x_{m-1}^{k+1})}_{\mathcal{I}_a}+\underbrace{\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_{m-1}^{k+1}(l)}\Psi(x_j^l-x_{m-1}^{k+1})}_{\mathcal{I}_b}\Big).
\end{aligned}
\tag{A.5}
$$

For the term $\mathcal{I}_a$, we can apply the same method in Lemma 3.4 to obtain

$$
\begin{aligned}
\mathcal{I}_a &= \sum_{j\in\mathcal{N}_{m-1}^{k+1}(k+1)}\Psi(x_j^{k+1}-x_{m-1}^{k+1}) \\
&= \sum_{\substack{j\in\mathcal{N}_{m-1}^{k+1}(k+1)\\ j>m-1}}\Psi(x_j^{k+1}-x_{m-1}^{k+1})+\sum_{\substack{j\in\mathcal{N}_{m-1}^{k+1}(k+1)\\ j\leq m-1}}\Psi(x_j^{k+1}-x_{m-1}^{k+1}) \\
&\leq (N_{k+1}-m+1)\Psi(x_{N_{k+1}}^{k+1}-x_{m-1}^{k+1})+\sum_{\substack{j\in\mathcal{N}_{m-1}^{k+1}(k+1)\\ j\leq m-1}}\Psi(x_j^{k+1}-x_{m-1}^{k+1}).
\end{aligned}
\tag{A.6}
$$

For the term $\mathcal{I}_b$, we observe that there are three possible orderings between x_j^l and x_{m-1}^{k+1}:

(1) If $x_j^l \leq x_{m-1}^{k+1}$, then $\Psi(x_j^l-x_{m-1}^{k+1})\leq 0$.
(2) If $x_{m-1}^{k+1}\leq x_j^l\leq x_{N_{k+1}}^{k+1}$, then $\Psi(x_j^l-x_{m-1}^{k+1})\leq\Psi(x_{N_{k+1}}^{k+1}-x_{m-1}^{k+1})$.
(3) If $x_{N_{k+1}}^{k+1}\leq x_j^l$, then $\Psi(x_j^l-x_{m-1}^{k+1})\leq\Psi(x_j^l-x_{N_{k+1}}^{k+1})+\Psi(x_{N_{k+1}}^{k+1}-x_{m-1}^{k+1})$.

Therefore, we apply the analysis in (A.4) again to conclude

$$
\mathcal{I}_b \leq g_k\Psi(x_{N_{k+1}}^{k+1}-x_{m-1}^{k+1})+g_k\Psi(\mathcal{D}_k).
\tag{A.7}
$$

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Now, we combine the estimates of (A.5), (A.6), (A.7) and Lemma 3.3 to get

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_{m-1}^{k+1} &= \frac{d}{dt}\Big(\frac{\bar{a}_{m-1}^{k+1}\bar{x}_m^{k+1}}{\bar{a}_{m-1}^{k+1}+1} + \frac{x_{m-1}^{k+1}}{\bar{a}_{m-1}^{k+1}+1}\Big) \\
&\leq \frac{\nu_{m-1}^{k+1}}{\bar{a}_{m-1}^{k+1}+1} + \sum_{i=m}^{N_{k+1}} \frac{\prod_{l=m-1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=m-1}^{i}(\bar{a}_l^{k+1}+1)} + \kappa g_k \Psi(\mathcal{D}_k) \\
&\quad + \frac{\kappa}{\bar{a}_{m-1}^{k+1}+1}\Big(\sum_{i=m-1}^{N_{k+1}} \min_{\substack{j\in\mathcal{N}_i^{k+1}(k+1) \\ j\leq i}} \Psi(x_j^{k+1}-x_i^{k+1})\Big).
\end{aligned}
$$

This finishes the proof of claim (A.1).

According to the node decomposition, we obtain

$$
\sum_{i=1}^{N_{k+1}} \min_{\substack{j\in\mathcal{N}_i^{k+1}(k+1) \\ j\leq i}} \Psi(x_j^{k+1}-x_i^{k+1}) \leq \Psi(x_1^{k+1}-x_{N_{k+1}}^{k+1}). \tag{A.8}
$$

We combine (A.1) and (A.8) to obtain the estimate of $\bar{x}_{k+1}$:

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_{k+1} &= \frac{d}{dt}\bar{x}_1^{k+1} \\
&\leq \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} + \kappa g_k\Psi(\mathcal{D}_k) + \frac{\kappa\Psi(x_1^{k+1}-x_{N_{k+1}}^{k+1})}{\bar{a}_1^{k+1}+1}.
\end{aligned}
\tag{A.9}
$$

Similarly, we can derive a differential inequality of $\underline{x}_{k+1}$:

$$
\begin{aligned}
\frac{d}{dt}\underline{x}_{k+1} &= \frac{d}{dt}\underline{x}_{N_{k+1}}^{k+1} \\
&\geq \frac{\nu_{N_{k+1}}^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_{N_{k+1}-i+1}^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} - \kappa g_k\Psi(\mathcal{D}_k) - \frac{\kappa\Psi(x_1^{k+1}-x_{N_{k+1}}^{k+1})}{\bar{a}_1^{k+1}+1}.
\end{aligned}
\tag{A.10}
$$

Finally, for (Case 2), we can combine (A.9) and (A.10) to conclude that for $t\in J_p$,

$$
\frac{d}{dt}Q^{k+1}(t) = \frac{d}{dt}(\bar{x}_{k+1}-\underline{x}_{k+1}) \leq C_1^{k+1} - C_2^{k+1}\Psi(\mathcal{X}^{k+1}) \leq C_1^{k+1} - C_2^{k+1}\Psi(Q^{k+1}).
$$

• (Case 3): Consider the case

$$
\max_{0\leq i\leq k+1}\{\bar{x}_i\} = \bar{x}_{k+1}, \quad \min_{0\leq i\leq k+1}\{\underline{x}_i\} = \min_{0\leq i\leq k}\{\underline{x}_i\} \quad \text{on } J_p.
$$

For this case, without loss of generality, we set

$$
\underline{x}_q = \min_{0\leq i\leq k}\{\underline{x}_i\},
$$

where $0 \leq q \leq k$. Further more, we assume

$$x_1^{k+1} \leq x_2^{k+1} \leq \cdots \leq x_{N_{k+1}}^{k+1}, \quad x_1^q \leq x_2^q \leq \cdots \leq x_{N_q}^q.$$

Then, we apply the same analysis in (Case 2) to find positive constant C_1^q and C_2^q such that

$$\frac{d}{dt}\underline{x}_q \geq -C_1^q - C_2^q \Psi(x_1^q - x_{N_q}^q). \tag{A.11}$$

$\diamond$ (Case 3.1): If $x_1^{k+1} \leq \max\limits_{1 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\}$, then we can apply (A.9) to have

$$
\begin{aligned}
\frac{d}{dt}Q^{k+1}(t) &= \frac{d}{dt}(\bar{x}_{k+1} - \underline{x}_q) \\
&\leq \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1} + 1} + \sum_{i=2}^{N_{k+1}} \frac{\prod_{l=1}^{i-1} \bar{a}_l^{k+1} \nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1} + 1)} + \kappa g_k \Psi(\mathcal{D}_k) + \frac{\kappa \Psi(x_1^{k+1} - x_{N_{k+1}}^{k+1})}{\bar{a}_1^{k+1} + 1} + C_1^q + C_2^q \Psi(x_1^q - x_{N_q}^q) \\
&\quad + \min\{\frac{\kappa}{\bar{a}_1^{k+1} + 1}, C_2^q\}\Psi(x_1^q - \max\limits_{1 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\}) - \min\{\frac{\kappa}{\bar{a}_1^{k+1} + 1}, C_2^q\}\Psi(x_1^q - \max\limits_{1 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\}) \\
&\leq C_1^{k+1} + \min\{\frac{\kappa}{\bar{a}_1^{k+1} + 1}, C_2^q\}\Psi(x_1^q - \max\limits_{1 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\}) + \min\{\frac{\kappa}{\bar{a}_1^{k+1} + 1}, C_2^q\}\Psi(x_1^{k+1} - x_{N_{k+1}}^{k+1}) \\
&\leq C_1^{k+1} + C_2^{k+1}\Psi(x_1^q - x_{N_{k+1}}^{k+1}) \\
&\leq C_1^{k+1} - C_2^{k+1}\Psi(Q^{k+1}).
\end{aligned}
$$

