

# VECTOR-VALUED LITTLEWOOD-PALEY-STEIN THEORY FOR SEMIGROUPS II

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ABSTRACT. Inspired by a recent work of Hytönen and Naor, we solve a problem left open in our previous work joint with Martínez and Torrea on the vector-valued Littlewood-Paley-Stein theory for symmetric diffusion semigroups. We prove a similar result in the discrete case, namely, for any  $T$  which is the square of a symmetric Markovian operator on a measure space  $(\Omega, \mu)$ . Moreover, we show that  $T \otimes \text{Id}_X$  extends to an analytic contraction on  $L_p(\Omega; X)$  for any  $1 < p < \infty$  and any uniformly convex Banach space  $X$ .

## 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space. By a *symmetric diffusion semigroup* on  $(\Omega, \mathcal{A}, \mu)$  in Stein's sense [24, section III.1], we mean a family  $\{T_t\}_{t>0}$  of linear maps satisfying the following properties:

- $T_t$  is a contraction on  $L_p(\Omega)$  for every  $1 \leq p \leq \infty$ ;
- $T_t T_s = T_{t+s}$ ;
- $\lim_{t \rightarrow 0} T_t f = f$  in  $L_2(\Omega)$  for every  $f \in L_2(\Omega)$ ;
- $T_t$  is positive (i.e. positivity preserving) and  $T_t 1 = 1$ ;
- $T_t$  is selfadjoint on  $L_2(\Omega)$ .

It is a classical fact that the orthogonal projection from  $L_2(\Omega)$  onto the fixed point subspace of  $\{T_t\}_{t>0}$  extends to a contractive projection on  $L_p(\Omega)$  for every  $1 \leq p \leq \infty$ . We will denote this projection by  $F$ . Then  $F$  is also positive and  $F(L_p(\Omega))$  is the fixed point subspace of  $\{T_t\}_{t>0}$  on  $L_p(\Omega)$  (cf. e.g. [4]).

Stein proved in [24, chapter IV] the following result which considerably extends the classical inequality on the Littlewood-Paley  $g$ -function in harmonic analysis: For every  $1 < p < \infty$

$$(1) \quad \|f - F(f)\|_{L_p(\Omega)} \approx \left\| \left( \int_0^\infty \left| t \frac{\partial}{\partial t} T_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega),$$

where the equivalence constants depend only on  $p$ .

The vector-valued Littlewood-Paley-Stein theory was developed in [26, 15]. Given a Banach space  $X$ , we denote by  $L_p(\Omega; X)$  the usual  $L_p$  space of strongly measurable functions from  $\Omega$  to  $X$ . It is a well known elementary fact that if  $T$  is a positive bounded operator on  $L_p(\Omega)$  with  $1 \leq p \leq \infty$ , then  $T \otimes \text{Id}_X$  is bounded on  $L_p(\Omega; X)$  with the same norm. For notational convenience, throughout this paper, we will denote  $T \otimes \text{Id}_X$  by  $T$  too. Thus  $\{T_t\}_{t>0}$  is also a semigroup of contractions on  $L_p(\Omega; X)$  for any Banach space  $X$ .

The one-sided vector-valued extension of (1) was obtained in [15] not for the semigroup  $\{T_t\}_{t>0}$  itself but for its subordinated Poisson semigroup  $\{P_t\}_{t>0}$  that is defined by

$$P_t f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} T_{\frac{t^2}{4s}} f ds.$$

$\{P_t\}_{t>0}$  is again a symmetric diffusion semigroup. Recall that if  $A$  denotes the negative infinitesimal generator of  $\{T_t\}_{t>0}$ , then  $P_t = e^{-\sqrt{A}t}$ .

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Let  $1 < q < \infty$ . Recall that a Banach space  $X$  is of *martingale cotype*  $q$  if there exists a positive constant  $C$  such that every finite  $X$ -valued  $L_q$ -martingale  $(f_n)$  defined on some probability space satisfies the following inequality

$$\sum_n \mathbb{E} \|f_n - f_{n-1}\|_X^q \leq C^q \sup_n \mathbb{E} \|f_n\|_X^q,$$

where  $\mathbb{E}$  denotes the expectation on the underlying probability space. We then must have  $q \geq 2$ .  $X$  is of *martingale type*  $q$  if the reverse inequality holds. It is easy to see that  $X$  is of martingale cotype  $q$  iff the dual space  $X^*$  is of martingale type  $q'$ , where  $q'$  denotes the conjugate index of  $q$ . We refer to [19, 20] for more information.

The following is the principal result of [15]. In the sequel, we will use the abbreviation  $\partial = \partial/\partial t$ .

**Theorem 1** (Martínez-Torrea-Xu). *Let  $1 < q < \infty$  and  $X$  be a Banach space.*

- (i)  *$X$  is of martingale cotype  $q$  iff for every  $1 < p < \infty$  (or equivalently, for some  $1 < p < \infty$ ) there exists a constant  $C$  such that every subordinated Poisson semigroup  $\{P_t\}_{t>0}$  as above satisfies the following inequality*

$$\left\| \left( \int_0^\infty \|t \partial P_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)} \leq C \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X).$$

- (ii)  *$X$  is of martingale type  $q$  iff for every  $1 < p < \infty$  (or equivalently, for some  $1 < p < \infty$ ) there exists a constant  $C$  such that every subordinated Poisson semigroup  $\{P_t\}_{t>0}$  as above satisfies the following inequality*

$$\|f\|_{L_p(\Omega; X)} \leq \|F(f)\|_{L_p(\Omega; X)} + C \left\| \left( \int_0^\infty \|t \partial P_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega; X).$$

Note that the above theorem for the Poisson semigroup of the torus  $\mathbb{T}$  was first proved in [26]. The main problem left open in [15] asks whether the theorem holds for the semigroup  $\{T_t\}_{t>0}$  itself instead of its subordinated Poisson semigroup  $\{P_t\}_{t>0}$  (see Problem 2 on page 447 of [15]). Very recently, Hytönen and Naor [8] proved that the answer is affirmative for the heat semigroup of  $\mathbb{R}^n$  and for  $p = q$ ; the resulting inequality plays a key role in their work on the approximation of Lipschitz functions by affine maps. Stimulated by their result and using a clever idea of them, we are able to resolve the problem in full generality.

**Theorem 2.** *Let  $X$  be a Banach space and  $k$  a positive integer.*

- (i) *If  $X$  is of martingale cotype  $q$  with  $2 \leq q < \infty$ , then for every symmetric diffusion semigroup  $\{T_t\}_{t>0}$  and for every  $1 < p < \infty$  we have*

$$\left\| \left( \int_0^\infty \|t^k \partial^k T_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)} \leq C \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X),$$

where  $C$  is a constant depending only on  $p, q, k$  and the martingale cotype  $q$  constant of  $X$ .

