

## NONCOMMUTATIVE DAVIS TYPE DECOMPOSITIONS AND APPLICATIONS

NARCISSE RANDRIANANTOANINA, LIAN WU, AND QUANHUA XU

ABSTRACT. We prove the noncommutative Davis decomposition for the column Hardy space  $\mathcal{H}_p^c$  for all  $0 < p \leq 1$ . A new feature of our Davis decomposition is a simultaneous control of  $\mathcal{H}_1^c$  and  $\mathcal{H}_q^c$  norms for any noncommutative martingale in  $\mathcal{H}_1^c \cap \mathcal{H}_q^c$  when  $q \geq 2$ . As applications, we show that the Burkholder/Rosenthal inequality holds for bounded martingales in a noncommutative symmetric space associated with a function space  $E$  that is either an interpolation of the couple  $(L_p, L_2)$  for some  $1 < p < 2$  or is an interpolation of the couple  $(L_2, L_q)$  for some  $2 < q < \infty$ . We also obtain the corresponding  $\Phi$ -moment Burkholder/Rosenthal inequality for Orlicz functions that are either  $p$ -convex and 2-concave for some  $1 < p < 2$  or are 2-convex and  $q$ -concave for some  $2 < q < \infty$ .

## 1. INTRODUCTION

This paper follows the current line of investigation on noncommutative martingale inequalities. Many classical results have been generalized to the noncommutative setting. One of them, directly relevant to the subject of the present paper is the so called Davis decomposition ([10]). The original Davis decomposition is fundamental in classical martingale theory and has been generalized to various contexts. For instance, the vector-valued case is nowadays well-known in the literature, a version of the Davis decomposition for a special class of martingales called Hardy martingales was studied recently in [33].

Recall that for the noncommutative setting, the Davis decomposition for the noncommutative martingale Hardy spaces  $\mathcal{H}_1$  was obtained in [35] using duality arguments. A constructive approach appeared in [24] for the space  $\mathcal{H}_p$  for  $1 \leq p < 2$ . The noncommutative Davis decomposition has proven to be a powerful tool in noncommutative martingale inequalities; for instance, it plays a prominent role in establishing various forms of Doob maximal inequalities in [19] as well as in the study of continuous time noncommutative martingale inequalities in [24].

It is our intention in this paper to investigate the case  $0 < p \leq 1$ . We provide a Davis type decomposition for certain class of sequences in the column- $L_p$ -spaces. This can be roughly described as splitting any adapted sequence in the column- $L_p$ -space into a diagonal part and an adapted sequence that belongs to the corresponding conditioned column- $L_p$ -space. Even for the commutative case, our result for  $0 < p < 1$  do not seem to be available in the literature. An important new feature of our Davis decomposition is that when applied to martingales, it gives a simultaneous control of the column Hardy spaces  $\mathcal{H}_1^c$  and  $\mathcal{H}_q^c$  norms when  $q \geq 2$ . More precisely, for any given  $q \geq 2$ , any martingale  $x$  in the intersection of Hardy spaces  $\mathcal{H}_1^c \cap \mathcal{H}_q^c$  can be written as a sum of two martingales  $y$  and  $z$  such that  $\|y\|_{\mathfrak{h}_p^d} + \|z\|_{\mathfrak{h}_p^c} \leq C\|x\|_{\mathcal{H}_p^c}$  for all  $1 \leq p \leq q$  where  $\mathfrak{h}_p^d$  and  $\mathfrak{h}_p^c$  denote the diagonal Hardy space and the column conditioned Hardy space respectively. Decompositions with such simultaneous control of norms are very useful for the study of noncommutative martingales in continuous time (see [24]). They are also very essential in the study of martingale Hardy spaces associated to noncommutative symmetric spaces ([40, 41]). Indeed, our primary motivation comes from the latter. The simultaneous nature of our Davis decomposition allows us to extend it through the use of interpolation to martingales in certain noncommutative symmetric spaces satisfying some natural conditions. This in turn provides a general framework to systematically transfer results involving square functions which are generally referred to as Burkholder-Gundy inequality to combinations of conditioned square functions and diagonal parts known as Burkholder/Rosenthal inequality.

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Recall the noncommutative Burkholder/Rosenthal inequalities from [25]. It asserted that if  $2 \leq p < \infty$  and  $x = (x_n)_{n \geq 1}$  is a noncommutative martingale that is  $L_p$ -bounded then

$$(1.1) \quad \|x\|_p \simeq_p \max \left\{ \|s_c(x)\|_p, \|s_r(x)\|_p, \left( \sum_{n \geq 1} \|dx_n\|_p^p \right)^{1/p} \right\},$$

where  $s_c(x)$  and  $s_r(x)$  denote the column and row versions of conditioned square functions which we refer to the next section for formal definitions. The corresponding inequalities for the range  $1 < p < 2$  dual to (1.1) reads as follows: if  $x = (x_n)_{n \geq 1}$  is a noncommutative martingale in  $L_2(\mathcal{M})$  then

$$(1.2) \quad \|x\|_p \simeq_p \inf \left\{ \|s_c(y)\|_p + \|s_r(z)\|_p + \left( \sum_{n \geq 1} \|dw_n\|_p^p \right)^{1/p} \right\},$$

where the infimum is taken over all  $x = y + z + w$  with  $y, z$ , and  $w$  martingales. The differences between the two cases  $1 < p < 2$  and  $2 \leq p < \infty$  are now well-understood in the field. The natural next step is to classify noncommutative symmetric spaces for which either (1.1) or (1.2) remains valid. Naturally, interpolation plays a significant role in this line of research. It was established in [11] that if a function space  $E$  is an interpolation space of the couple  $(L_p, L_q)$  for  $2 < p < q < \infty$ , then

$$(1.3) \quad \|x\|_{E(\mathcal{M})} \simeq_E \max \left\{ \|s_c(x)\|_{E(\mathcal{M})}, \|s_r(x)\|_{E(\mathcal{M})}, \|(dx_n)_{n \geq 1}\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \right\}.$$

On the other hand, the dual result was proved in [40] which states that if  $E$  is a symmetric space that is an interpolation space of the couple  $(L_p, L_q)$  for  $1 < p < q < 2$  then

$$(1.4) \quad \|x\|_{E(\mathcal{M})} \simeq_E \inf \left\{ \|s_c(y)\|_{E(\mathcal{M})} + \|s_r(z)\|_{E(\mathcal{M})} + \|(dw_n)_{n \geq 1}\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} \right\},$$

where as in (1.2), the infimum is taken over all decompositions  $x = y + z + w$  with  $y, z$ , and  $w$  martingales in  $E(\mathcal{M}, \tau)$ . The situation at the endpoints were left open in [40]. We solve this problem positively. More precisely, we obtain that (1.3) and (1.4) remain valid for  $E$  being an interpolation of the couple  $(L_2, L_q)$  for  $2 < q < \infty$ , respectively,  $(L_p, L_2)$  for  $1 < p < 2$ . As noted earlier, our new Davis decomposition provides the decisive ingredient in our argument.