$\diamond$ (Case 3.2): If $x_1^{k+1} \geq \max\limits_{1 \leq i \leq k} \max\limits_{1 \leq j \leq N_i} \{x_j^i\}$, similar to Case 2, we can apply the induction principle to prove that for $1 \leq m \leq N_{k+1}$,

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_m^{k+1} &\leq \frac{\nu_m^{k+1}}{\bar{a}_m^{k+1} + 1} + \sum_{i=m+1}^{N_{k+1}} \frac{\prod_{l=m}^{i-1} \bar{a}_l^{k+1} \nu_i^{k+1}}{\prod_{l=m}^{i}(\bar{a}_l^{k+1} + 1)} + \frac{\kappa}{\bar{a}_m^{k+1} + 1} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_m^{k+1}(l)} \Psi(x_j^l - x_m^{k+1}) \\
&\quad + \sum_{i=m+1}^{N_{k+1}} \frac{\prod_{l=m}^{i-1} \bar{a}_l^{k+1} \kappa}{\prod_{l=m}^{i}(\bar{a}_l^{k+1} + 1)} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_i^{k+1}(l)} \Psi(x_j^l - x_i^{k+1}) \\
&\quad + \frac{\kappa}{\bar{a}_m^{k+1} + 1}\Big(\sum_{i=m}^{N_{k+1}} \min\limits_{\substack{j \in \mathcal{N}_i^{k+1}(k+1) \\ j \leq i}} \Psi(x_j^{k+1} - x_i^{k+1})\Big).
\end{aligned}
\tag{A.12}
$$

Since the proof of (A.12) is similar to the proof of (A.1), we will omit the details.

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Then, we set $m = 1$ and apply Lemma 3.3 to have

$$
\frac{d}{dt}\bar{x}_{k+1} \le \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)}
$$

$$
\underbrace{+ \frac{\kappa}{\bar{a}_1^{k+1}+1}\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_1^{k+1}(l)}\Psi(x_j^l - x_1^{k+1}) + \sum_{i=2}^{N_{k+1}}\frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\kappa}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)}\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_i^{k+1}(l)}\Psi(x_j^l - x_i^{k+1})}_{\mathcal{I}_c}
$$

$$
+ \frac{\kappa}{\bar{a}_1^{k+1}+1}\Psi(x_1^{k+1} - x_{N_{k+1}}^{k+1}).
$$

Since

$$
x_1^{k+1} \ge \max_{1\le i\le k}\max_{1\le j\le N_i}\{x_j^i\},
$$

we note that all $\Psi(x_j^l - x_i^{k+1})$ in $\mathcal{I}_c$ are non-positive. Therefore, we have

$$
\frac{d}{dt}\bar{x}_{k+1} \le \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}}\frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)}
$$

$$
+ \frac{\kappa}{\bar{a}_1^{k+1}+1}\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_1^{k+1}(l)}\Psi(\max_{1\le i\le k}\max_{1\le j\le N_i}\{x_j^i\} - x_1^{k+1})
$$

$$
+ \sum_{i=2}^{N_{k+1}}\frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\kappa}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)}\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_i^{k+1}(l)}\Psi(\max_{1\le i\le k}\max_{1\le j\le N_i}\{x_j^i\} - x_i^{k+1}) \tag{A.13}
$$

$$
+ \frac{\kappa}{\bar{a}_1^{k+1}+1}\Psi(x_1^{k+1} - x_{N_{k+1}}^{k+1}).
$$

Moreover, according to the existence of spanning tree in $\mathcal{G}$ and the node decomposition,

$$
\bigcup_{i=1}^{N_{k+1}}\bigcup_{l=0}^{k}\mathcal{N}_i^{k+1}(l) \ne \emptyset.
$$

Therefore, (A.13) implies

$$
\frac{d}{dt}\bar{x}_{k+1} \le \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}}\frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)}
$$

$$
+ \frac{\kappa}{\bar{a}_1^{k+1}+1}\left|\bigcup_{i=1}^{N_{k+1}}\bigcup_{l=0}^{k}\mathcal{N}_i^{k+1}(l)\right|\Psi(\max_{1\le i\le k}\max_{1\le j\le N_i}\{x_j^i\} - x_1^{k+1}) + \frac{\kappa}{\bar{a}_1^{k+1}+1}\Psi(x_1^{k+1} - x_{N_{k+1}}^{k+1})
$$

$$
\le \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}}\frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} + \frac{\kappa}{\bar{a}_1^{k+1}+1}\Psi(\max_{1\le i\le k}\max_{1\le j\le N_i}\{x_j^i\} - x_{N_{k+1}}^{k+1}). \tag{A.14}
$$

We combine (A.11) and (A.14) to obtain

$$
\begin{aligned}
\frac{d}{dt}Q^{k+1} &= \frac{d}{dt}(\bar{x}_{k+1} - \underline{x}_q) \\
&\leq \frac{\nu_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{\prod_{l=1}^{i-1}\bar{a}_l^{k+1}\nu_i^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} + C_1^q \\
&\quad + C_2^q\Psi(x_1^q - x_{N_q}^q) + \frac{\kappa}{\bar{a}_1^{k+1}+1}\Psi(\max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\} - x_{N_{k+1}}^{k+1}) \\
&\quad + \min\{C_2^q, \frac{\kappa}{\bar{a}_1^{k+1}+1}\}\Psi(x_{N_q}^q - \max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\}) + \min\{C_2^q, \frac{\kappa}{\bar{a}_1^{k+1}+1}\}\Psi(\mathcal{D}_k) \\
&\leq C_1^{k+1} + C_2^{k+1}\Psi(x_1^q - x_{N_{k+1}}^{k+1}) \\
&\leq C_1^{k+1} + C_2^{k+1}\Psi(Q^{k+1}).
\end{aligned}
$$

- (Case 4): Consider the case

$$
\max_{0\leq i\leq k+1}\{\bar{x}_i\} = \max_{0\leq i\leq k}\{\bar{x}_i\}, \qquad \min_{0\leq i\leq k+1}\{\underline{x}_i\} = \underline{x}_{k+1} \quad \text{on } J_p.
$$

The proof is similar to Case 3. Thus we omit the details.