- (ii) *If  $X$  is of martingale type  $q$  with  $1 < q \leq 2$ , then for every symmetric diffusion semigroup  $\{T_t\}_{t>0}$  and for every  $1 < p < \infty$  we have*

$$\|f\|_{L_p(\Omega; X)} \leq \|F(f)\|_{L_p(\Omega; X)} + C \left\| \left( \int_0^\infty \|t^k \partial^k T_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega; X),$$

where  $C$  is a constant depending only on  $p, q, k$  and the martingale type  $q$  constant of  $X$ .

**Remark 3.** Applied to the heat semigroup  $\{H_t\}_{t>0}$  of  $\mathbb{R}^n$ , the above theorem implies a dimension free estimate for the  $g$ -function associated to  $\{H_t\}_{t>0}$ :

$$\left\| \left( \int_0^\infty \|t \partial H_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \leq C \|f\|_{L_p(\mathbb{R}^n; X)}, \quad \forall f \in L_p(\mathbb{R}^n; X)$$

when  $X$  is of martingale cotype  $q$ . Compare this with [8, Theorem 17] (and the paragraph thereafter).

**Remark 4.** Theorem 2 allows one to improve some recent results of Hong and Ma on vector-valued variational inequalities associated to symmetric diffusion semigroups. For instance, using it, one can extend [6, Theorem 5.2] to any Banach space  $X$  of martingale cotype  $q_0$ . See also [5] for related results in the Banach lattice case.

Theorem 2 admits a discrete analogue. First recall that a power bounded operator  $R$  on a Banach space  $Y$  is said to be *analytic* if

$$\sup_{n \geq 1} n \|R^n(R-1)\| < \infty,$$

where the norm is the operator norm on  $Y$ . It is known that the analyticity of  $R$  is equivalent to

$$\sup_{z \in \mathbb{C}, |z| > 1} |1-z| \|(z-R)^{-1}\| < \infty.$$

Moreover, if  $R$  is analytic, its spectrum  $\sigma(R)$  is contained in  $\overline{B_\gamma}$  for some  $0 < \gamma < \pi/2$ , where  $B_\gamma$  denotes the Stolz domain which is the interior of the convex hull of 1 and the disc  $D(0, \sin \gamma)$  (see figure 1). We refer to [2, 17] for more information.

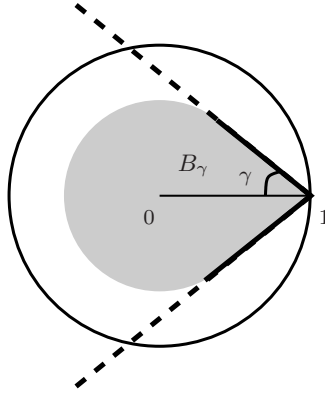


FIGURE 1.

Now consider a *symmetric Markovian operator*  $T$  on  $(\Omega, \mathcal{A}, \mu)$ , that is,  $T$  satisfies the following conditions:

- $T$  is a linear contraction on  $L_p(\Omega)$  for every  $1 \leq p \leq \infty$ ;
- $T$  is positivity preserving and  $T1 = 1$ ;
- $T$  is a selfadjoint operator on  $L_2(\Omega)$ .

With a slight abuse of notation, we use again  $F$  to denote the projection on the fixed point subspace of  $T$ . Both  $T$  and  $F$  extend to contractions on  $L_p(\Omega; X)$  for any Banach space  $X$ . In the following two theorems,  $T = S^2$  with  $S$  a symmetric Markovian operator, so  $T$  is a symmetric Markovian operator too. The following is the discrete analogue of a theorem of Pisier [21] for semigroups.

**Theorem 5.** *Let  $T = S^2$  with  $S$  a symmetric Markovian operator,  $1 < p < \infty$  and  $X$  be a uniformly convex Banach space. Then the extension of  $T$  to  $L_p(\Omega; X)$  is analytic. More precisely, there exist constants  $C$  and  $\gamma \in (0, \pi/2)$ , depending only on  $p$  and the modulus of uniform convexity of  $X$ , such that*

$$(2) \quad \sigma(T) \subset \overline{B_\gamma} \quad \text{and} \quad \|(z-T)^{-1}\| \leq \frac{C}{|1-z|}, \quad \forall z \in \mathbb{C} \setminus \overline{B_\gamma}.$$

The discrete analogue of Theorem 2 is the following

**Theorem 6.** *Let  $T = S^2$  be as above and  $1 < p < \infty$ .*

- (i) *If  $X$  is of martingale cotype  $q$  with  $2 \leq q < \infty$ , then*

$$\left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|(T^n - T^{n-1})f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)} \leq C \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X),$$

where the constant  $C$  depends only on  $p, q$  and the martingale cotype  $q$  constant of  $X$ .

(ii) If  $X$  is of martingale type  $q$  with  $1 < q \leq 2$ , then

$$\|f\|_{L_p(\Omega; X)} \leq C \left\| \left( \|Ff\|_X^q + \sum_{n=1}^{\infty} n^{q-1} \|(T^n - T^{n-1})f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega; X),$$

where the constant  $C$  depends only on  $p, q$  and the martingale type  $q$  constant of  $X$ .

**Remark 7.** If the inequality in Theorem 6 (i) holds for every positive symmetric Markovian operator  $T$ , then the corresponding inequality of Theorem 1 holds for every subordinated Poisson semigroup  $\{P_t\}_{t>0}$ . Thus  $X$  is of martingale cotype  $q$ . Therefore, the validity of the inequality in Theorem 6 (i) characterizes the martingale cotype  $q$  of  $X$ . A similar remark applies to part (ii).

**Remark 8.** It is worth to note that all constants involved in the preceding theorems are independent of the semigroup  $\{T_t\}_{t>0}$  or contraction  $T$  in consideration. They depend only on the indices  $p, q$  and the relevant geometric constants of the space  $X$ .

The preceding three theorems will be proved in the next three sections. The proofs of Theorem 2 and Theorem 6 follow the same pattern although the latter one is more involved. The last section contains some open problems.

We will use the symbol  $\lesssim$  to denote an inequality up to a constant factor; all constants will depend only on  $X, p, q$ , etc. but never on the function  $f$  in consideration.

## 2. A SPECTRAL ESTIMATE

This section contains a spectral estimate for positive symmetric Markovian operators. Let  $X$  be a uniformly convex Banach space and  $1 < p < \infty$ . Then  $Y = L_p(\Omega; X)$  is uniformly convex too. By Pisier's renorming theorem [19], we can assume that  $Y$  is uniformly convex of power type  $q$  for some  $2 \leq q < \infty$ , namely,

$$(3) \quad \left\| \frac{x+y}{2} \right\|^q + \delta \left\| \frac{x-y}{2} \right\|^q \leq \frac{1}{2} (\|x\|^q + \|y\|^q), \quad \forall x, y \in Y$$

for some positive constant  $\delta$ . Note that the above inequality implies the martingale cotype  $q$  of  $Y$ . Conversely, if  $Y$  is of martingale cotype  $q$ , then it admits an equivalent norm which satisfies (3). Let  $T = S^2$  with  $S$  a symmetric Markovian operator on  $(\Omega, \mathcal{A}, \mu)$ . We extend  $T$  to a contraction on  $Y$ , still denoted by  $T$ . In the following the norm and spectrum of  $T$  is taken for  $T$  viewed as an operator on  $Y$ .