In the last part of the paper, we consider the noncommutative Burkholder/Rosenthal inequalities using moments associated with Orlicz spaces. These moments are generally referred to in the literature as  $\Phi$ -moment inequalities. For the classical setting, this topic goes back to [7, 8]. For noncommutative martingales, this line of research was initiated by Bekjan and Chen in [1] where they provided several  $\Phi$ -moment inequalities such as  $\Phi$ -moment versions of the noncommutative Khintchine inequalities and noncommutative Burkholder-Gundy inequality among other closely related results. Subsequently,  $\Phi$ -moment analogues of other inequalities were also considered (see for instance, [2, 12, 14]). Recently, the sharpest result for the  $\Phi$ -moment analogue of the noncommutative Burkholder-Gundy inequalities was obtained by Jiao *et al.* (see [22, Theorem 7.2]). Using our general approach, we extend their result to the  $\Phi$ -moment analogues of the noncommutative Burkholder inequalities. More precisely, if the Orlicz function is  $p$ -convex and 2-concave (for some  $1 < p < 2$ ), respectively 2-convex and  $q$ -concave (for some  $2 < q < \infty$ ), then the  $\Phi$ -moment analogue of (1.3), respectively (1.4), holds. Our results in this part solve some problems left open in [41].

The paper is organized as follows. In the next section, we collect notions and notation from noncommutative symmetric spaces and noncommutative martingales necessary for the whole paper. Section 3 is devoted to the statements and proofs of our version of noncommutative Davis decompositions for the full range  $0 < p \leq 1$  (Theorem 3.1 and Theorem 3.3) along with some immediate corollaries. We also provide in this section an extension of the Davis decomposition to the case of noncommutative symmetric spaces (Theorem 3.9). In the last section, we give the main applications in the forms of various Burkholder/Rosenthal inequalities for martingales in noncommutative symmetric spaces and their modular versions.

## 2. PRELIMINARIES

**2.1. Noncommutative symmetric spaces.** Throughout this paper,  $\mathcal{M}$  will always denote a semifinite von Neumann algebra equipped with a faithful normal semifinite trace  $\tau$ .  $L_0(\mathcal{M}, \tau)$  denotes the associated

topological  $*$ -algebra of measurable operators and  $\mu(x)$  the generalized singular number of an element  $x \in L_0(\mathcal{M}, \tau)$ . If  $\mathcal{M}$  is the abelian von Neumann algebra  $L_\infty(0, \infty)$  with the trace given by integration with respect to Lebesgue measure,  $L_0(\mathcal{M}, \tau)$  becomes the space of those measurable complex functions on  $(0, \infty)$  which are bounded except on a set of finite measure and for  $f \in L_0(\mathcal{M}, \tau)$ ,  $\mu(f)$  is the usual decreasing rearrangement of  $f$ . We refer to [36] for more information on noncommutative integration.

A Banach function space  $(E, \|\cdot\|_E)$  of measurable functions on the interval  $(0, \infty)$  is called *symmetric* if for any  $g \in E$  and any  $f \in L_0(0, \infty)$  with  $\mu(f) \leq \mu(g)$ , we have  $f \in E$  and  $\|f\|_E \leq \|g\|_E$ . For such a space  $E$ , we define the corresponding noncommutative space by setting:

$$E(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) : \mu(x) \in E\}.$$

Equipped with the norm  $\|x\|_{E(\mathcal{M}, \tau)} := \|\mu(x)\|_E$ ,  $E(\mathcal{M}, \tau)$  becomes a complex Banach space ([28, 43]) and is usually referred to as the *noncommutative symmetric space* associated with  $\mathcal{M}$  and  $E$ . An extensive discussion of the various properties of such spaces can be found in [15, 16, 17, 37, 43]. We remark that if  $1 \leq p < \infty$  and  $E = L_p(0, \infty)$ , then  $E(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau)$  where  $L_p(\mathcal{M}, \tau)$  is the usual noncommutative  $L_p$ -space associated with  $(\mathcal{M}, \tau)$ .

In this paper, we will only consider symmetric spaces that are interpolations of the couple  $(L_p, L_q)$  for  $1 \leq p < q \leq \infty$ . For a given compatible Banach couple  $(X, Y)$ , we recall that a Banach space  $Z$  is called an *interpolation space* if  $X \cap Y \subseteq Z \subseteq X + Y$  and whenever a bounded linear operator  $T : X + Y \rightarrow X + Y$  is such that  $T(X) \subseteq X$  and  $T(Y) \subseteq Y$ , we have  $T(Z) \subseteq Z$  and  $\|T : Z \rightarrow Z\| \leq C \max\{\|T : X \rightarrow X\|, \|T : Y \rightarrow Y\|\}$  for some constant  $C$ . In this case, we write  $Z \in \text{Int}(X, Y)$ . We refer to [5, 6, 27] for more unexplained definitions and terminology from interpolation. We record here two facts that we will use repeatedly. The first is the fact that interpolation lifts to noncommutative symmetric spaces. More precisely, we have:

**Lemma 2.1** ([37]). *Let  $1 \leq p < q \leq \infty$ . Assume that  $E \in \text{Int}(L_p, L_q)$  and  $\mathcal{M}$  and  $\mathcal{N}$  are semifinite von Neumann algebras. Let  $T : L_p(\mathcal{M}) + L_q(\mathcal{M}) \rightarrow L_p(\mathcal{N}) + L_q(\mathcal{N})$  be a linear operator such that  $T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{N})$  and  $T : L_q(\mathcal{M}) \rightarrow L_q(\mathcal{N})$  are bounded. Then  $T$  maps  $E(\mathcal{M})$  into  $E(\mathcal{N})$  and the resulting operator  $T : E(\mathcal{M}) \rightarrow E(\mathcal{N})$  is bounded and satisfies*

$$\|T : E(\mathcal{M}) \rightarrow E(\mathcal{N})\| \leq C \max\{\|T : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{N})\|, \|T : L_q(\mathcal{M}) \rightarrow L_q(\mathcal{N})\|\},$$

where  $C$  is the interpolation constant of  $E$  relative to the couple  $(L_p, L_q)$ .