Finally, we combine all analysis from (Case 1) to (Case 4) to conclude that if there exist positive constants C_1^k, C_2^k and C_4^k such that

$$
\frac{d}{dt}Q^k(t) \leq C_1^k - C_2^k\Psi(Q^k), \quad \mathcal{D}_k \leq C_4^k.
$$

Then we can find positive constants C_1^{k+1} and C_2^{k+1} such that

$$
\frac{d}{dt}Q^{k+1}(t) \leq C_1^{k+1} - C_2^{k+1}\Psi(Q^{k+1}). \tag{A.15}
$$

$\square$

A.2. *Proof of Step 2*

In fact, if (3.32) holds, we can follow (Step 3) in Lemma 3.5 to construct the positive constant M_{k+1} such that

$$
Q^{k+1} \leq M_{k+1}.
$$

On the other hand, according to the definition (3.31) of Q^{k+1}, we have

$$
Q^{k+1} \geq \bar{x}_{k+1} - \underline{x}_{k+1}.
$$

Therefore, similar to the proof of (ii) in Lemma 3.4, we can apply (3.29) and (3.30) to have

$$
\begin{aligned}
Q^{k+1}(t) &\geq \bar{x}_{k+1} - \underline{x}_{k+1} \\
&= \frac{x_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{x_i^{k+1}\prod_{l=1}^{i-1}\bar{a}_l^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} - \frac{x_{N_{k+1}}^{k+1}}{\underline{a}_{N_{k+1}}^{k+1}+1} - \sum_{i=1}^{N_{k+1}-1} \frac{x_i^{k+1}\prod_{l=i+1}^{N_{k+1}}\underline{a}_l^{k+1}}{\prod_{l=i}^{N_{k+1}}(\underline{a}_l^{k+1}+1)} \\
&= \frac{x_1^{k+1}}{\bar{a}_1^{k+1}+1} + \sum_{i=2}^{N_{k+1}} \frac{x_i^{k+1}\prod_{l=1}^{i-1}\bar{a}_l^{k+1}}{\prod_{l=1}^{i}(\bar{a}_l^{k+1}+1)} - \frac{x_{N_{k+1}}^{k+1}}{\bar{a}_1^{k+1}+1} - \sum_{i=1}^{N_{k+1}-1} \frac{x_i^{k+1}\prod_{l=1}^{N_{k+1}-i}\bar{a}_l^{k+1}}{\prod_{l=1}^{N_{k+1}-i+1}(\bar{a}_l^{k+1}+1)} \\
&= \frac{1}{\bar{a}_1^{k+1}+1} \sum_{i=1}^{N} x_i^{k+1}\Big(\frac{(N_{k+1}+g_k)!}{(N_{k+1}-i+1+g_k)!} - \frac{(N_{k+1}+g_k)!}{(i+g_k)!} \Big) \\
&\geq \frac{x_{N_{k+1}}^{k+1}-x_1^{k+1}}{\bar{a}_1^{k+1}+1}\Big(\frac{(N_{k+1}+g_k)!}{(1+g_k)!} - 1 \Big) \\
&\geq (x_{N_{k+1}}^{k+1}-x_1^{k+1})\frac{(N_{k+1}+g_k)!}{2(\bar{a}_1^{k+1}+1)(1+g_k)!} \\
&\geq \frac{(x_{N_{k+1}}^{k+1}-x_1^{k+1})}{4} = \frac{\mathcal{X}^{k+1}(t)}{4}.
\end{aligned}
\tag{A.16}
$$

Therefore we immediately have

$$
\mathcal{X}^{k+1}(t) \leq 4Q^{k+1}(t) \leq 4M_{k+1} = C_3^{k+1}.
\tag{A.17}
$$

Next, we show that $\mathcal{G}_{k+1}$ cannot be far away from the set $\bigcup_{i=0}^{k}\mathcal{G}_i$. For this aim, we recall that the set of neighbors of z_i^{k+1} can be denoted by $\bigcup_{j=0}^{k+1}\mathcal{N}_i^{k+1}(j)$, where $\mathcal{N}_i^{k+1}(j)$ represents the neighbors of z_i^{k+1} in $\mathcal{G}_j$. The node decomposition and spanning tree in $\mathcal{G}$ guarantee that for any k with $0 \leq k \leq d-1$, there exists x_p^{k+1} such that $\bigcup_{j=0}^{k}\mathcal{N}_p^{k+1}(j) \neq \emptyset$ (see Remark 2.1). Then we consider the quantity $(x_p^{k+1} - \max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\})$ and it is obvious that this quantity is Lipschitz continuous. We fix t at which $(x_M^{k+1} - \max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\})$ is differentiable. Then we consider the rate of change for this quantity as below

$$
\begin{aligned}
\frac{d}{dt}\Big(x_p^{k+1} - \max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\}\Big) &\leq \max\{|\nu_i|\} + \kappa \sum_{j\in\mathcal{N}_p^{k+1}(k+1)} \Psi(x_j^{k+1}-x_p^{k+1}) \\
&+ \kappa \sum_{m=0}^{k}\sum_{j\in\mathcal{N}_p^{k+1}(m)} \Psi(x_j^m-x_p^{k+1}) - \frac{d}{dt}\Big(\max_{1\leq i\leq k}\max_{1\leq j\leq N_i}\{x_j^i\}\Big).
\end{aligned}
\tag{A.18}
$$

Note that $(\max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\})$ is only affected by the particles in $\bigcup\limits_{i=0}^{k} \mathcal{G}_i$, whose relative positions are uniformly bounded by C_4^k. Therefore, we have

$$
\begin{aligned}
\frac{d}{dt}(\max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}) &\ge -\max\{|\nu_i|\} + \kappa \sum_{m=0}^{k} \sum_{j \in \mathcal{N}_M^k(m)} \Psi(x_j^m - \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}) \\
&\ge -\max\{|\nu_i|\} - \kappa g_k \Psi(C_4^k),
\end{aligned}
\tag{A.19}
$$

where g_k is defined in (3.29). Moreover, we also have a uniform bound for $\mathcal{X}^{k+1}$ in (A.17). Therefore, we combine (A.17), (A.18) and (A.19) to get

$$
\begin{aligned}
\frac{d}{dt}(x_p^{k+1} &- \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}) \\
&\le \underbrace{2\max\{|\nu_i|\} + \kappa g_k \Psi(C_4^k) + \kappa N_{k+1}\Psi(C_3^{k+1})}_{B_{k+1}} + \kappa \sum_{m=0}^{k} \sum_{j \in \mathcal{N}_p^{k+1}(m)} \Psi(x_j^m - x_p^{k+1}).
\end{aligned}
\tag{A.20}
$$

Now we claim:

$$
\begin{aligned}
x_p^{k+1} &- \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\} \\
&\le \max\left\{ (x_p^{k+1}(0) - \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}(0)), \quad \Psi^{-1}\left(\frac{B_{k+1}}{\kappa}\right) \right\} = C_6^{k+1}.
\end{aligned}
\tag{A.21}
$$

Proof of the claim (A.21)**:** We prove it by contradiction. Suppose not, then we assume (A.21) does not hold at $\bar{t}$:

$$
x_p^{k+1}(\bar{t}) - \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}(\bar{t}) > \max\left\{ (x_p^{k+1}(0) - \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}(0)), \quad \Psi^{-1}\left(\frac{B_{k+1}}{\kappa}\right) \right\}.
$$

As in Lemma 3.5, we set

$$
\mathcal{M}_{k+1} := \{t \mid t < \bar{t}, \ (\text{A.21}) \text{ holds}\},
$$

which is obviously non-empty. Then we can define

$$
t_{k+1}^* := \sup \mathcal{M}_{k+1}.
$$

On the other hand, since $\bigcup\limits_{j=0}^{k} \mathcal{N}_p^{k+1}(j) \ne \emptyset$ and the fact that

$$
x_p^{k+1} \ge \max\limits_{1 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\} \quad \text{for } t_{k+1}^* \le t \le \bar{t},
$$

we can simplify (A.20) as below:

$$
\frac{d}{dt}(x_p^{k+1} - \max\limits_{0 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\}) \le B_{k+1} + \kappa \Psi(\max\limits_{0 \le i \le k} \max\limits_{1 \le j \le N_i} \{x_j^i\} - x_p^{k+1}).
\tag{A.22}
$$