**Lemma 9.** *Under the above assumptions we have*

- (i)  $\|1 - T\| \leq \min\left(\frac{3}{2}, 2\left(1 - \frac{\delta}{2^q}\right)^{1/q}\right) < 2$ ;
- (ii) *the spectrum of  $T$  is contained in a Stolz domain  $\overline{B}_\gamma$  for some  $\gamma \in (0, \pi/2)$  depending only on  $\delta$  and  $q$  in (3).*

Part (i) above is already contained in [21] (see, in particular, Remark 1.8 there). In fact, our proof below is modeled on that of [21, Lemma 1.5]. As in [21], We will need the following one step version of Rota's dilation theorem for positive symmetric Markovian operators. We refer to [24, Chapter IV] for its proof as well as its full version.

**Lemma 10** (Rota). *Let  $T = S^2$  with  $S$  a symmetric Markovian operator on  $(\Omega, \mathcal{A}, \mu)$ . Then there exist a larger measure space  $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$  containing  $(\Omega, \mathcal{A}, \mu)$ , and a  $\sigma$ -subalgebra  $\mathcal{B}$  of  $\tilde{\mathcal{A}}$  such that*

$$Tf = \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{B}} f, \quad \forall f \in L_p(\Omega, \mathcal{A}, \mu),$$

where  $\mathbb{E}_{\mathcal{A}}$  denotes the conditional expectation relative to  $\mathcal{A}$  (and similarly for  $\mathbb{E}_{\mathcal{B}}$ ).

*Proof of Lemma 9.* Rota's dilation extends to  $X$ -valued functions:

$$T = \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathcal{B}} \Big|_Y.$$

Here we have used our usual convention that  $\mathbb{E}_{\mathcal{A}} \otimes \text{Id}_X$  and  $\mathbb{E}_{\mathcal{B}} \otimes \text{Id}_X$  are abbreviated to  $\mathbb{E}_{\mathcal{A}}$  and  $\mathbb{E}_{\mathcal{B}}$ , respectively. Thus for any  $\lambda \in \mathbb{C}$  (with  $P = \mathbb{E}_{\mathcal{B}}$ )

$$\lambda + T = \mathbb{E}_{\mathcal{A}}(\lambda + P) \Big|_Y.$$

Let  $y$  be a unit vector in  $Y$ . Using (3), we get

$$\left\| \frac{\lambda y + Py}{2} \right\|^q + \delta \left\| \frac{\lambda y - Py}{2} \right\|^q \leq \frac{1}{2} (|\lambda|^q + 1).$$

However (noting that  $P$  is a contractive projection),

$$\|\lambda y - Py\| \geq |1 - \lambda| \|Py\| \geq |1 - \lambda| (\|\lambda y + Py\| - |\lambda|) \geq |1 - \lambda| (\|\lambda y + Ty\| - |\lambda|).$$

When  $\|\lambda y + Ty\|$  approaches  $\|\lambda + T\|$ , we then deduce

$$(4) \quad \left\| \frac{\lambda + T}{2} \right\|^q + \delta |1 - \lambda|^q \left( \frac{\|\lambda + T\| - |\lambda|}{2} \right)^q \leq \frac{1}{2} (|\lambda|^q + 1).$$

In particular, for  $\lambda = -1$  we obtain

$$\|1 - T\|^q + \delta 2^q (\|1 - T\| - 1)^q \leq 2^q,$$

which implies

$$\|1 - T\| \leq \min \left( \frac{3}{2}, 2 \left(1 - \frac{\delta}{2^q}\right)^{1/q} \right).$$

This is part (i). On the other hand, if  $\lambda \in \sigma(T)$ , then (4) yields

$$|\lambda|^q + \delta |1 - \lambda|^q |\lambda|^q \leq \frac{1}{2} (|\lambda|^q + 1),$$

whence

$$|1 - \lambda| |\lambda| \leq \left( \frac{q}{2\delta} \right)^{1/q} (1 - |\lambda|).$$

The last inequality implies (in fact, is equivalent to) that  $\lambda \in \overline{B}_\gamma$  for some  $\gamma \in (0, \pi/2)$  depending only on the constant  $(q/(2\delta))^{1/q}$ . The proof of the lemma is thus complete.  $\square$

Lemma 9 (i) implies the following result which is [21, Remark 1.8].

**Lemma 11.** *Let  $X$  and  $p$  be as above and  $\{T_t\}_{t>0}$  be a symmetric diffusion semigroup on  $(\Omega, \mathcal{A}, \mu)$ . Then the extension of  $\{T_t\}_{t>0}$  to  $Y = L_p(\Omega; X)$  is analytic. Consequently,  $\{t\partial T_t\}_{t>0}$  is a uniformly bounded family of operators on  $Y$ , namely,*

$$(5) \quad \sup_{t>0} \|t\partial T_t\| \leq C,$$

where  $C$  is a constant depending only on  $\delta$  and  $q$  in (3).

*Proof.* Applying Lemma 9 to  $T = T_t$ , we get

$$\sup_{t>0} \|1 - T_t\| \leq \min \left( \frac{3}{2}, 2 \left(1 - \frac{\delta}{2^q}\right)^{1/q} \right) < 2.$$

Then using Kato's characterization of analytic semigroups in [10], we deduce (5).  $\square$

### 3. PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2. Let us first note that assertion (ii) follows easily from (i) by duality. Indeed, let  $\{e_\lambda\}$  be the resolution of the identity of  $\{T_t\}_{t>0}$  on  $L_2(\Omega)$ :

$$T_t f = \int_0^\infty e^{-\lambda t} de_\lambda f, \quad f \in L_2(\Omega).$$

Then

$$\partial^k T_t f = (-1)^k \int_0^\infty \lambda^k e^{-\lambda t} de_\lambda f.$$

It thus follows that

$$\begin{aligned}
\int_{\Omega} \int_0^{\infty} |t^k \partial^k T_t f|^2 \frac{dt}{t} d\mu &= \int_0^{\infty} \int_{\Omega} t^{2k} \lambda^{2k} e^{-2\lambda t} d\langle e_{\lambda} f, f \rangle \frac{dt}{t} \\
&= 4^{-k} \int_0^{\infty} \int_{\Omega} t^{2k} e^{-t} \frac{dt}{t} d\langle e_{\lambda} f, f \rangle \\
&= 4^{-k} (2k-1)! \int_0^{\infty} d\langle e_{\lambda} f, f \rangle \\
&= 4^{-k} (2k-1)! \int_{\Omega} |f - F(f)|^2 d\mu.
\end{aligned}$$

By polarization, for  $f, g \in L_2(\Omega)$  we have

$$\int_{\Omega} (f - F(f))(g - F(g)) d\mu = \frac{4^k}{(2k-1)!} \int_{\Omega} \int_0^{\infty} (t^k \partial^k T_t f) (t^k \partial^k T_t g) \frac{dt}{t} d\mu.$$