The second is the fact that any function space  $E \in \text{Int}(L_p, L_q)$  can be described by a concrete interpolation method involving the notions of  $K$ -functionals and  $J$ -functionals. We only describe here a version that we need. First, we recall that for a compatible couple  $(X, Y)$ , the  $J$ -functional of  $z \in X \cap Y$  is given by

$$J(x, t; X, Y) = \max\{\|z\|_X, t\|z\|_Y\}, \quad t > 0.$$

The dual functional called  $K$ -functional of  $z \in X + Y$  is given by

$$K(z, t; X, Y) = \inf\{\|x\|_X + t\|y\|_Y : z = x + y\}, \quad t > 0.$$

Fix  $(X, Y)$  and a symmetric Banach function space  $F$  on  $(0, \infty)$ . For  $x \in X + Y$ , let  $x = \sum_{\nu \in \mathbb{Z}} u_\nu$  be a (discrete) representation of  $x$  and set:

$$\underline{j}(\{u_\nu\}_\nu, t) = \sum_{\gamma \geq \nu+1} 2^{-\gamma} J(u_\gamma, 2^\gamma) \quad \text{for } t \in [2^\nu, 2^{\nu+1}).$$

We define the interpolation space  $(X, Y)_{F, \underline{j}}$  to be the space of elements  $x \in X + Y$  such that

$$\|x\|_{F, \underline{j}} := \inf\left\{\left\|\underline{j}(\{u_\nu\}_\nu, \cdot)\right\|_F\right\} < \infty$$

with the infimum being taken over all representations of  $x$  as above. By combining results of Brudnyi and Krugliak (see [27, Theorem 6.3]), [4], and [37, Corollary 2.2], we derive the following general result:

**Lemma 2.2.** *Let  $1 \leq p < q \leq \infty$  and  $E$  be a symmetric Banach function space on  $(0, \infty)$  with  $E \in \text{Int}(L_p, L_q)$ . There exists a symmetric Banach function space  $F$  on  $(0, \infty)$  so that for every semifinite von Neumann algebra  $(\mathcal{N}, \sigma)$ ,*

$$E(\mathcal{N}) = (L_p(\mathcal{N}), L_q(\mathcal{N}))_{F, \underline{j}},$$

with equivalent norms depending only on  $E$ ,  $p$ , and  $q$ .

**2.2. Martingales and Hardy spaces.** We now briefly describe the general setup for martingales in non-commutative symmetric spaces. Denote by  $(\mathcal{M}_n)_{n \geq 1}$  an increasing sequence of von Neumann subalgebras of  $\mathcal{M}$  whose union is weak\*-dense in  $\mathcal{M}$ . For  $n \geq 1$ , we assume that there exists a trace preserving conditional expectation  $\mathcal{E}_n$  from  $\mathcal{M}$  onto  $\mathcal{M}_n$ . It is well-known that if  $\tau_n$  denotes the restriction of  $\tau$  on  $\mathcal{M}_n$ , then  $\mathcal{E}_n$  extends to a contractive projection from  $L_p(\mathcal{M}, \tau)$  onto  $L_p(\mathcal{M}_n, \tau_n)$  for all  $1 \leq p \leq \infty$ . More generally, if  $E$  is a symmetric Banach function space on  $(0, \infty)$  that belongs to  $\text{Int}(L_1, L_\infty)$ , then for every  $n \geq 1$ ,  $\mathcal{E}_n$  is bounded from  $E(\mathcal{M}, \tau)$  onto  $E(\mathcal{M}_n, \tau_n)$ .

**Definition 2.3.** A sequence  $x = (x_n)_{n \geq 1}$  in  $L_1(\mathcal{M}) + \mathcal{M}$  is called a *noncommutative martingale* with respect to  $(\mathcal{M}_n)_{n \geq 1}$  if  $\mathcal{E}_n(x_{n+1}) = x_n$  for every  $n \geq 1$ .

If in addition, all  $x_n$ 's belong to  $E(\mathcal{M})$  then  $x$  is called an  $E(\mathcal{M})$ -martingale. In this case, we set

$$\|x\|_{E(\mathcal{M})} = \sup_{n \geq 1} \|x_n\|_{E(\mathcal{M})}.$$

If  $\|x\|_{E(\mathcal{M})} < \infty$ , then  $x$  is called a bounded  $E(\mathcal{M})$ -martingale.

Let  $x = (x_n)$  be a noncommutative martingale with respect to  $(\mathcal{M}_n)_{n \geq 1}$ . Define  $dx_n = x_n - x_{n-1}$  for  $n \geq 1$  with the usual convention that  $x_0 = 0$ . The sequence  $dx = (dx_n)$  is called the *martingale difference sequence* of  $x$ . A martingale  $x$  is called a *finite martingale* if there exists  $N$  such that  $dx_n = 0$  for all  $n \geq N$ . In the sequel, for any operator  $x \in L_1(\mathcal{M}) + \mathcal{M}$ , we denote  $x_n = \mathcal{E}_n(x)$  for  $n \geq 1$ . We observe that conversely, if  $E \in \text{Int}(L_p, L_q)$  for  $1 < p \leq q < \infty$  and satisfies the Fatou property, then any bounded  $E(\mathcal{M})$ -martingale  $x = (x_n)_{n \geq 1}$  is of the form  $(\mathcal{E}_n(x_\infty))_{n \geq 1}$  where  $x_\infty \in E(\mathcal{M})$  satisfying  $\|x\|_{E(\mathcal{M})} \approx_E \|x_\infty\|_{E(\mathcal{M})}$ , with equality if  $E$  is an exact interpolation space.

Let us now review the definitions of the square functions and Hardy spaces of noncommutative martingales. Following [36], we define the following column square functions relative to a martingale  $x = (x_n)$ :

$$S_{c,n}(x) = \left( \sum_{k=1}^n |dx_k|^2 \right)^{1/2}, \quad S_c(x) = \left( \sum_{k=1}^{\infty} |dx_k|^2 \right)^{1/2}.$$

For  $0 < p < \infty$ , the column martingale Hardy space  $\mathcal{H}_p^c(\mathcal{M})$  is defined to be the space of all martingales  $x$  for which  $S_c(x)$  belongs to  $L_p(\mathcal{M}, \tau)$ . More generally, if  $E$  is a symmetric Banach function space, we define  $\mathcal{H}_E^c(\mathcal{M})$  to be the space of all martingales  $x = (x_n)_{n \geq 1}$  in  $E(\mathcal{M})$  for which  $S_c(x)$  belongs to  $E(\mathcal{M})$ .  $\mathcal{H}_E^c(\mathcal{M})$  becomes a Banach space when equipped with the norm

$$\|x\|_{\mathcal{H}_E^c} = \|S_c(x)\|_{E(\mathcal{M})}.$$

We now consider the conditioned version of  $\mathcal{H}_p^c$  developed in [25]. Let  $x = (x_n)_{n \geq 1}$  be a martingale in  $L_2(\mathcal{M}) + \mathcal{M}$ . We set (with the convention that  $\mathcal{E}_0 = \mathcal{E}_1$ ):

$$s_{c,n}(x) = \left( \sum_{k=1}^n \mathcal{E}_{k-1} |dx_k|^2 \right)^{1/2}, \quad s_c(x) = \left( \sum_{k=1}^{\infty} \mathcal{E}_{k-1} |dx_k|^2 \right)^{1/2}.$$