Now, we apply to (A.22) the same argument in (Step 3) of the proof of Lemma 3.5 to finish the proof of claim. Then we set $x_M^{k+1} := \max\{x_j^{k+1}\}$, and we have

$$x_M^{k+1} - \max_{0 \le i \le k} \max_{1 \le j \le N_i}\{x_j^i\}) = x_M^{k+1} - x_p^{k+1} + x_p^{k+1} - \max_{0 \le i \le k}\max_{1 \le j \le N_i}\{x_j^i\}) \le \mathcal{X}^{k+1} + C_6^{k+1}. \tag{A.23}$$

Similarly, we set $x_m^{k+1} := \min\{x_j^{k+1}\}$ and conclude that there exists a positive constant $\bar{C}_6^{k+1}$ such that

$$\min_{0 \le i \le k}\min_{1 \le j \le N_i}\{x_j^i\} - x_m^{k+1} \le \mathcal{X}^{k+1} + \bar{C}_6^{k+1}. \tag{A.24}$$

We combine (A.23) and (A.24) to obtain that there exists a positive constant C_4^{k+1} such that

$$\begin{aligned}
\mathcal{D}_{k+1} &:= \Big(\max_{1 \le i \le k+1}\max_{1 \le j \le N_i}\{x_j^i\} - \min_{1 \le i \le k+1}\min_{1 \le j \le N_i}\{x_j^i\} \Big) \\
&\le \mathcal{D}_k + \bar{C}_6^{k+1} + C_6^{k+1} + 2\mathcal{X}^{k+1} \\
&\le C_4^k + \bar{C}_6^{k+1} + C_6^{k+1} + 2C_3^{k+1} = C_4^{k+1}.
\end{aligned} \tag{A.25}$$

□

Appendix B. Proof of Theorem 1.1

In this section, we will provide the details of the two steps in the proof of Theorem 1.1 in Section 4.

B.1. *Proof of Step 1*

The proof is similar to Section A.1. We set

$$[0, T] := \bigcup_{i=1}^{q} J_i,$$

where $J_i = [t_{i-1}, t_i]$ and $0 = t_0 < t_1 \cdots < t_q = T$, such that the order of both $\{\bar{v}_j\}_{j=0}^{k+1}$ and $\{\underline{v}_j\}_{j=0}^{k+1}$ are preserved in each J_i. Now we pick out any J_p, where $1 \le p \le q$, and consider four cases depending on the relative positions between $\bigcup_{i=0}^{k} \mathcal{G}_i$ and $\mathcal{G}_{k+1}$.

- (Case 1): Consider the case

$$\max_{0 \le i \le k+1} \{\bar{v}_i\} = \max_{0 \le i \le k} \{\bar{v}_i\}, \quad \min_{0 \le i \le k+1} \{\underline{v}_i\} = \min_{0 \le i \le k} \{\underline{v}_i\} \quad \text{on } J_p.$$

In this case, we immediately have $Y^{k+1}(t) = Y^k(t)$ and

$$\frac{d}{dt} Y^{k+1}(t) = \frac{d}{dt} Y^k(t) \le -C_5^k Y^k + C_8^k \mathcal{R}_{k-1} \le -C_5^k Y^{k+1} + C_8^k \mathcal{R}_k, \quad t_{p-1} < t < t_p,$$

where we use the fact that $\mathcal{R}_{k-1} \le \mathcal{R}_k$, which is obvious according to the definition of $\mathcal{R}_k$ in the proof of the main theorem in Section 4. Therefore, we simply set

$$C_5^{k+1} = C_5^k \quad \text{and} \quad C_8^{k+1} = C_8^k$$

to obtain (4.17).

- (Case 2): Consider the case

$$\max_{0 \le i \le k+1} \{\bar{v}_i\} = \bar{v}_{k+1}, \quad \min_{0 \le i \le k+1} \{\underline{v}_i\} = \underline{v}_{k+1} \quad \text{on } J_p.$$

In this case, we assume

$$v_1^{k+1} \le v_2^{k+1} \le \cdots \le v_{N_{k+1}}^{k+1}, \quad \bar{v}_j^{k+1} := \text{Proj}_v \bar{\mathcal{L}}_j^{N_{k+1}}(\bar{C}_{j,N_{k+1}}), \quad \underline{v}_j^{k+1} := \text{Proj}_v \underline{\mathcal{L}}_1^j(\underline{C}_{1,j}).$$

It is clear to see that

$$\bar{v}_1^{k+1} = \bar{v}_{k+1} \quad \text{and} \quad \underline{v}_{N_{k+1}}^{k+1} = \underline{v}_{k+1}.$$

We claim: for $1 \le m \le N_{k+1}$, we have

$$\frac{d}{dt} \bar{v}_m^{k+1} \le \kappa g_k \psi_0 \mathcal{R}_k + \frac{\kappa \psi_m}{\bar{b}_m^{k+1} + 1} \Big(\min_{j \in \bigcup_{i=m}^{N_{k+1}} \mathcal{N}_i^{k+1}(k+1)} \{v_j^{k+1}\} - v_{N_{k+1}}^{k+1} \Big). \tag{B.1}$$

Proof of claim: We will prove the claim by induction.

$\diamond$ (Step 1): We verify that (B.1) holds for $m = N_{k+1}$. In fact, we have the differential equation of $\bar{v}^{k+1}_{N_{k+1}}$:

$$
\begin{aligned}
\frac{d}{dt}\bar{v}^{k+1}_{N_{k+1}} = \frac{d}{dt}v^{k+1}_{N_{k+1}} = \kappa \sum_{j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \psi(||x^{k+1}_j - x^{k+1}_{N_{k+1}}||)(v^{k+1}_j - v^{k+1}_{N_{k+1}}) \\
+ \kappa \sum_{l=0}^{k} \sum_{j \in \mathcal{N}^{k+1}_{N_{k+1}}(l)} \psi(||x^l_j - x^{k+1}_{N_{k+1}}||)(v^l_j - v^{k+1}_{N_{k+1}}).
\end{aligned}
\tag{B.2}
$$

According to the well ordered assumption, we obtain that

$$
(v^{k+1}_j - v^{k+1}_{N_{k+1}}) \le 0 \quad \text{for all } j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1).
$$

Therefore, we apply above inequality to obtain

$$
\sum_{j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1)} \psi(||x^{k+1}_j - x^{k+1}_{N_{k+1}}||)(v^{k+1}_j - v^{k+1}_{N_{k+1}}) \le \psi_m \min_{j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1)} (v^{k+1}_j - v^{k+1}_{N_{k+1}}).
\tag{B.3}
$$

On the other hand, as $\max_{0 \le i \le k+1}\{\bar{v}_i\} = \bar{v}_{k+1}$, we immediately have

$$
v^{k+1}_{N_{k+1}} \ge \min_{0 \le i \le k} \min_{1 \le j \le N_i}\{v^i_j\}.
$$

Otherwise, the convex combination $\bar{v}_{k+1}$ will be strictly smaller than all $\bar{v}_i$ for $0 \le i \le k$. Therefore, we have

$$
v^l_j - v^{k+1}_{N_{k+1}} \le v^l_j - \min_{0 \le i \le k} \min_{1 \le j \le N_i}\{v^i_j\} \le \mathcal{R}_k.
\tag{B.4}
$$

We combine (B.2), (B.3) and (B.4) to derive

$$
\frac{d}{dt}\bar{v}^{k+1}_{N_{k+1}} \le \kappa \psi_m \min_{j \in \mathcal{N}^{k+1}_{N_{k+1}}(k+1)} (v^{k+1}_j - v^{k+1}_{N_{k+1}}) + \kappa g_k \psi_0 \mathcal{R}_k.
$$

Therefore, (B.1) holds for $\bar{x}^{k+1}_{N_{k+1}}$.