We then deduce that for any  $f \in L_1(\Omega) \cap L_{\infty}(\Omega) \otimes X$  and  $g \in L_1(\Omega) \cap L_{\infty}(\Omega) \otimes X^*$

$$\int_{\Omega} \langle g - F(g), f - F(f) \rangle d\mu = \frac{4^k}{(2k-1)!} \int_{\Omega} \int_0^{\infty} \langle t^k \partial^k T_t g, t^k \partial^k T_t f \rangle \frac{dt}{t} d\mu,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality bracket between  $X$  and  $X^*$ . Hence

$$\begin{aligned}
\left| \int_{\Omega} \langle g - F(g), f - F(f) \rangle d\mu \right| &\leq \frac{4^k}{(2k-1)!} \left\| \left( \int_0^{\infty} \|t^k \partial^k T_t g\|_{X^*}^{q'} \frac{dt}{t} \right)^{1/q'} \right\|_{L_{p'}(\Omega)} \\
&\quad \cdot \left\| \left( \int_0^{\infty} \|t^k \partial^k T_t f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)},
\end{aligned}$$

where  $r'$  is the conjugate index of  $r$ . Under the assumption of (ii) and by duality, we have that  $X^*$  is of martingale cotype  $q'$ . Therefore, (i) implies

$$\left\| \left( \int_0^{\infty} \|t^k \partial^k T_t g\|_{X^*}^{q'} \frac{dt}{t} \right)^{1/q'} \right\|_{L_{p'}(\Omega)} \leq \frac{4^k C}{(2k-1)!} \|g\|_{L_{p'}(\Omega; X^*)}.$$

Combining the previous inequalities and taking the supremum over all  $g$  in the unit ball of  $L_{p'}(\Omega; X^*)$ , we derive assertion (ii).

Thus we are left to showing assertion (i). In the rest of this section, we will assume that  $X$  is a Banach space of martingale cotype  $q$  with  $2 \leq q < \infty$ . The following lemma, due to Hytönen and Naor [8, Lemma 24], will play an important role in our argument.

**Lemma 12** (Hytönen-Naor). *For any  $f \in L_q(\Omega; X)$  we have*

$$\left( \int_0^{\infty} \|(T_t - T_{3t})f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \lesssim \|f\|_{L_q(\Omega; X)}, \quad \forall f \in L_q(\Omega; X).$$

Based on Rota's dilation theorem quoted in the previous section, the proof is simple. Below is the main idea. First write

$$\begin{aligned}
\int_0^{\infty} \|(T_t - T_{3t})f\|_{L_q(\Omega; X)}^q \frac{dt}{t} &= \sum_{k \in \mathbb{Z}} \int_{3^k}^{3^{k+1}} \|(T_t - T_{3t})f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \\
&= \int_1^3 \sum_{k \in \mathbb{Z}} \|(T_{3^k t} - T_{3^{k+1} t})f\|_{L_q(\Omega; X)}^q \frac{dt}{t}.
\end{aligned}$$

Then Rota's dilation theorem allows us to turn  $\{T_{3^k t} - T_{3^{k+1} t}\}_k$  for each fixed  $t$  into a martingale difference sequence.

The following lemma shows Theorem 2 in the case of  $p = q$ .

**Lemma 13.** *Let  $k$  be a positive integer. Then*

$$(6) \quad \left( \int_0^{\infty} \|t^k \partial^k T_t f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \lesssim \|f\|_{L_q(\Omega; X)}, \quad \forall f \in L_q(\Omega; X),$$

where the relevant constant depends on  $k$  and the martingale cotype  $q$  constant of  $X$ .

*Proof.* We will use the idea of the proof of Theorem 17 of [8]. By virtue of the identity  $\partial T_{t+s} = \partial T_t T_s$ , we write

$$\partial T_t f = \sum_{k=-1}^{\infty} (\partial T_{2^{k+1}t} - \partial T_{2^{k+2}t}) f = \sum_{k=-1}^{\infty} \partial T_{2^k t} (T_{2^k t} - T_{3 \cdot 2^k t}) f.$$

Then by the triangle inequality we get

$$\begin{aligned} \left( \int_0^{\infty} \|t \partial T_t f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} &\leq \sum_{k=-1}^{\infty} \left( \int_0^{\infty} \|t \partial T_{2^k t} (T_{2^k t} - T_{3 \cdot 2^k t}) f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \\ &= \sum_{k=-1}^{\infty} 2^{-k} \left( \int_0^{\infty} \|t \partial T_t (T_t - T_{3t}) f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \\ &= 4 \left( \int_0^{\infty} \|t \partial T_t (T_t - T_{3t}) f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

We are now in a position of using Lemma 11 with  $p = q$ . Indeed, since  $X$  is of martingale cotype  $q$ , so is  $Y = L_q(\Omega; X)$ . Then by [19],  $Y$  can be renormalized into a uniformly convex space of power type  $q$ , that is,  $Y$  admits an equivalent norm satisfying (3). Thus we have (5); moreover, the constant  $C$  there depends only on  $q$  and the martingale cotype  $q$  constant of  $X$ .

Therefore,

$$\|t \partial T_t (T_t - T_{3t}) f\|_{L_q(\Omega; X)} \lesssim \|(T_t - T_{3t}) f\|_{L_q(\Omega; X)}, \quad \forall t > 0.$$

Combining the above inequalities together with Lemma 12, we deduce

$$\left( \int_0^{\infty} \|t \partial T_t f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \lesssim \left( \int_0^{\infty} \|(T_t - T_{3t}) f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \right)^{1/q} \lesssim \|f\|_{L_q(\Omega; X)}.$$

This is (6) for  $k = 1$ . To handle a general  $k$ , by the semigroup identity  $T_{t+s} = T_t T_s$  once more, we have

$$t^k \partial^k T_t = k^k \left( \frac{t}{k} \partial T_{\frac{t}{k}} \right)^k.$$

Thus, by (5) and the already proved inequality, we obtain

$$\begin{aligned} \int_0^{\infty} \|t^k \partial^k T_t f\|_{L_q(\Omega; X)}^q \frac{dt}{t} &= k^k \int_0^{\infty} \|(t \partial T_t)^k f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \\ &\lesssim \int_0^{\infty} \|t \partial T_t f\|_{L_q(\Omega; X)}^q \frac{dt}{t} \lesssim \|f\|_{L_q(\Omega; X)}^q. \end{aligned}$$

The lemma is thus proved.  $\square$

To show Theorem 2 for any  $1 < p < \infty$ , we will use Stein's complex interpolation machinery. To that end, we will need the fractional integrals. For a (nice) function  $\varphi$  on  $(0, \infty)$  define

$$I^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds, \quad t > 0.$$

The integral in the right hand side is well defined for any  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha > 0$ ; moreover,  $I^\alpha \varphi$  is analytic in the right half complex plane  $\operatorname{Re} \alpha > 0$ . Using integration by parts, Stein showed in [24, section III.3] that  $I^\alpha \varphi$  has an analytic continuation to the whole complex plane, which satisfies the following properties

- $I^\alpha I^\beta \varphi = I^{\alpha+\beta} \varphi$  for any  $\alpha, \beta \in \mathbb{C}$ ;
- $I^0 \varphi = \varphi$ ;
- $I^{-k} = \partial^k \varphi$  for any positive integer  $k$ .