For  $2 \leq p < \infty$ , the column conditioned martingale Hardy space  $\mathfrak{h}_p^c(\mathcal{M})$  is defined to be the space of all martingale  $x$  for which  $s_c(x)$  belongs to  $L_p(\mathcal{M})$  equipped with the norm  $\|x\|_{\mathfrak{h}_p^c} = \|s_c(x)\|_p$ . More generally, if  $E$  is a symmetric Banach function space with the Fatou property and  $E \subseteq L_2 + L_\infty$ , we define  $\mathfrak{h}_E^c(\mathcal{M})$  to be the set of all martingale  $x = (x_n)_{n \geq 1}$  in  $E(\mathcal{M})$  for which  $s_c(x)$  belongs to  $E(\mathcal{M})$ .  $\mathfrak{h}_E^c(\mathcal{M})$  becomes a Banach space when equipped with the norm

$$\|x\|_{\mathfrak{h}_E^c} = \|s_c(x)\|_{E(\mathcal{M})}.$$

For  $0 < p < 2$  or  $E \not\subseteq L_2 + L_\infty$ , the definition is more involved. In this range, we define  $\mathfrak{h}_p^c(\mathcal{M})$  to be the completion of the linear space of finite martingales in  $L_p(\mathcal{M}) \cap \mathcal{M}$  under the (quasi) norm  $\|x\|_{\mathfrak{h}_p^c} = \|s_c(x)\|_p$ . We postpone the description of  $\mathfrak{h}_E^c(\mathcal{M})$  until after the next discussion.

In the sequel, we will use more general versions of these spaces by considering arbitrary sequences in place of martingale difference sequences. For  $0 < p \leq \infty$ , and a finite sequence  $a = (a_n)_{n \geq 1}$  in  $L_p(\mathcal{M})$ , we set

$$\|a\|_{L_p(\mathcal{M}; \ell_2^c)} = \left\| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \right\|_p.$$

The completion (relative to the  $w^*$ -topology for  $p = \infty$ ) of the space of finite sequences in  $L_p(\mathcal{M})$  equipped with the (quasi) norm  $\|\cdot\|_{L_p(\mathcal{M}; \ell_2^c)}$  will be denoted by  $L_p(\mathcal{M}; \ell_2^c)$ . We will also need the conditioned  $L_p$ -spaces which is defined as follows: for  $0 < p \leq \infty$  and a finite sequence  $a = (a_n)_{n \geq 1}$  in  $L_p(\mathcal{M}) \cap \mathcal{M}$ , we set

$$\|a\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} = \left\| \left( \sum_{n \geq 1} \mathcal{E}_{n-1}(a_n^* a_n) \right)^{1/2} \right\|_p.$$

For  $0 < p < \infty$ , the completion of the space of finite sequences in  $L_p(\mathcal{M}) \cap \mathcal{M}$  equipped with the (quasi) norm  $\|\cdot\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)}$  will be denoted by  $L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)$ . For  $p = \infty$ , we may define  $L_\infty^{\text{cond}}(\mathcal{M}; \ell_2^c)$  as the set of all sequences  $a = (a_n)_{n \geq 1}$  in  $\mathcal{M}$  for which the increasing sequence  $(\sum_{k=1}^n \mathcal{E}_{k-1}(a_k^* a_k))_{n \geq 1}$  is bounded in  $\mathcal{M}$ . In this case,

$$\|a\|_{L_\infty^{\text{cond}}(\mathcal{M}; \ell_2^c)} = \sup_{n \geq 1} \left\| \left( \sum_{k=1}^n \mathcal{E}_{k-1}(a_k^* a_k) \right)^{1/2} \right\|_\infty.$$

A very crucial result of Junge [23] states that there exists an isometric embedding of  $\mathfrak{h}_p^c(\mathcal{M})$  into a noncommutative  $L_p$ -space. Namely, for  $0 < p \leq \infty$ , we have an isometry

$$U : L_p^{\text{cond}}(\mathcal{M}; \ell_2^c) \rightarrow L_p(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))$$

with the property that if  $a = (a_n) \in L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)$  and  $b = (b_n) \in L_q^{\text{cond}}(\mathcal{M}; \ell_2^c)$  with  $1/p + 1/q \leq 1$ , then

$$U(a)^* U(b) = \left( \sum_{n \geq 1} \mathcal{E}_{n-1}(a_n^* b_n) \right) \otimes e_{1,1} \otimes e_{1,1},$$

where  $(e_{i,j})_{i,j \geq 1}$  denotes the unit matrices in  $B(\ell_2(\mathbb{N}))$ . If we denote by  $\mathcal{D}_c : \mathfrak{h}_p^c(\mathcal{M}) \rightarrow L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)$  the natural map  $x \mapsto (dx_n)$ , then its composition with  $U$  induces the isometric embedding:

$$U \mathcal{D}_c : \mathfrak{h}_p^c(\mathcal{M}) \rightarrow L_p(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))$$

with the property that if  $x \in \mathfrak{h}_p^c(\mathcal{M})$ ,  $y \in \mathfrak{h}_q^c(\mathcal{M})$ , and  $1/p + 1/q \leq 1$  then

$$U \mathcal{D}_c(x)^* U \mathcal{D}_c(y) = \left( \sum_{n \geq 1} \mathcal{E}_{n-1}(dx_n^* dy_n) \right) \otimes e_{1,1} \otimes e_{1,1}.$$

In particular, for  $x \in \mathfrak{h}_2^c(\mathcal{M})$ , we have

$$(2.1) \quad |U \mathcal{D}_c(x)|^2 = (s_c(x))^2 \otimes e_{1,1} \otimes e_{1,1}.$$

Now let  $E$  be a symmetric Banach function space with the Fatou property. We define  $\mathfrak{h}_E^c(\mathcal{M})$  to be the set of all martingales  $x = (x_n)_{n \geq 1} \in \mathfrak{h}_1^c(\mathcal{M}) + \mathfrak{h}_\infty^c(\mathcal{M})$  for which  $U \mathcal{D}_c(x) \in E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))$ . Then for  $x \in \mathfrak{h}_E^c(\mathcal{M})$ , we set

$$\|x\|_{\mathfrak{h}_E^c} := \|U \mathcal{D}_c(x)\|_{E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))}.$$

Equipped with  $\|\cdot\|_{\mathfrak{h}_E^c}$ ,  $\mathfrak{h}_E^c(\mathcal{M})$  is a Banach space and  $U \mathcal{D}_c$  extends to an isometric embedding of  $\mathfrak{h}_E^c(\mathcal{M})$  into  $E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))$ . We note that if  $E \subseteq L_2 + L_\infty$ , then the two definitions of  $\mathfrak{h}_E^c(\mathcal{M})$  coincide.

All definitions and statements above admit corresponding row versions by passing to adjoints. For instance, the row square function of a martingale  $x$  is defined as  $S_r(x) = S_c(x^*)$ , and the row Hardy space  $\mathcal{H}_p^r(\mathcal{M})$  consists of all martingales  $x$  such that  $x^* \in \mathcal{H}_p^c(\mathcal{M})$ .