$\diamond$ (Step 2): Suppose (B.1) holds for m where $2 \le m \le N_{k+1}$. Next, we verify the relation (B.1) also holds for $m-1$. Following the same argument in Lemma 3.6, we

can derive

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{\bar{b}_{m-1}^{k+1}\bar{v}_m^{k+1}}{\bar{b}_{m-1}^{k+1}+1} + \frac{v_{m-1}^{k+1}}{\bar{b}_{m-1}^{k+1}+1}\Big) &\leq \frac{\bar{b}_{m-1}^{k+1}}{\bar{b}_{m-1}^{k+1}+1}\kappa g_k\psi_0\mathcal{R}_k \\
&\quad + \frac{\kappa(N_{k+1}-m+2+g_k)\psi_0}{\bar{b}_{m-1}^{k+1}+1}\Big(\min_{j\in\bigcup\limits_{i=m}^{N_{k+1}}\mathcal{N}_i^{k+1}(k+1)}\{v_j^{k+1}\} - v_{N_{k+1}}^{k+1}\Big) \\
&\quad + \frac{\kappa}{\bar{b}_{m-1}^{k+1}+1}\underbrace{\sum_{j\in\mathcal{N}_{m-1}^{k+1}(k+1)}\psi(\|x_j^{k+1}-x_{m-1}^{k+1}\|)(v_j^{k+1}-v_{m-1}^{k+1})}_{\mathcal{I}_a} \\
&\quad + \frac{\kappa}{\bar{b}_{m-1}^{k+1}+1}\underbrace{\sum_{l=0}^{k}\sum_{j\in\mathcal{N}_{m-1}^{k+1}(l)}\psi(\|x_j^l-x_{m-1}^{k+1}\|)(v_j^l-v_{m-1}^{k+1})}_{\mathcal{I}_b}.
\end{aligned}
\tag{B.5}
$$

For the term $\mathcal{I}_a$, we can apply the same method in Lemma 3.4 to obtain

$$
\begin{aligned}
\mathcal{I}_a &\leq (N_{k+1}-m+1)\psi_0(v_{N_{k+1}}^{k+1}-v_{m-1}^{k+1}) + \sum_{\substack{j\in\mathcal{N}_{m-1}^{k+1}(k+1)\\ j\leq m-1}}\psi_m(v_j^{k+1}-v_{m-1}^{k+1}) \\
&\leq (N_{k+1}-m+1)\psi_0(v_{N_{k+1}}^{k+1}-v_{m-1}^{k+1}) + \psi_m\min_{j\in\mathcal{N}_{m-1}^{k+1}(k+1)}(v_j^{k+1}-v_{m-1}^{k+1}).
\end{aligned}
\tag{B.6}
$$

For the term $\mathcal{I}_b$, we observe that there are three possible order relations between v_j^l and v_{m-1}^{k+1}:

(1) If $v_j^l \leq v_{m-1}^{k+1}$, then $(v_j^l - v_{m-1}^{k+1}) \leq 0$.
(2) If $v_{m-1}^{k+1} \leq v_j^l \leq v_{N_{k+1}}^{k+1}$, then $v_j^l - v_{m-1}^{k+1} \leq v_{N_{k+1}}^{k+1} - v_{m-1}^{k+1}$.
(3) If $v_{N_{k+1}}^{k+1} \leq v_j^l$, then $v_j^l - v_{m-1}^{k+1} \leq (v_j^l - v_{N_{k+1}}^{k+1}) + (v_{N_{k+1}}^{k+1} - v_{m-1}^{k+1})$.

Therefore, we again apply the analysis in (B.4) to conclude that

$$
\mathcal{I}_b \leq g_k\psi_0(v_{N_{k+1}}^{k+1} - v_{m-1}^{k+1}) + g_k\psi_0\mathcal{R}_k.
\tag{B.7}
$$

Moreover, since $\mathcal{G}_{k+1}$ is a node, we have

$$
\Big(\min_{j\in\bigcup\limits_{i=m}^{N_{k+1}}\mathcal{N}_i^{k+1}(k+1)}\{v_j^{k+1}\} - v_{N_{k+1}}^{k+1}\Big) \leq (v_{m-1}^{k+1} - v_{N_{k+1}}^{k+1}).
\tag{B.8}
$$

Now we combine (B.5), (B.6), (B.7) and (B.8) to get

$$
\begin{aligned}
\frac{d}{dt}\bar{v}_{m-1}^{k+1} &= \frac{d}{dt}\Big(\frac{\bar{b}_{m-1}^{k+1}\bar{v}_m^{k+1}}{\bar{b}_{m-1}^{k+1}+1} + \frac{v_{m-1}^{k+1}}{\bar{b}_{m-1}^{k+1}+1}\Big) \\
&\leq \kappa g_k\psi_0\mathcal{R}_k + \frac{\kappa\psi_m}{\bar{b}_m^{k+1}+1}\Big(\min_{j\in\bigcup\limits_{i=m-1}^{N_{k+1}}\mathcal{N}_i^{k+1}(k+1)}\{v_j^{k+1}\} - v_{N_{k+1}}^{k+1}\Big).
\end{aligned}
\tag{B.9}
$$

This completes the proof of the claim.

Due to the fact $\mathcal{G}_{k+1}$ is a node, we obtain

$$\left(\min_{j \in \bigcup\limits_{i=1}^{N_{k+1}} \mathcal{N}_i^{k+1}(k+1)} \{v_j^{k+1}\} - v_{N_{k+1}}^{k+1} \right) = v_1^{k+1} - v_{N_{k+1}}^{k+1}. \tag{B.10}$$

Therefore, we further combine (B.9) and (B.10) to derive the estimate of $\bar{v}_{k+1}$:

$$\frac{d}{dt}\bar{v}_{k+1} = \frac{d}{dt}\bar{v}_1^{k+1} \le \kappa g_k \psi_0 \mathcal{R}_k + \frac{\kappa \psi_m (v_1^{k+1} - v_{N_{k+1}}^{k+1})}{\bar{b}_1^{k+1} + 1}. \tag{B.11}$$

Similarly, we can derive the differential inequality of $\underline{v}_{k+1}$ as

$$\frac{d}{dt}\underline{v}_{k+1} = \frac{d}{dt}\underline{v}_{N_{k+1}}^{k+1} \ge -\kappa g_k \psi_0 \mathcal{R}_k - \frac{\kappa \psi_m (v_1^{k+1} - v_{N_{k+1}}^{k+1})}{\bar{b}_1^{k+1} + 1}. \tag{B.12}$$

Finally, for (Case 2), we can combine (B.11) and (B.12) to conclude for $t \in J_p$ that

$$\frac{d}{dt}Y^{k+1}(t) = \frac{d}{dt}(\bar{v}_{k+1} - \underline{v}_{k+1}) \le C_8^{k+1}\mathcal{R}_k - C_5^{k+1}\mathcal{V}^{k+1} \le -C_5^{k+1}Y^{k+1} + C_8^{k+1}\mathcal{R}_k.$$

• (Case 3): Consider the case

$$\max_{0 \le i \le k+1}\{\bar{v}_i\} = \bar{v}_{k+1}, \quad \min_{0 \le i \le k+1}\{\underline{v}_i\} = \min_{0 \le i \le k}\{\underline{v}_i\} \quad \text{on } J_p.$$

In this case, without loss of generality, we set

$$\underline{v}_q := \min_{0 \le i \le k}\{\underline{v}_i\}, \quad \text{where } 0 \le q \le k.$$

Further more, we assume

$$v_1^{k+1} \le v_2^{k+1} \le \cdots \le v_{N_{k+1}}^{k+1}, \quad v_1^q \le v_2^q \le \cdots \le v_{N_q}^q.$$