We will apply  $I^\alpha$  to  $\varphi$  defined by  $\varphi(s) = T_s f$  for a given function  $f$  in  $L_p(\Omega; X)$  and set

$$M_t^\alpha f = t^{-\alpha} I^\alpha \varphi(t) \quad \text{with} \quad \varphi(s) = T_s f.$$

Note that

$$M_t^1 f = \frac{1}{t} \int_0^t T_s f ds, \quad M_t^0 f = T_t f \quad \text{and} \quad M_t^{-k} f = t^k \partial^k T_t f \quad \text{for} \quad k \in \mathbb{N}.$$

The following lemma is [15, Theorem 2.3].

**Lemma 14.** *Let  $q$  and  $X$  be as in Theorem 2. Then for any  $1 < p < \infty$  we have*

$$\left\| \left( \int_0^\infty \|t \partial M_t^1 f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X).$$

**Lemma 15.** *Let  $\alpha$  and  $\beta$  be complex numbers such that  $\operatorname{Re} \alpha > \operatorname{Re} \beta > -1$ . Then for any positive integer  $k$*

$$\left( \int_0^\infty \|t^k \partial^k M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} \leq C e^{\pi |\operatorname{Im}(\alpha - \beta)|} \left( \int_0^\infty \|t^k \partial^k M_t^\beta f\|_X^q \frac{dt}{t} \right)^{1/q} \quad \text{on } \Omega,$$

where  $C$  is a constant depending only on  $\operatorname{Re} \alpha$  and  $\operatorname{Re} \beta$ .

*Proof.* Using  $I^\alpha = I^{\alpha - \beta} I^\beta$ , we write

$$M_t^\alpha f = \frac{t^{-\alpha}}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} s^\beta M_s^\beta f ds = \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} s^\beta M_{ts}^\beta f ds.$$

Thus

$$t^k \partial^k M_t^\alpha f = \frac{1}{\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha - \beta - 1} s^\beta (ts)^k \partial^k M_{ts}^\beta f ds,$$

which implies

$$\begin{aligned} \left( \int_0^\infty \|t^k \partial^k M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} &\leq \frac{1}{|\Gamma(\alpha - \beta)|} \int_0^1 (1 - s)^{\operatorname{Re}(\alpha - \beta) - 1} s^{\operatorname{Re} \beta} ds \left( \int_0^\infty \|t^k \partial^k M_t^\beta f\|_X^q \frac{dt}{t} \right)^{1/q} \\ &\lesssim \frac{1}{|\Gamma(\alpha - \beta)|} \left( \int_0^\infty \|t^k \partial^k M_t^\beta f\|_X^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

Then the desired inequality follows from the following well known estimate on the  $\Gamma$ -function:

$$\forall x, y \in \mathbb{R}, \quad |\Gamma(x + iy)| \sim e^{-\frac{\pi}{2}|y|} |y|^{x - \frac{1}{2}} \quad \text{as } y \rightarrow \pm\infty$$

(see [25, p. 151]). □

Combining Lemma 14 and Lemma 15 with  $k = \beta = 1$ , we get

**Lemma 16.** *For any  $1 < p < \infty$  and  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha > 1$*

$$\left\| \left( \int_0^\infty \|t \partial M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)} \leq C e^{\pi |\operatorname{Im} \alpha|} \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X),$$

where  $C$  depends on  $\operatorname{Re} \alpha$ ,  $p$  and the martingale cotype  $q$  constant of  $X$ .

**Lemma 17.** *For any  $\alpha \in \mathbb{C}$*

$$(7) \quad \left\| \left( \int_0^\infty \|t \partial M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_q(\Omega)} \leq C e^{\pi |\operatorname{Im} \alpha|} \|f\|_{L_q(\Omega; X)}, \quad \forall f \in L_p(\Omega; X),$$

where  $C$  depends on  $\operatorname{Re} \alpha$  and the martingale cotype  $q$  constant of  $X$ .

*Proof.* Combining Lemma 13 and Lemma 15 with  $\beta = 0$ , we deduce that for a positive integer  $k$  and  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha > 0$

$$(8) \quad \left\| \left( \int_0^\infty \|t^k \partial^k M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_q(\Omega)} \leq C e^{\pi |\operatorname{Im} \alpha|} \|f\|_{L_q(\Omega; X)}, \quad \forall f \in L_q(\Omega; X),$$

where  $C$  depends on  $k$ ,  $\operatorname{Re} \alpha$  and the martingale cotype  $q$  constant of  $X$ . In particular, when  $k = 1$ , we get (7) for any  $\alpha$  such that  $\operatorname{Re} \alpha > 0$ .

To deal with the general case, we will use an iteration procedure. Noting that for any  $\alpha \in \mathbb{C}$

$$\partial M_t^\alpha = -\alpha t^{-1} M_t^\alpha + t^{-1} M_t^{\alpha - 1},$$

we have

$$(9) \quad t^k \partial^k M_t^{\alpha - 1} = (k + \alpha) t^k \partial^k M_t^\alpha + t^{k+1} \partial^{k+1} M_t^\alpha.$$



This shows that if (8) holds for  $M^\alpha$ , so does it for  $M^{\alpha-1}$  instead of  $M^\alpha$  (with a different constant). Therefore, by what already proved, we deduce that (8) holds for any  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha > -1$ . Repeating this argument, we obtain (8) for any  $\alpha \in \mathbb{C}$ . In particular for  $k = 1$ , we have (7).  $\square$

Now we are ready to show Theorem 2 (i).

*Proof of Theorem 2 (i).* We will prove the following more general statement: Under the assumption of assertion (i), we have for any  $\alpha \in \mathbb{C}$

$$(10) \quad \left\| \left( \int_0^\infty \|t^k \partial^k M_t^\alpha f\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X).$$

Assertion (i) corresponds to (10) for  $\alpha = 0$ .