A third type of Hardy spaces that we will use in the sequel are the diagonal Hardy spaces. For  $0 < p \leq \infty$ , we recall that the diagonal Hardy space  $\mathfrak{h}_p^d(\mathcal{M})$  is the subspace of  $\ell_p(L_p(\mathcal{M}))$  consisting of

martingale difference sequences. This definition can be easily extended to the case of symmetric spaces by setting  $\mathfrak{h}_E^d(\mathcal{M})$  as the space of all martingales whose martingale difference sequences belong to  $E(\mathcal{M} \overline{\otimes} \ell_\infty)$ , equipped with the norm  $\|x\|_{\mathfrak{h}_E^d} := \|(dx_n)\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)}$ . We will denote by  $\mathcal{D}_d$  the isometric embedding of  $\mathfrak{h}_E^d$  into  $E(\mathcal{M} \overline{\otimes} \ell_\infty)$  given by  $x \mapsto (dx_n)_{n \geq 1}$ .

We will also make use of another type of diagonal spaces developed in [24]. For  $0 < p < 2$ , a sequence  $x = (x_n)$  belongs to  $L_p(\mathcal{M}; \ell_1^c)$  if there exist  $b_{k,n} \in L_2(\mathcal{M})$  and  $a_{k,n} \in L_q(\mathcal{M})$  where  $1/p = 1/2 + 1/q$  such that for every  $n \geq 1$ ,

$$(2.2) \quad x_n = \sum_{k \geq 1} b_{k,n}^* a_{k,n},$$

$\sum_{k,n \geq 1} |b_{k,n}|^2 \in L_1(\mathcal{M})$ , and  $\sum_{k,n \geq 1} |a_{k,n}|^2 \in L_{q/2}$ . We equip  $L_p(\mathcal{M}; \ell_1^c)$  with the (quasi) norm:

$$\|x\|_{L_p(\mathcal{M}; \ell_1^c)} = \inf \left\{ \left( \sum_{k,n \geq 1} \|b_{k,n}\|_2^2 \right)^{1/2} \left\| \left( \sum_{k,n \geq 1} |a_{k,n}|^2 \right)^{1/2} \right\|_q \right\},$$

where the infimum is taken over all factorizations (2.2). As in [24, Lemma 6.1.2], the unit ball of  $L_p(\mathcal{M}; \ell_1^c)$  coincides with the set of all sequences  $(\beta_n \alpha_n)$  satisfying the following inequality:

$$\left( \sum_{n \geq 1} \|\beta_n\|_2^2 \right)^{1/2} \left\| \left( \sum_{n \geq 1} |\alpha_n|^2 \right)^{1/2} \right\|_q \leq 1.$$

The following facts are clear from the definitions:  $L_1(\mathcal{M}; \ell_1^c) = \ell_1(L_1(\mathcal{M}))$ ,  $L_p(\mathcal{M}, \ell_1^c) \subseteq \ell_p(L_p(\mathcal{M}))$  for  $1 < p < 2$ , and  $\ell_p(L_p(\mathcal{M})) \subseteq L_p(\mathcal{M}; \ell_1^c)$  for  $0 < p < 1$ . The diagonal space  $\mathfrak{h}_p^{1c}(\mathcal{M})$  is the subspace of  $L_p(\mathcal{M}; \ell_1^c)$  consisting of martingale difference sequences.

### 3. DAVIS-TYPE DECOMPOSITIONS

The primary goal of this section is to provide extensions of Davis' decomposition for adapted sequences in  $L_p(\mathcal{M}; \ell_2^c)$  for all  $0 < p < 2$ . Our first result deals with the case  $2/3 \leq p < 2$ . In this range, we obtain a decomposition with simultaneous control of norms.

**Theorem 3.1.** *Let  $2/3 \leq p < 2$  and  $\xi = (\xi_n)_{n \geq 1}$  be an adapted sequence that belongs to  $L_p(\mathcal{M}; \ell_2^c) \cap L_\infty(\mathcal{M}; \ell_2^c)$ . Then there exist two adapted sequences  $y = (y_n)_{n \geq 1}$  and  $z = (z_n)_{n \geq 1}$  such that:*

- (i)  $\xi = y + z$ ;
- (ii)  $\|y\|_{\ell_p(L_p(\mathcal{M}))} + \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq 2 \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}$ ;
- (iii)  $\|y\|_{L_q(\mathcal{M}; \ell_2^c)} + \|z\|_{L_q(\mathcal{M}; \ell_2^c)} \leq 3 \|\xi\|_{L_q(\mathcal{M}; \ell_2^c)}$  for every  $2 \leq q \leq \infty$ .

For the proof of the theorem, we will need the following lemma which is an extension of [36, Lemma 1.1].

**Lemma 3.2.** *Let  $2 \leq p \leq \infty$  and  $1/p = 1/q + 1/r$ . For any sequence  $a = (a_n)_{n \geq 1}$  in  $L_q(\mathcal{M})$  and any  $A \in L_r(\mathcal{M})$ , we set  $B(a, A) = (a_n A)_{n \geq 1}$ . Then*

$$(3.1) \quad \left\| B(a, A) \right\|_{L_p(\mathcal{M}; \ell_2^c)} \leq \max \left\{ \|a\|_{L_q(\mathcal{M}; \ell_2^c)}, \|a\|_{L_q(\mathcal{M}; \ell_2^c)} \right\} \|A\|_r.$$

*Proof.* Clearly,  $\|B(a, A)\|_{L_p(\mathcal{M}; \ell_2^c)} = \left\| \sum_{n \geq 1} a_n A A^* a_n^* \right\|_{p/2}^{1/2}$ . Denote by  $s$  the conjugate index of  $p/2$ . By duality, we may fix  $B \in L_s(\mathcal{M})$  with  $B \geq 0$ ,  $\|B\|_s = 1$ , and such that

$$(3.2) \quad \psi(B) = \left\| \sum_{n \geq 1} a_n A A^* a_n^* \right\|_{p/2} = \tau \left( \sum_{n \geq 1} a_n A A^* a_n^* B \right).$$

Set  $\alpha = (r/2)/(q/2)'$  and  $\beta = s/(q/2)'$  where  $(q/2)'$  denotes the conjugate index of  $q/2$ . One can readily verify that  $(1 - \alpha^{-1})\beta = 1$ . We will apply the three lines lemma to the analytic function  $F$  defined for  $0 \leq \text{Re}(z) \leq 1$  by

$$F(z) = \tau \left( \sum_{n \geq 1} a_n (A A^*)^{\alpha z} a_n^* B^{\beta(1-z)} \right).$$

Let  $\theta = \alpha^{-1}$  so that  $1 - \theta = \beta^{-1}$ . Then  $0 \leq \theta \leq 1$  and  $F(\theta) = \psi(B)$ . Hence, by the three lines lemma, we have

$$(3.3) \quad |\psi(B)| = |F(\theta)| \leq \left( \sup_{t \in \mathbb{R}} |F(it)| \right)^{(1-\theta)} \left( \sup_{t \in \mathbb{R}} |F(1+it)| \right)^\theta.$$