Then, we apply the same analysis in (Case 2) to find positive constant C_5^q and C_8^q such that

$$\frac{d}{dt}\underline{v}_q \ge -C_5^q(v_1^q - v_{N_q}^q) - C_8^q\mathcal{R}_{q-1}. \tag{B.13}$$

⋄ (Case 3.1): If $v_1^{k+1} \le \max\limits_{0 \le i \le k} \max\limits_{1 \le j \le N_i}\{v_j^i\}$, then we can apply (B.11) and (B.13) to

have

$$
\frac{d}{dt}Y^{k+1}(t) = \frac{d}{dt}(\bar{v}_{k+1} - \underline{v}_q)
$$

$$
\leq \kappa g_k \psi_0 \mathcal{R}_k + \frac{\kappa \psi_m(v_1^{k+1} - v_{N_{k+1}}^{k+1})}{\bar{b}_1^{k+1} + 1} + C_5^q(v_1^q - v_{N_q}^q) + C_8^q \mathcal{R}_{q-1}
$$

$$
\leq (\kappa g_k \psi_0 + C_8^q)\mathcal{R}_k + C_5^q\big(\max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i}\{v_j^i\} - v_{N_q}^q\big) + C_5^q(v_1^q - \max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i}\{v_j^i\})
$$

$$
\leq (\kappa g_k \psi_0 + C_8^q + C_5^q)\mathcal{R}_k + C_5^q(v_1^q - v_{N_{k+1}}^{k+1})
$$

$$
\leq -C_5^{k+1}Y^{k+1} + C_8^{k+1}\mathcal{R}_k.
$$

$$(B.14)$$

$\diamond$ (Case 3.2): If $v_1^{k+1} \geq \max\limits_{0 \leq i \leq k} \max\limits_{1 \leq j \leq N_i}\{v_j^i\}$, similar to the (Case 2), we can apply the induction principle to derive that for $1 \leq m \leq N_{k+1}$,

$$
\frac{d}{dt}\bar{v}_m^{k+1} \leq \frac{\kappa}{\bar{b}_m^{k+1} + 1} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_m^{k+1}(l)} \psi(\|x_j^l - x_m^{k+1}\|)(v_j^l - v_m^{k+1})
$$

$$
+ \sum_{i=m+1}^{N_{k+1}} \frac{\prod_{l=m}^{i-1} \bar{b}_l^{k+1} \kappa}{\prod_{l=m}^{i}(\bar{b}_l^{k+1} + 1)} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_i^{k+1}(l)} \psi(\|x_j^l - x_i^{k+1}\|)(v_j^l - v_i^{k+1})
$$

$$
+ \frac{\kappa}{\bar{b}_m^{k+1} + 1} \psi_m\Big(\min_{j \in \bigcup_{i=m}^{N_{k+1}} \mathcal{N}_i^{k+1}(k+1)}\{v_j^{k+1}\} - v_{N_{k+1}}^{k+1}\Big).
$$

$$(B.15)$$

Since the proof of (B.15) is similar to the proof of (B.1), we omit the details. Next, due to the relation

$$
v_1^{k+1} \geq \max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i}\{v_j^i\},
$$

we note all $(v_j^l - v_i^{k+1})$ are non-positive if $0 \leq l \leq k$. Then, we set $m = 1$ and apply the connectivity of node $\mathcal{G}_{k+1}$ to have

$$
\frac{d}{dt}\bar{v}_{k+1} \leq \frac{\kappa}{\bar{b}_m^{k+1} + 1} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_m^{k+1}(l)} \psi_m(v_j^l - v_m^{k+1})
$$

$$
+ \sum_{i=m+1}^{N_{k+1}} \frac{\prod_{l=m}^{i-1} \bar{b}_l^{k+1} \kappa}{\prod_{l=m}^{i}(\bar{b}_l^{k+1} + 1)} \sum_{l=0}^{k} \sum_{j \in \mathcal{N}_i^{k+1}(l)} \psi_m(v_j^l - v_i^{k+1}) \qquad (B.16)
$$

$$
+ \frac{\kappa}{\bar{b}_1^{k+1} + 1} \psi_m(v_1^{k+1} - v_{N_{k+1}}^{k+1}).
$$

Moreover, according to the existence of a spanning tree in $\mathcal{G}$ and the node decomposition, we have

$$
\bigcup_{i=1}^{N_{k+1}} \bigcup_{l=0}^{k} \mathcal{N}_i^{k+1}(l) \neq \emptyset.
$$

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Therefore, (B.16) implies that

$$
\begin{aligned}
\frac{d}{dt}\bar{v}_{k+1} &\leq \frac{\kappa}{\bar{b}_1^{k+1}+1}\Big|\bigcup_{i=1}^{N_{k+1}}\bigcup_{l=0}^{k}\mathcal{N}_i^{k+1}(l)\Big|\psi_m\Big(\max_{0\leq i\leq k}\max_{1\leq j\leq N_i}\{v_j^i\}-v_1^{k+1}\Big)\\
&\quad + \frac{\kappa}{\bar{b}_1^{k+1}+1}\psi_m(v_1^{k+1}-v_{N_{k+1}}^{k+1})\\
&\leq \frac{\kappa}{\bar{b}_1^{k+1}+1}\psi_m\big(\max_{0\leq i\leq k}\max_{1\leq j\leq N_i}\{v_j^i\}-v_{N_{k+1}}^{k+1}\big).
\end{aligned}
\tag{B.17}
$$

We combine (B.13) and (B.17) to obtain

$$
\begin{aligned}
\frac{d}{dt}Y^{k+1} &= \frac{d}{dt}(\bar{v}_{k+1}-\underline{v}_q)\\
&\leq \frac{\kappa}{\bar{b}_1^{k+1}+1}\psi_m\big(\max_{0\leq i\leq k}\max_{1\leq j\leq N_i}\{v_j^i\}-v_{N_{k+1}}^{k+1}\big) + C_5^q(v_1^q-v_{N_q}^q) + C_8^q\mathcal{R}_{q-1}\\
&\leq \min\{\frac{\kappa\psi_m}{\bar{b}_1^{k+1}+1},C_5^q\}(v_1^q-v_{N_{k+1}}^{k+1}) + \min\{\frac{\kappa\psi_m}{\bar{b}_1^{k+1}+1},C_5^q\}\big(\max_{0\leq i\leq k}\max_{1\leq j\leq N_i}\{v_j^i\}-v_{N_q}^q\big)\\
&\quad + C_8^q\mathcal{R}_{q-1}\\
&\leq -\min\{\frac{\kappa\psi_m}{\bar{b}_1^{k+1}+1},C_5^q\}Y^{k+1} + (\min\{\frac{\kappa\psi_m}{\bar{b}_1^{k+1}+1},C_5^q\}+C_8^q)\mathcal{R}_{q-1}\\
&\leq -C_5^{k+1}Y^{k+1} + C_8^{k+1}\mathcal{R}_k.
\end{aligned}
$$

• (Case 4): Consider the case

$$
\max_{0\leq i\leq k+1}\{\bar{v}_i\} = \max_{0\leq i\leq k}\{\bar{v}_i\}, \quad \min_{0\leq i\leq k+1}\{\underline{v}_i\} = \underline{v}_{k+1} \quad \text{on } J_p.
$$

The proof is similar to Case 3, hence we omit the details.