Fix  $\alpha \in \mathbb{C}$ . Choose  $\theta \in (0, 1)$ ,  $r \in (1, \infty)$ ,  $\alpha_0, \alpha_1 \in \mathbb{C}$  such that

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r}, \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \operatorname{Re} \alpha_1 > 1 \text{ and } \operatorname{Im} \alpha_0 = \operatorname{Im} \alpha_1 = \operatorname{Im} \alpha.$$

Then by the classical complex interpolation on vector-valued  $L_p$ -spaces (cf. [1]), we have

$$L_p(\Omega; X) = (L_q(\Omega; X), L_r(\Omega; X))_\theta.$$

Thus for any  $f \in L_p(\Omega; X)$  with norm less than 1 there exists a continuous function  $F$  from the closed strip  $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$  to  $L_q(\Omega; X) + L_r(\Omega; X)$ , which is analytic in the interior and satisfies

$$F(\theta) = f, \quad \sup_{y \in \mathbb{R}} \|F(iy)\|_{L_q(\Omega; X)} < 1 \text{ and } \sup_{y \in \mathbb{R}} \|F(1+iy)\|_{L_r(\Omega; X)} < 1.$$

Define

$$\mathcal{F}_t(z) = e^{z^2 - \theta^2} t \partial M_t^{(1-z)\alpha_0 + z\alpha_1} F(z).$$

Viewed as a function of  $z$  on the strip  $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ ,  $\mathcal{F}$  takes values in  $L_q(\Omega; L_q(\mathbb{R}_+; X)) + L_r(\Omega; L_q(\mathbb{R}_+; X))$ , where  $\mathbb{R}_+$  is equipped with the measure  $\frac{dt}{t}$ . By the analyticity of  $M^{(1-z)\alpha_0 + z\alpha_1}$  in  $z$ , we see that  $\mathcal{F}$  is analytic in the interior of the strip. Moreover, by Lemma 17

$$\left\| \left( \int_0^\infty \|\mathcal{F}_t(iy)\|_X^q \frac{dt}{t} \right)^{1/q} \right\|_{L_q(\Omega)} \leq C'_0 e^{-y^2 - \theta^2} e^{\pi(|\operatorname{Im} \alpha| + |\operatorname{Re}(\alpha_1 - \alpha_0)|y)}, \quad \forall y \in \mathbb{R},$$

where  $C'_0$  is a constant depending on  $\alpha, \alpha_0, \alpha_1$  and  $X$ . Hence

$$\sup_{y \in \mathbb{R}} \|\mathcal{F}(iy)\|_{L_q(\Omega; L_q(\mathbb{R}_+, \frac{dt}{t}); X)} \leq C_0.$$

Similarly, Lemma 16 implies

$$\sup_{y \in \mathbb{R}} \|\mathcal{F}(1+iy)\|_{L_r(\Omega; L_q(\mathbb{R}_+, \frac{dt}{t}); X)} \leq C_1.$$

We then deduce that  $\mathcal{F}(\theta)$  belongs to the complex interpolation space

$$(L_q(\Omega; L_q(\mathbb{R}_+; X)), L_r(\Omega; L_q(\mathbb{R}_+; X)))_\theta$$

with norm majorized by  $C_0^{1-\theta} C_1^\theta$ . However, the latter space coincides with  $L_p(\Omega; L_q(\mathbb{R}_+; X))$  isometrically. Since

$$\mathcal{F}_t(\theta) = t \partial M_t^\alpha F(\theta) = t \partial M_t^\alpha f,$$

we get (10) for  $k = 1$ . Then using (9) and an induction argument, we derive (10) for any  $k$ . Thus the theorem is completely proved.  $\square$

## 4. PROOFS OF THEOREM 5 AND THEOREM 6

The main part of Theorem 5 is already contained in Lemma 9. Armed with that lemma, we can easily show Theorem 5. Let us first recall the following well known characterization of the analyticity of power bounded operators (cf. [2, Theorem 2.3] and [17, Theorem 4.5.4]). Let  $\mathbb{D}$  denote the open unit disc of the complex plane and  $\mathbb{T}$  the boundary of  $\mathbb{D}$ .

**Lemma 18.** *Let  $T$  be a power bounded linear operator on a Banach space  $Y$ . Then  $T$  is analytic iff the semigroup  $\{e^{t(T-1)}\}_{t>0}$  is analytic and  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ .*

*Proof of Theorem 5.* Note that  $\{e^{t(T-1)}\}_{t>0}$  is a symmetric diffusion semigroup on  $(\Omega, \mathcal{A}, \mu)$ . Thus, by Lemma 11, its extension to  $Y = L_p(\Omega; X)$  is analytic. Then Theorem 5 immediately follows from Lemmas 9 and 18.  $\square$

The difficult part ( Lemma 9) of the above proof concerns the quantitative dependence on the geometry of  $X$  of the angle  $\gamma$  of the Stolz domain which contains the spectrum of the operator  $T$ . If we only need to show the analyticity of  $T$  on  $Y$ , the proof can be largely shortened by virtue of the following simple fact which, together with Lemma 10, ensures that  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ .

**Remark 19.** Let  $P$  be a contractive linear projection on a uniformly convex Banach space  $Y$ . Then  $\|\lambda - P\| < 2$  for any  $\lambda \in \mathbb{T} \setminus \{-1\}$ .

This remark is a weaker form of Lemma 9. Let  $\lambda \in \mathbb{T}$  such that  $\|\lambda - P\| = 2$ . Choose a sequence  $\{y_k\}$  of unit vectors in  $Y$  such that  $\|y_k - Py_k\| \rightarrow 2$  as  $k \rightarrow \infty$ . Then the uniform convexity of  $Y$  implies  $\|\lambda y_k + Py_k\| \rightarrow 0$ . However,

$$|\lambda + 1| \|Py_k\| = \|P(\lambda + P)y_k\| \leq \|(\lambda + P)y_k\| \quad \text{and} \quad \|Py_k\| \geq \|\lambda y_k - Py_k\| - 1 \rightarrow 1.$$

It thus follows that  $|\lambda + 1| = 0$ , that is,  $\lambda = -1$ .

Now we turn to the proof of Theorem 6. We first deduce assertion (ii) from assertion (i) by duality as in the continuous case. Under the assumption of Theorem 6 and Pisier's renorming theorem [19], we can assume that  $X$  is uniformly convex.

*Proof of Theorem 6 (ii).* Using the spectral resolution of the identity of  $T$  on  $L_2(\Omega)$ , we obtain

$$\|f - F(f)\|_{L_2(\Omega)}^2 = \sum_{n=1}^{\infty} n \|T^{n-1}(1 - T^2)f\|_{L_2(\Omega)}^2, \quad f \in L_2(\Omega).$$

Polarizing this identity, we deduce, for  $f \in L_1(\Omega) \cap L_\infty(\Omega) \otimes X$  and  $g \in L_1(\Omega) \cap L_\infty(\Omega) \otimes X^*$ , that

$$\begin{aligned} \left| \int_{\Omega} \langle f - F(f), g - F(g) \rangle d\mu \right| &\leq \left\| \left( \sum_{n=1}^{\infty} n^{q'-1} \|T^{n-1}(1 - T^2)g\|_{X^*}^{q'} \right)^{1/q'} \right\|_{L_{p'}(\Omega)} \\ &\quad \cdot \left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|T^{n-1}(1 - T^2)f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)} \\ &\leq 4 \left\| \left( \sum_{n=1}^{\infty} n^{q'-1} \|T^{n-1}(1 - T)g\|_{X^*}^{q'} \right)^{1/q'} \right\|_{L_{p'}(\Omega)} \\ &\quad \cdot \left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|T^{n-1}(1 - T)f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)}. \end{aligned}$$

Thus under the assumption of (ii) and admitting (i), we obtain

$$\|f - F(f)\|_{L_p(\Omega; X)} \lesssim \left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|T^{n-1}(1 - T)f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)}.$$

Thus assertion (ii) is proved.  $\square$

We will need some preparations on the  $H^\infty$  functional calculus for the proof of Theorem 6 (i). Our reference for the latter subject is [3]. Let  $A$  be a sectorial operator on a Banach space  $Y$  with angle  $\gamma$  and  $\omega > \gamma$ . Define  $H_0^\infty(\Sigma_\omega)$  to be the space of all bounded analytic functions  $\varphi$  on the sector  $\Sigma_\omega$  for which there exist two positive constants  $s$  and  $C$  such that

$$|\varphi(z)| \leq C \min\{|z|^s, |z|^{-s}\}, \quad \forall z \in \Sigma_\omega.$$

For any  $\varphi \in H_0^\infty(\Sigma_\omega)$ , we define

$$\varphi(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} \varphi(z)(z - A)^{-1} dz,$$

where  $\theta \in (\gamma, \omega)$  and  $\Gamma_\theta$  is the boundary  $\partial\Sigma_\theta$  oriented counterclockwise. Then  $\varphi(A)$  is a bounded operator on  $Y$ .