Using Hölder's inequality, we have

$$(3.4) \quad \sup_{t \in \mathbb{R}} |F(it)| \leq \sup \left\{ \left\| \sum_{n \geq 1} a_n U a_n^* \right\|_{q/2} : U \in \mathcal{M}, \|U\| \leq 1 \right\}.$$

On the other hand, using the tracial property of  $\tau$ , we also have

$$(3.5) \quad \sup_{t \in \mathbb{R}} |F(1+it)| \leq \sup \left\{ \left\| \sum_{n \geq 1} a_n^* U a_n^* \right\|_{q/2} : U \in \mathcal{M}, \|U\| \leq 1 \right\} \cdot \|A\|_r^{2/\theta}.$$

As already noted in [36], for every operator  $U \in \mathcal{M}$  with  $\|U\| \leq 1$ , we have

$$\left\| \sum_{n \geq 1} a_n^* U a_n \right\|_{q/2} \leq \left\| \sum_{n \geq 1} a_n^* a_n \right\|_{q/2} \quad \text{and} \quad \left\| \sum_{n \geq 1} a_n U a_n^* \right\|_{q/2} \leq \left\| \sum_{n \geq 1} a_n a_n^* \right\|_{q/2}.$$

Therefore, by combining (3.2) - (3.5), the desired inequality (3.1) follows.  $\square$

*Proof of Theorem 3.1.* For  $n \geq 1$ , let  $\zeta_n^2 = \sum_{j=1}^n |\xi_j|^2$  and  $\zeta^2 = \sum_{j \geq 1} |\xi_j|^2$ . Then  $(\zeta_n)$  is an adapted sequence. Let  $\alpha = 1 - \frac{p}{2}$ . For  $n \geq 1$ , we let  $w_n = \zeta_n^\alpha$ . By approximation, we assume that the  $w_n$ 's are invertible. As in the case of martingales, the following inequality holds:

$$(3.6) \quad \sum_{n \geq 1} \|\xi_n w_n^{-1}\|_2^2 \leq \frac{2}{p} \|\xi\|_{L_p(\mathcal{M}; \ell_2^{\frac{p}{2}})}^p.$$

This is implicit in [3]. Indeed,

$$\|\xi_n w_n^{-1}\|_2^2 = \tau(|\xi_n|^2 w_n^{-2}) = \tau(\zeta_n^{p-2}(\zeta_n^2 - \zeta_{n-1}^2)) \leq \frac{2}{p} \tau(\zeta_n^p - \zeta_{n-1}^p),$$

where the last inequality comes from [3].

Consider now the following decomposition of  $\xi$ : for  $n \geq 1$ , we set

$$\begin{cases} y_n = \xi_n w_n^{-1} (w_n - w_{n-1}), \\ z_n = \xi_n w_n^{-1} w_{n-1}, \end{cases}$$

where we have taken  $w_0 = 0$ . Clearly,  $y = (y_n)$  and  $z = (z_n)$  are adapted sequences and for every  $n \geq 1$ , we have  $\xi_n = y_n + z_n$ . We claim that  $y$  and  $z$  satisfy (ii) and (iii). The argument for (ii) is similar to the proof of [24, Proposition 6.1.2].

We begin with the diagonal part. We will verify that  $\|y\|_{\ell_p(L_p(\mathcal{M}))} \leq \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^{\frac{p}{2}})}$ . For this, let  $1/p = 1/2 + 1/r$ . By Hölder's inequality and (3.6), we have

$$\begin{aligned} \left( \sum_{n \geq 1} \|y_n\|_p^p \right)^{1/p} &\leq \left( \sum_{n \geq 1} \|\xi_n w_n^{-1} (w_n - w_{n-1})\|_p^p \right)^{1/p} \\ &\leq \left( \sum_{n \geq 1} \|\xi_n w_n^{-1}\|_2^2 \right)^{1/2} \left( \sum_{n \geq 1} \|w_n - w_{n-1}\|_r^r \right)^{1/r} \\ &\leq \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^{\frac{p}{2}})}^{p/2} \left\| \left( \sum_{n \geq 1} (w_n - w_{n-1})^r \right)^{1/r} \right\|_r. \end{aligned}$$

The crucial fact here is that when  $2/3 \leq p < 2$ , we have  $r \geq 1$  and therefore by [42], we get

$$\left\| \left( \sum_{n \geq 1} (w_n - w_{n-1})^r \right)^{1/r} \right\|_r \leq \left\| \sum_{n \geq 1} w_n - w_{n-1} \right\|_r = \|\zeta^\alpha\|_r.$$

Moreover, as  $r = \frac{2p}{2-p} = \frac{p}{\alpha}$ , we have  $\|\varsigma^\alpha\|_r = \|\varsigma\|_p^\alpha = \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^\alpha$ . Thus, the above estimate becomes

$$(3.7) \quad \|y\|_{\ell_p(L_p(\mathcal{M}))} \leq \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}.$$

Let us now show that  $z \in L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)$ . We have,

$$\begin{aligned} \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} &\leq \left\| \left( \sum_{n \geq 1} \mathcal{E}_{n-1} |\xi_n w_n^{-1} w_{n-1}|^2 \right)^{1/2} \right\|_p \\ &= \left\| \left( \sum_{n \geq 1} w_{n-1} \mathcal{E}_{n-1} |\xi_n w_n^{-1}|^2 w_{n-1} \right)^{1/2} \right\|_p. \end{aligned}$$

Write  $w_{n-1} = v_{n-1} \varsigma^\alpha$  for some contraction  $v_{n-1}$ . Then

$$\|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq \left\| \varsigma^\alpha \left( \sum_{n \geq 1} v_{n-1}^* \mathcal{E}_{n-1} |\xi_n w_n^{-1}|^2 v_{n-1} \right) \varsigma^\alpha \right\|_{p/2}^{1/2}.$$

Using Hölder's inequality and (3.6), we deduce that

$$\begin{aligned} \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)}^2 &\leq \|\varsigma^\alpha\|_{p/\alpha} \left\| \sum_{n \geq 1} v_{n-1}^* \mathcal{E}_{n-1} |\xi_n w_n^{-1}|^2 v_{n-1} \right\|_1 \|\varsigma^\alpha\|_{p/\alpha} \\ &\leq \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^{2\alpha} \sum_{n \geq 1} \|\xi_n w_n^{-1}\|_2^2 \\ &\leq \frac{2}{p} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^2. \end{aligned}$$

This shows that  $\|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}$ . Combining the last inequality with (3.7), we obtain (ii).