We combine all analysis from Case 1 to Case 4 to conclude that if there exist positive constants C_5^k, C_6^k, C_7^k and C_8^k such that

$$
\frac{d}{dt}Y^k(t) \leq -C_5^k Y^k + C_8^k\mathcal{R}_{k-1}(t), \qquad \mathcal{R}_k(t) \leq C_6^k e^{-C_7^k t}\mathcal{R}_k(0),
$$

then we can find positive constants C_5^{k+1} and C_8^{k+1} such that

$$
\frac{d}{dt}Y^{k+1}(t) \leq -C_5^{k+1}Y^k + C_8^{k+1}\mathcal{R}_k(t).
$$

$\square$

B.2. *Proof of Step 2*

Suppose that we have (4.18):

$$
\frac{d}{dt}Y^{k+1}(t) \leq -C_5^{k+1}Y^{k+1} + C_8^{k+1}\mathcal{R}_k(t), \quad \mathcal{R}_k(t) \leq C_6^k e^{-C_7^k t}\mathcal{R}_k(0), \quad 0\leq k\leq d-1.
$$

Then, since Y^{k+1} is Lipschitz continuous, we immediately obtain

$$Y^{k+1}(t) \leq e^{-C_5^{k+1}t}Y^{k+1}(0) + \frac{C_8^{k+1}C_6^k}{C_5^{k+1}}\mathcal{R}_k(0)e^{-C_7^k t} + \frac{C_8^{k+1}C_6^k}{C_7^k}\mathcal{R}_k(0)e^{-C_5^{k+1}t}. \quad \text{(B.18)}$$

By definition of $Y^{k+1}(t)$, it is easy to see

$$\mathcal{V}^{k+1}(t) \leq \frac{4\psi_0}{2\psi_0 - \psi_m}Y^{k+1}(t) \leq \frac{4\psi_0}{2\psi_0 - \psi_m}\mathcal{R}_{k+1}(t), \quad t \geq 0.$$

Therefore, we can rewrite (B.18) as

$$Y^{k+1}(t) \leq C_9^{k+1}e^{-C_{10}^{k+1}t}\mathcal{R}_{k+1}(0).$$

Now we need to show $Y^{k+1}(t)$ and $\mathcal{R}_{k+1}(t)$ are of the same order. For this aim, we consider the four quantities

$$v_1^{k+1}, \quad v_{N_{k+1}}^{k+1}, \quad \min_{0 \leq i \leq k}\min_{1 \leq j \leq N_i}\{v_j^i\}, \quad \max_{0 \leq i \leq k}\max_{1 \leq j \leq N_i}\{v_j^i\}.$$

Similar to the previous analysis, we can set

$$[0, T] := \bigcup_{i=1}^{q} J_i, \quad \text{where } J_i = [t_{i-1}, t_i] \text{ and } 0 = t_0 < t_1 \cdots < t_q = T,$$

such that the order of above four quantities are preserved in each J_i.

• (Case 1): Consider the case

$$\max_{0 \leq i \leq k+1}\max_{1 \leq j \leq N_i}\{v_j^i\} = v_{N_{k+1}}^{k+1}, \qquad \min_{0 \leq i \leq k+1}\min_{1 \leq j \leq N_i}\{v_j^i\} = v_1^{k+1}.$$

In this case, we simply have

$$\mathcal{R}_{k+1}(t) = \mathcal{V}^{k+1}(t) \leq \frac{4\psi_0}{2\psi_0 - \psi_m}Y^{k+1}(t) \leq \frac{4\psi_0 C_9^{k+1}\mathcal{R}_{k+1}(0)}{2\psi_0 - \psi_m}e^{-C_{10}^{k+1}t}.$$

• (Case 2): Consider the case

$$\max_{0 \leq i \leq k+1}\max_{1 \leq j \leq N_i}\{v_j^i\} = \max_{0 \leq i \leq k}\max_{1 \leq j \leq N_i}\{v_j^i\}, \qquad \min_{0 \leq i \leq k+1}\min_{1 \leq j \leq N_i}\{v_j^i\} = \min_{0 \leq i \leq k}\min_{1 \leq j \leq N_i}\{v_j^i\}.$$

In this case, we have

$$\mathcal{R}_{k+1}(t) = \mathcal{R}_k(t) \leq C_6^k e^{-C_7^k t}\mathcal{R}_k(0) \leq C_6^k e^{-C_7^k t}\mathcal{R}_{k+1}(0).$$

• (Case 3): Consider the case

$$\max_{0 \leq i \leq k+1}\max_{1 \leq j \leq N_i}\{v_j^i\} = v_{N_{k+1}}^{k+1}, \qquad \min_{0 \leq i \leq k+1}\min_{1 \leq j \leq N_i}\{v_j^i\} = \min_{0 \leq i \leq k}\min_{1 \leq j \leq N_i}\{v_j^i\}.$$

In this case, we will discuss in two subcases:

$\diamond$ (Case 3.1): Suppose that

$$\max_{0 \leq i \leq k+1} \max_{1 \leq j \leq N_i} \{v_j^i\} \geq v_1^{k+1}.$$

In this case, we directly obtain

$$
\begin{aligned}
\mathcal{R}_{k+1}(t) &\leq (v_{N_{k+1}}^{k+1} - v_1^{k+1}) + \big(\max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i} \{v_j^i\} - \min_{0 \leq i \leq k} \min_{1 \leq j \leq N_i} \{v_j^i\} \big) \\
&\leq \mathcal{V}^{k+1}(t) + \mathcal{R}_k(t) \leq \frac{4\psi_0 C_9^{k+1} \mathcal{R}_{k+1}(0)}{2\psi_0 - \psi_m} e^{-C_{10}^{k+1} t} + C_6^k e^{-C_7^k t} \mathcal{R}_{k+1}(0).
\end{aligned}
$$

$\diamond$ (Case 3.2): Suppose that

$$\max_{0 \leq i \leq k+1} \max_{1 \leq j \leq N_i} \{v_j^i\} \leq v_1^{k+1}.$$

In this case, due to the spanning tree in $\mathcal{G}$, there must be some v_p^{k+1} and v_q^l, where $0 \leq l \leq k$, such that v_p^{k+1} is affected by v_q^l. Therefore, the differential equation of v_p^{k+1} can be written as

$$
\begin{aligned}
\frac{d}{dt} v_p^{k+1} &= \kappa \sum_{j \in \mathcal{N}_p^{k+1}(k+1)} \psi(\|x_j^{k+1} - x_p^{k+1}\|) \big(v_j^{k+1} - v_p^{k+1} \big) \\
&\quad + \kappa \sum_{r=0}^{k} \sum_{j \in \mathcal{N}_p^{k+1}(r)} \psi(\|x_j^r - x_p^{k+1}\|) \big(v_j^r - v_p^{k+1} \big) \\
&\leq \kappa N_{k+1} \psi_0 \mathcal{V}^{k+1} + \kappa \psi_m (v_q^l - v_p^{k+1}).
\end{aligned}
\tag{B.19}
$$

On the other hand, it is clear to see

$$\frac{d}{dt} v_q^l \geq -\kappa \psi_0 b_k \mathcal{R}_k. \tag{B.20}$$

We combine (B.19) and (B.20) to get

$$\frac{d}{dt}(v_p^{k+1} - v_q^l) \leq \kappa \psi_m (v_q^l - v_p^{k+1}) + \kappa N_{k+1} \psi_0 \mathcal{V}^{k+1} + \kappa \psi_0 b_k \mathcal{R}_k.$$

Then we can find positive constants C_{11}^{k+1} and C_{12}^{k+1} such that

$$v_p^{k+1} - v_q^l \leq C_{11}^{k+1} e^{-C_{12}^{k+1} t} \mathcal{R}_{k+1}(0).$$