The following result is a variant of [16, Theorem 5]. The proof there works equally for the present setting without change. This was pointed to us by Christian Le Merdy (see [13, page 719]).

**Lemma 20.** *Let  $1 < q < \infty$  and  $\varphi, \psi \in H_0^\infty(\Sigma_\omega)$  with*

$$\int_0^\infty \psi(t) \frac{dt}{t} \neq 0.$$

*Then there exists a positive constant  $C$ , depending only on  $\varphi, \psi$  and  $q$ , such that*

$$\left( \int_0^\infty \|\varphi(tA)y\|^q \frac{dt}{t} \right)^{1/q} \leq C \left( \int_0^\infty \|\psi(tA)y\|^q \frac{dt}{t} \right)^{1/q}, \quad \forall y \in Y.$$

*Proof of Theorem 6 (i).* We will follow the pattern set up in the proof of Theorem 2. The main difficulty is to prove the following discrete analogue of Lemma 13:

$$(11) \quad \sum_{n=1}^{\infty} n^{q-1} \|T^n(T-1)f\|_{L_q(\Omega; X)}^q \lesssim \|f\|_{L_q(\Omega; X)}^q, \quad \forall f \in L_q(\Omega; X).$$

Contrary to Lemma 13, the proof of the above inequality is much more involved. We will adapt the proof of [14, Proposition 3.2] which is based on the  $H^\infty$  functional calculus.

By Theorem 5,  $T$  is analytic as an operator on  $Y = L_q(\Omega; X)$  and we have (2). Let  $A = 1 - T$ . Then  $A$  is a sectorial operator on  $Y$  with angle  $\gamma$ . Fix  $\theta \in (\gamma, \pi/2)$ . Let  $L_\theta$  be the boundary of  $1 - B_\theta$  oriented counterclockwise (see figure 2).

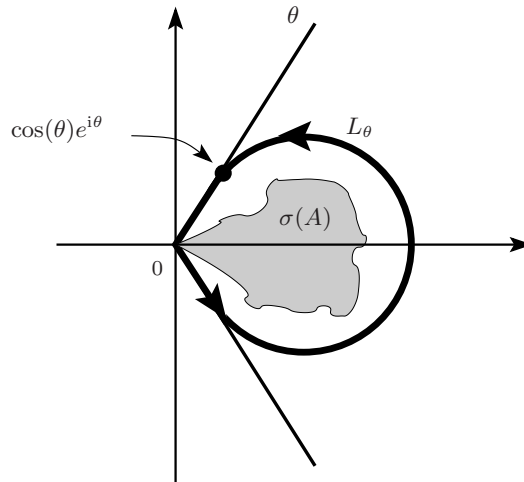


FIGURE 2.

Let  $\varphi_n(z) = n^{1/q'} z(1-z)^n$ . Then by the Dunford functional calculus

$$\frac{1}{2\pi i} \int_{L_\theta} \varphi_n(z)(z - A)^{-1} dz = \varphi_n(A) \quad \text{and} \quad \frac{1}{2\pi i} \int_{L_\theta} \varphi_n(z)(z + A)^{-1} dz = 0.$$

Thus

$$n^{1/q'} T^n(1-T) = \varphi_n(A) = \frac{1}{\pi i} \int_{L_\theta} \varphi_n(z) A(z-A)^{-1}(z+A)^{-1} dz.$$

Fix  $f$  in the unit ball of  $Y$ . Then

$$\sum_{n=1}^{\infty} n^{q-1} \|T^n(T-1)f\|_Y^q \lesssim \int_{L_\theta} \sum_{n=1}^{\infty} |\varphi_n(z)|^q \|A(z-A)^{-1}(z+A)^{-1}f\|_Y^q |dz|.$$

Note that for any  $z \in L_\theta$ , an elementary calculation shows that

$$\sum_{n=1}^{\infty} |\varphi_n(z)|^q \leq \sup_{\lambda \in B_\theta} \sum_{n=1}^{\infty} n^{q-1} |\lambda|^{nq} |1-\lambda|^q \lesssim \sup_{\lambda \in B_\theta} \frac{|1-\lambda|^q}{(1-|\lambda|)^q} \lesssim 1,$$

where the relevant constants depend only on  $q$  and  $\theta$ . On the other hand, by the  $H^\infty$  functional calculus,  $A^{1/q}(z+A)^{-1}$  is a bounded operator on  $Y$ . Then we deduce

$$\sum_{n=1}^{\infty} n^{q-1} \|T^n(T-1)f\|_Y^q \lesssim \int_{L_\theta} \|A^{1/q'}(z-A)^{-1}f\|_Y^q |dz|.$$

The contour  $L_\theta$  is the juxtaposition of a part  $L_{\theta,1}$  of  $\Gamma_\theta$  (recalling that  $\Gamma_\theta$  is the boundary of the sector  $\Sigma_\theta$ ) and the curve  $L_{\theta,2}$  going from  $\cos(\theta)e^{-i\theta}$  to  $\cos(\theta)e^{i\theta}$  counterclockwise along the circle of center 1 and radius  $\sin\theta$ . Accordingly,

$$\int_{L_\theta} \|A^{1/q'}(z-A)^{-1}f\|_Y^q |dz| = \int_{L_{\theta,1}} \|A^{1/q'}(z-A)^{-1}f\|_Y^q |dz| + \int_{L_{\theta,2}} \|A^{1/q'}(z-A)^{-1}f\|_Y^q |dz|.$$