For (iii), we will only need to verify that for  $2 \leq q \leq \infty$ ,  $y \in L_q(\mathcal{M}; \ell_2^c)$ . To that end, we observe that since for  $n \geq 1$ ,  $|\xi_n|^2 \leq \varsigma_n^2 = w_n^{2/\alpha}$ , it is clear that  $w_n^{2/\alpha} \leq \varsigma^2$ . This fact implies that for every  $n \geq 1$ ,  $w_n^{-1} |\xi_n|^2 w_n^{-1} \leq w_n^{(2/\alpha)-2} = (w_n^{2/\alpha})^{1-\alpha} \leq \varsigma^{2-2\alpha} = \varsigma^p$ . Therefore, the following estimate clearly follows:

$$\begin{aligned} \left\| \left( \sum_{n \geq 1} |\xi_n w_n^{-1} (w_n - w_{n-1})|^2 \right)^{1/2} \right\|_q &\leq \left\| \left( \sum_{n \geq 1} (w_n - w_{n-1}) \varsigma^p (w_n - w_{n-1}) \right)^{1/2} \right\|_q \\ &= \left\| \{(w_n - w_{n-1}) \varsigma^{p/2}\}_{n \geq 1} \right\|_{L_q(\mathcal{M}; \ell_2^r)}. \end{aligned}$$

Using Lemma 3.2 and [42], we deduce that

$$\begin{aligned} \|y\|_{L_q(\mathcal{M}; \ell_2^c)} &\leq \|\varsigma^{p/2}\|_{q/(1-\alpha)} \|(w_n - w_{n-1})_{n \geq 1}\|_{L_{q/\alpha}(\mathcal{M}; \ell_2^c)} \\ &\leq \|\varsigma^{p/2}\|_{q/(1-\alpha)} \left\| \sum_{n \geq 1} (w_n - w_{n-1}) \right\|_{q/\alpha} \\ &\leq \|\xi\|_{L_q(\mathcal{M}; \ell_2^c)}^{1-\alpha} \|\varsigma^\alpha\|_{q/\alpha} \\ &\leq \|\xi\|_{L_q(\mathcal{M}; \ell_2^c)}. \end{aligned}$$

This proves that  $\|y\|_{L_q(\mathcal{M}; \ell_2^c)} \leq \|\xi\|_{L_q(\mathcal{M}; \ell_2^c)}$ . The corresponding inequality for  $z$  easily follows from triangle inequality. Thus the proof is complete.  $\square$

We remark that for  $1 < p < 2$ , the norm used for the diagonal part  $\|\cdot\|_{\ell_p(L_p(\mathcal{M}))}$  in item (ii) can be improved to  $\|\cdot\|_{L_p(\mathcal{M}; \ell_1^c)}$ . This was already known to [24].

We now consider the remaining case  $0 < p < 2/3$ . At the time of this writing, we are unable to provide a simultaneous control of norms in the spirit of Theorem 3.1. We also do not know if the quasi norm  $\|\cdot\|_{L_p(\mathcal{M}; \ell_1^c)}$  used below can be improved to  $\|\cdot\|_{\ell_p(L_p(\mathcal{M}))}$ .



**Theorem 3.3.** *Let  $0 < p < 2/3$  and  $\xi = (\xi_n)_{n \geq 1}$  be an adapted sequence that belongs to  $L_p(\mathcal{M}; \ell_2^c)$ . Then there exist two adapted sequences  $y = (y_n)_{n \geq 1}$  and  $z = (z_n)_{n \geq 1}$  such that:*

- (i)  $\xi = y + z$ ;
- (ii)  $\|y\|_{L_p(\mathcal{M}; \ell_1^c)} + \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq 2\left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}$ .

*Proof.* We follow the notation from the proof of Theorem 3.1. This time the decomposition is given as follows:

$$(3.8) \quad \begin{cases} y_n = \xi_n w_n^{-2} (w_n^2 - w_{n-1}^2), \\ z_n = \xi_n w_n^{-2} w_{n-1}^2. \end{cases}$$

Note that compared to the previous one, we use  $w_n^2$  in place of  $w_n$ . This adjustment is mainly needed for the diagonal part. As before, we clearly have that  $y = (y_n)$  and  $z = (z_n)$  are adapted sequences with  $\xi_n = y_n + z_n$  for  $n \geq 1$ . We claim that  $y$  and  $z$  satisfy (ii). The argument is an adaptation of that used in the proof of item (ii) from Theorem 3.1, so we only present a sketch.

We verify first that  $y \in L_p(\mathcal{M}; \ell_1^c)$ . Define  $r$  by  $1/p = 1/2 + 1/r$ . For  $n \geq 1$ , write  $y_n = \beta_n \alpha_n$  where  $\beta_n = \xi_n w_n^{-2} (w_n^2 - w_{n-1}^2)^{1/2}$  and  $\alpha_n = (w_n^2 - w_{n-1}^2)^{1/2}$ . It follows that

$$\begin{aligned} \|y\|_{L_p(\mathcal{M}; \ell_1^c)} &= \|(\beta_n \alpha_n)_{n \geq 1}\|_{L_p(\mathcal{M}; \ell_1^c)} \\ &\leq \left( \sum_{n \geq 1} \|\xi_n w_n^{-2} (w_n^2 - w_{n-1}^2)^{1/2}\|_2^2 \right)^{1/2} \left\| \left( \sum_{n \geq 1} |(w_n^2 - w_{n-1}^2)^{1/2}|^2 \right)^{1/2} \right\|_r \\ &= \left( \sum_{n \geq 1} \|\xi_n w_n^{-1} [w_n^{-1} (w_n^2 - w_{n-1}^2)^{1/2}]\|_2^2 \right)^{1/2} \left\| \left( \sum_{n \geq 1} (w_n^2 - w_{n-1}^2) \right)^{1/2} \right\|_r. \end{aligned}$$

Observing that  $\{w_n^{-1} (w_n^2 - w_{n-1}^2)^{1/2}\}_{n \geq 1}$  is a sequence of contractions, we have

$$\|y\|_{L_p(\mathcal{M}; \ell_1^c)} \leq \left( \sum_{n \geq 1} \|\xi_n w_n^{-1}\|_2^2 \right)^{1/2} \left\| \left( \sum_{n \geq 1} (w_n^2 - w_{n-1}^2) \right)^{1/2} \right\|_r.$$

Moreover, as  $r = \frac{2p}{2-p} = \frac{p}{\alpha}$ , we have

$$\left\| \left( \sum_{n \geq 1} w_n^2 - w_{n-1}^2 \right)^{1/2} \right\|_r = \|\varsigma^\alpha\|_r = \|\varsigma\|_p^\alpha = \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^\alpha.$$

Using (3.6), the above estimate leads to:

$$\|y\|_{L_p(\mathcal{M}; \ell_1^c)} \leq \left(\frac{2}{p}\right)^{1/2} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}.$$

On the other hand, we may also write:

$$\begin{aligned} \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)} &\leq \left\| \left( \sum_{n \geq 1} \mathcal{E}_{n-1} |\xi_n w_n^{-2} w_{n-1}^2|^2 \right)^{1/2} \right\|_p \\ &= \left\| \left( \sum_{n \geq 1} w_{n-1} \mathcal{E}_{n-1} |\xi_n w_n^{-2} w_{n-1}|^2 w_{n-1} \right)^{1/2} \right\|_p. \end{aligned}$$

As before, we may deduce that

$$\|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)}^2 \leq \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^{2\alpha} \sum_{n \geq 1} \|\xi_n w_n^{-2} w_{n-1}\|_2^2.$$

Using (3.6) and the fact that  $(w_n^{-1} w_{n-1})_{n \geq 1}$  is a sequence of contractions, we conclude that

$$\|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)}^2 \leq \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^{2\alpha} \sum_{n \geq 1} \|\xi_n w_n^{-1}\|_2^2 \leq \frac{2}{p} \|\xi\|_{L_p(\mathcal{M}; \ell_2^c)}^2$$

The proof is complete.  $\square$

The following noncommutative Davis decomposition easily follows from Theorem 3.1 when  $1 \leq p < 2$ . A notable new feature of this decomposition is a simultaneous control on the  $\mathcal{H}_p^c$  and  $\mathcal{H}_q^c$  norms for  $2 \leq q < \infty$ . Decompositions of such a nature are important for some aspects of noncommutative martingale theory, for instance, for the analytic theory of quantum stochastic integrals and the study of noncommutative martingales in symmetric spaces.