Therefore, we have

$$
\begin{aligned}
\mathcal{R}_{k+1} &= v_{N_{k+1}}^{k+1} - \min_{0 \leq i \leq k} \min_{1 \leq j \leq N_i} \{v_j^i\} \\
&= (v_{N_{k+1}}^{k+1} - v_p^{k+1}) + (v_p^{k+1} - v_q^l) + (v_q^l - \min_{0 \leq i \leq k} \min_{1 \leq j \leq N_i} \{v_j^i\}) \\
&\leq \mathcal{V}^{k+1} + (v_p^{k+1} - v_q^l) + \mathcal{R}_k \\
&\leq \frac{4\psi_0 C_9^{k+1} \mathcal{R}_{k+1}(0)}{2\psi_0 - \psi_m} e^{-C_{10}^{k+1} t} + C_{11}^{k+1} e^{-C_{12}^{k+1} t} \mathcal{R}_{k+1}(0) + C_6^k e^{-C_7^k t} \mathcal{R}_{k+1}(0).
\end{aligned}
$$

- (Case 4): Consider the case

$$\min_{0 \leq i \leq k+1} \min_{1 \leq j \leq N_i} \{v_j^i\} = v_1^{k+1}, \qquad \max_{0 \leq i \leq k+1} \max_{1 \leq j \leq N_i} \{v_j^i\} = \max_{0 \leq i \leq k} \max_{1 \leq j \leq N_i} \{v_j^i\}.$$

This case can be treated similar to Case 3, and thus we omit the analysis. Now we combine all the cases to obtain that there exist positive constants C_6^{k+1} and C_7^{k+1} such that

$$\mathcal{R}_{k+1}(t) \leq C_6^{k+1} e^{-C_7^{k+1}t} \mathcal{R}_{k+1}(0).$$

□

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References

1. F. Achleitner, A. Arnold and D. St*ü*rzer, Large-time behavior in non-symmetric Fokker-Planck equations, *Riv. Math. Univ. Parma (N.S.)* **6(1)** (2015) 1–68.
2. G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato, and J. Soler, Vehicular traffic, crowds, and swarms: From kinetic theory and multiscale methods to applications and research perspectives, *Math. Mod. Methods Appl. Sci.* **29** (2019) 1901–2005
3. S. Ahn, H. Choi, S.-Y. Ha and H. Lee, On the collision avoiding initial-configurations to the Cucker-Smale type flocking models, *Commun. Math. Sci.* **10** (2012) 625–643.
4. S. Ahn and S.-Y. Ha, Stochastic flocking dynamics of the Cucker-Smale model with multiplicative white noises, *J. Math. Phys.* **51** (2010) 103301.
5. A. Arnold and C. Schmeiser, Relative entropies and hypocoercivity revisited, *Technical report, Homepage of Christian Schmeiser* (2017).
6. J. A. Carrillo, M. R. D' Orsogna and V. Panferov, Double milling in self-propelled swarms from kinetic theory, *Kinetic Relat. Models* **2** (2009) 363–378.
7. J. A. Carrillo, M. Fornasier, J. Rosado and G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, *SIAM J. Math. Anal.* **42** (2010) 218–236.
8. J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko, Emergence of bi-cluster flocking for the Cucker-Smale model, *Math. Models Methods Appl. Sci.* **26** (2016) 1191–1218.
9. J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko, Emergence of bi-cluster flocking for agent-based models with unit speed constraint, *Anal. Appl.* **14** (2016) 1–35.
10. Y.-P. Choi, S.-Y. Ha and Z. Li, Emergent dynamics of the Cucker-Smale flocking model and its variants, *Active Particles, Vol. 1: Theory, Models, Applications, eds. N. Bellomo, P. Degond and E. Tadmor, Modeling and Simulation in Science and Technology (Birkhäuser, 2017)* 299–331.

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11. F. Cucker and J.-G. Dong, A General collision-avoiding flocking framework, *IEEE Trans. Autom. Control* **56** (2011) 1124–1129.
12. F. Cucker and J.-G. Dong, On the critical exponent for flocks under hierarchical leadership, *Math. Models Methods Appl. Sci.* **19** (2009) 1391–1404.
13. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control* **52** (2007) 852–862.
14. P. Degond and S. Motsch, Macroscopic limit of self-driven particles with orientation interaction, *C.R. Math. Acad. Sci. Paris* **345** (2007) 555–560.
15. P. Degond and S. Motsch, Large-scale dynamics of the persistent Turing Walker model of fish behavior, *J. Stat. Phys.* **131** (2008) 989–1022.
16. P. Degond and S. Motsch, Continuum limit of self-driven particles with orientation interaction, *Math. Mod. Meth. Appl. Sci.* **18** (2008) 1193–1215.
17. H. Dietert, R. Shvydkoy, On Cucker-Smale dynamical systems with degenerate communication, *ArXiv:1903.00094.*
18. J.-G. Dong and L. Qiu, Flocking of the Cucker-Smale model on general digraphs, *IEEE Trans. Automat. Control* **62** (2017) 5234–5239.
19. R. Duan, M. Fornasier and G. Toscani, A kinetic flocking model with diffusion, *Commun. Math. Phys.* **300** (2010) 95–145.
20. S.-Y. Ha, T. Ha and J. Kim, Asymptotic flocking dynamics for the Cucker-Smale model with the Rayleigh friction, *J. Phys. A: Math. Theor.* **43** (2010) 315201.
21. S.-Y. Ha, K. Lee and D. Levy, Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system, *Commun. Math. Sci.* **7** (2009) 453–469.
22. S.-Y. Ha and J.-G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, *Commun. Math. Sci.* **7** (2009) 297–325.
23. S.-Y. Ha, J. Kim, J. Park and X. Zhang, Complete cluster predictability of the Cucker-Smale flocking model on the real line, *Arch. Ration. Mech. Anal.* **31** (2019) 319–365.
24. S.-Y. Ha, J. Park and X. Zhang, On the first-order reduction of the Cucker-Smale model and its clustering dynamics, *Commun. Math. Sci.* **16** (2018) 1907–1931.
25. S.-Y. Ha and E. Tadmor, From particle to kinetic and hydrodynamic description of flocking, *Kinetic Relat. Models* **1** (2008) 415–435.
26. I. Ipsen, T. Selee, Ergodicity coefficients defined by vector norms, *SIAM Matrix Anal, Appl.* **32** (1) (2011) 152–200.
27. C. Jin, Flocking of the Motsch-Tadmor model with a cut-off interaction function, *J. Stat. Phys.* **171** (2018) 345–360.
28. U. Krause, A dynamical model for the process of sharing, *Advances in Complex Systems* **21** (2018) 6–7.
29. N. E. Leonard, D. A. Paley, F. Lekien, R. Sepulchre, D. M. Fratantoni and R. E. Davis, Collective motion, sensor networks and ocean sampling, *Proc. IEEE* **95** (2007) 48–74.
30. Z. Li and X. Xue, Cucker-Smale flocking under rooted leadership with fixed and switching topologies, *SIAM J. Appl. Math.* **70** (2010) 3156–3174.
31. S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior, *J. Stat. Phys.* **144** (2011) 923–947.
32. S. Mostch and E. Tadmor, Heterophilious dynamics enhances consensus, *SIAM REV.* **56** (4) (2014) 577–621.
33. D. A. Paley, N. E. Leonard, R. Sepulchre, D. Grunbaum and J. K. Parrish, Oscillator models and collective motion, *IEEE Control Sys.* **27** (2007) 89–105.
34. J. Park, H. Kim and S.-Y. Ha, Cucker-Smale flocking with inter-particle bonding forces, *IEEE Tran. Automat. Control* **55** (2010) 2617–2623.
35. J. Toner and Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking,

Phys. Rev. E **58** (1998) 4828–4858.

36. C. M. Topaz and A. L. Bertozzi, Swarming patterns in a two-dimensional kinematic model for biological groups, *SIAM J. Appl. Math.* **65** (2004) 152–174.
37. T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Schochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.* **75** (1995) 1226–1229.
38. A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.* **16** (1967) 15–42.