Since  $L_{\theta,2} \cap \sigma(A) = \emptyset$ , the function  $z \mapsto \|A^{1/q'}(z-A)^{-1}\|$  is bounded on  $L_{\theta,2}$ . Thus the second integral in the right hand side above is majorized by a constant independent of  $f$  (recalling that  $\|f\|_Y \leq 1$ ). For the first one, we have

$$\begin{aligned} \int_{L_{\theta,1}} \|A^{1/q'}(z-A)^{-1}f\|_Y^q |dz| &\leq \sum_{\varepsilon=\pm 1} \int_0^\infty \|A^{1/q'}(te^{\varepsilon i\theta} - A)^{-1}f\|_Y^q dt \\ &= \sum_{\varepsilon=\pm 1} \int_0^\infty \|(tA)^{1/q'}(e^{\varepsilon i\theta} - tA)^{-1}f\|_Y^q \frac{dt}{t} \\ &= \sum_{\varepsilon=\pm 1} \int_0^\infty \|\varphi_\varepsilon(tA)f\|_Y^q \frac{dt}{t}, \end{aligned}$$

where

$$\varphi_\varepsilon(z) = \frac{z^{1/q'}}{e^{\varepsilon i\theta} - z}, \quad \varepsilon = \pm 1.$$

Note that  $\varphi_\varepsilon \in H_0^\infty(\Sigma_\omega)$  for  $\omega \in (\theta, \pi/2)$ . On the other hand, the function  $\psi$  defined by  $\psi(z) = ze^{-z}$  belongs to  $H_0^\infty(\Sigma_\omega)$  too. Thus applying Lemma 20, we get

$$\int_0^\infty \|\varphi_\varepsilon(tA)f\|_Y^q \frac{dt}{t} \lesssim \int_0^\infty \|\psi(tA)f\|_Y^q \frac{dt}{t} = \int_0^\infty \|t \partial T_t f\|_Y^q \frac{dt}{t},$$

where  $\{T_t\}_{t>0} = \{e^{-tA}\}_{t>0}$  is the semigroup already used at the beginning of the proof of Theorem 5. Thus by Lemma 13,

$$\int_0^\infty \|\psi(tA)f\|_Y^q \frac{dt}{t} \lesssim \|f\|_Y \lesssim 1.$$

Combining all preceding inequalities, we finally get

$$\sum_{n=1}^{\infty} n^{q-1} \|T^n(T-1)f\|_{L_q(\Omega; X)}^q \lesssim 1$$

for any  $f$  in the unit ball of  $Y$ . This yields (11) by homogeneity.

Armed with (11), we can finish the proof of Theorem 6 (i) by Stein's complex interpolation machinery as in the continuous case. To that end, first recall that Lemma 14 is deduced by approximation from its discrete analogue in [15]. Thus, although not explicitly stated there, the discrete analogue of Lemma 14 is indeed obtained during the proof of [15, Theorem 2.3]. Then the interpolation arguments in the previous section can be modified to the present discrete setting. We

refer the reader to [23] for the necessary ingredients. However, note that the presentation of [23] is quite brief, it is developed in detail in [9]. We leave the details to the reader. Thus the proof of Theorem 6 is complete.  $\square$

## 5. OPEN PROBLEMS

We conclude this article by some open problems. The first one concerns Theorem 6. Note that in that theorem the contraction  $T$  is assumed to be the square of another symmetric Markovian operator. Compared with the continuous case, this assumption is natural since every operator in a symmetric diffusion semigroup is automatically the square of a symmetric Markovian operator. However, a less restrictive assumption would be that  $T$  is a selfadjoint contraction on  $L_2(\Omega)$  and its spectrum does not contain  $-1$ . Under this assumption,  $T$  is analytic. If in addition  $T$  is a contraction on  $L_p(\Omega)$  for every  $1 \leq p \leq \infty$ , then  $T$  is also analytic on  $L_p(\Omega)$  for every  $1 < p < \infty$ .

**Problem 21.** *Let  $T$  be a positive contraction on  $L_p(\Omega)$  for every  $1 \leq p \leq \infty$  with  $T1 = 1$ . Assume that  $T$  is selfadjoint on  $L_2(\Omega)$  and its spectrum does not contain  $-1$ .*

- (i) *Let  $X$  be a uniformly convex Banach space. Is the extension of  $T$  to  $L_p(\Omega; X)$  analytic for every  $1 < p < \infty$  (or equivalently, for one  $1 < p < \infty$ )?*
- (ii) *Let  $X$  be a Banach space of martingale cotype  $q$  and  $1 < p < \infty$ . Does one have*

$$\left\| \left( \sum_{n=1}^{\infty} n^{q-1} \|(T^n - T^{n-1})f\|_X^q \right)^{1/q} \right\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X)?$$

An affirmative answer to part (i) would imply the same for part (ii). In the spirit of [21], one can ask a similar question as part (i) for  $K$ -convex  $X$ . In fact, we do not know whether Theorem 5 holds for  $K$ -convex targets (see [12] for related results). This is the discrete analogue of Problem 11 (i) of [26] for symmetric diffusion semigroups.

**Problem 22.** *Does Theorem 5 remain true if  $X$  is assumed  $K$ -convex?*

**Remark 23.** The answers to Problem 21 (i) and Problem 22 are both positive if  $X$  is a complex interpolation space between a Hilbert space and a Banach space. This is the case if  $X$  is a  $K$ -convex Banach lattice thanks to [22]. More generally, let  $(X_0, X_1)$  be a compatible pair of Banach spaces, and let  $X = (X_0, X_1)_\theta$  with  $0 < \theta < 1$ . Assume that  $T$  is a contraction on both  $X_0$  and  $X_1$ , and  $T$  is analytic on  $X_1$ . Then  $T$  is analytic on  $X$  too.

Indeed, since the semigroup  $\{e^{(T-1)t}\}_{t>0}$  is analytic on  $X_1$ , by Stein's complex interpolation, it is analytic on  $X$  too. Thus by Lemma 18, it remains to show that as an operator on  $X$ , the spectrum of  $T$  intersects  $\mathbb{T}$  at most at the point 1. The latter is equivalent to  $\lim_{n \rightarrow \infty} \|T^n(T-1) : X \rightarrow X\| = 0$ , thanks to Katznelson and Tzafriri's theorem [11]. Using the analyticity of  $T$  on  $X_1$  and interpolation, we get

$$\|T^n(T-1) : X \rightarrow X\| \lesssim \frac{1}{n^\theta}.$$

So we are done.

Hytönen [7] studied another variant of Stein's inequality (1) in the vector-valued setting. Like [15], his main theorem deals with the Poisson semigroup subordinated to a symmetric diffusion semigroup for a general UMD space  $X$ , except when  $X$  is a complex interpolation space between a Hilbert space and another UMD space. In the same spirit of this article, one may ask whether the main result of [7] remains true for any symmetric diffusion semigroup and any UMD space  $X$ . It is easier to formulate this problem in the discrete case as follows. Let  $T$  be as in Theorem 6.

**Problem 24.** *Let  $T$  be as in Theorem 6,  $X$  be a UMD space and  $1 < p < \infty$ . Does one have*

$$\mathbb{E} \left\| \sum_{n=1}^{\infty} \varepsilon_n \sqrt{n} (T^n - T^{n-1})f \right\|_{L_p(\Omega; X)} \approx \|f - F(f)\|_{L_p(\Omega; X)}, \quad \forall f \in L_p(\Omega; X)?$$

Here  $\{\varepsilon_n\}$  is a sequence of symmetric random variables taking values  $\pm 1$  on a probability space and  $\mathbb{E}$  is the corresponding expectation.

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