**Corollary 3.4.** *Let  $1 \leq p < 2 \leq q < \infty$ . Then every martingale  $x \in \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_q^c(\mathcal{M})$  admits a decomposition into two martingales  $x^c$  and  $x^d$  such that:*

- (i)  $x = x^c + x^d$ ;
- (ii)  $\|x^c\|_{\mathcal{H}_p^c} + \|x^d\|_{\mathcal{H}_p^d} \leq 2^{5/2} \|x\|_{\mathcal{H}_p^c}$ ;
- (iii)  $\|x^c\|_{\mathcal{H}_q^c} + \|x^d\|_{\mathcal{H}_q^d} \leq Cq \|x\|_{\mathcal{H}_q^c}$  with an absolute constant  $C$ .

*Proof.* Let  $\xi = (dx_n)_{n \geq 1}$ . It is enough to take martingales  $x^c$  and  $x^d$  with  $dx_n^c = z_n - \mathcal{E}_{n-1}(z_n)$  and  $dx_n^d = y_n - \mathcal{E}_{n-1}(y_n)$ , where  $y$  and  $z$  are the adapted sequences from Theorem 3.1. Clearly,  $x = x^d + x^c$ . From the facts that  $\mathcal{E}_{n-1}$ 's are contractive projections on  $L_p(\mathcal{M})$ , we get that  $\|x^d\|_{\mathcal{H}_p^d} \leq 2\|y\|_{\ell_p(L_p(\mathcal{M}))}$ . Also, since for every  $n \geq 1$ ,  $\mathcal{E}_{n-1}|dx_n^c|^2 \leq \mathcal{E}_{n-1}|z_n|^2$ , we immediately get  $\|x^c\|_{\mathcal{H}_p^c} \leq \|z\|_{L_p^{\text{cond}}(\mathcal{M}; \ell_2^c)}$ . For the  $\mathcal{H}_q^c$ -norm of  $x^c$ , we have from the noncommutative Stein inequality and Theorem 3.1(iii) that

$$\|x^c\|_{\mathcal{H}_q^c} \leq \|z\|_{L_q^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq \gamma_q \|z\|_{L_q(\mathcal{M}; \ell_2^c)} \leq 3\gamma_q \|x\|_{\mathcal{H}_q^c}.$$

The order of  $\gamma_q$  is from [26]. The estimate on  $\|x^d\|_{\mathcal{H}_q^d}$  follows from the fact that for  $q \geq 2$ , the identity map from  $L_q(\mathcal{M}; \ell_2^c)$  into  $\ell_q(L_q(\mathcal{M}))$  is a contraction.  $\square$

*Remark 3.5.* Corollary 3.4 goes beyond the version of the noncommutative Davis' decomposition of Perrin ([35, Theorem 2.1]) since for any given  $2 \leq q_0 < \infty$ , it provides through the use of interpolation a decomposition that works simultaneously for all  $1 \leq p \leq q_0$  with universal constants. This fact is essential for the applications in the next section.

An application of the proof of Theorem 3.3 to the case  $p = 1$  provides uniform previsible estimates. The next result should be compared with [18, Theorem III.3.5] and [33, Theorem 4.4].

**Corollary 3.6.** *Every martingale  $x \in \mathcal{H}_1^c(\mathcal{M})$  admits a decomposition into two martingales  $x^c$  and  $x^d$  such that:*

- (i)  $x = x^c + x^d$ ;
- (ii)  $\|x^c\|_{\mathcal{H}_1^c} + \|x^d\|_{\mathcal{H}_1^d} \leq 2^{5/2} \|x\|_{\mathcal{H}_1^c}$ ;
- (iii) *the martingale  $x^c$  satisfies the previsible uniform estimates:*

$$|dx_n^c|^2 \leq 2S_{c,n-1}^2(x), \quad n \geq 1.$$

*Proof.* It suffices to apply the decomposition (3.8) to  $p = 1$ . That is, we set for  $n \geq 1$ ,

$$\begin{cases} dx_n^d = dx_n S_{c,n}^{-1} (S_{c,n} - S_{c,n-1}) - \mathcal{E}_{n-1}[dx_n S_{c,n}^{-1} (S_{c,n} - S_{c,n-1})] \\ dx_n^c = dx_n S_{c,n}^{-1} S_{c,n-1} - \mathcal{E}_{n-1}[dx_n S_{c,n}^{-1} S_{c,n-1}]. \end{cases}$$

The verification of (ii) follows exactly the proof of Theorem 3.3. For the previsible estimates, we have for  $n \geq 1$ ,

$$|dx_n^c|^2 \leq 2(|dx_n S_{c,n}^{-1} S_{c,n-1}|^2 + |\mathcal{E}_{n-1}[dx_n S_{c,n}^{-1} S_{c,n-1}]|^2).$$

Since

$$|dx_n S_{c,n}^{-1} S_{c,n-1}|^2 = S_{c,n-1} S_n^{-1} (S_{c,n}^2 - S_{c,n-1}^2) S_{c,n}^{-1} S_{c,n-1} \leq S_{c,n-1}^2,$$

we clearly get the desired estimate.  $\square$

*Remark 3.7.* Corollaries 3.4 and 3.6 are not valid for the case  $0 < p < 1$  even for classical martingales. Indeed, if Corollary 3.4 was valid for  $0 < p < 1$ , we would then have  $\mathcal{H}_p^{1c}(\mathcal{M}) + \mathcal{H}_p^c(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M})$ . On the other hand, one can easily see that  $\mathcal{H}_p^{1c}(\mathcal{M}) \subseteq L_p(\mathcal{M})$  and for classical martingales, we also have

$\mathfrak{h}_p^c(\mathcal{M}) \subseteq L_p(\mathcal{M})$ . Thus, there would exist a constant  $C_p$  such that  $\|x\|_p \leq C_p \|x\|_{\mathcal{H}_p}$  for all (classical) martingales  $x \in \mathcal{H}_p$ . However, such constant  $C_p$  does not exist (see [9, Example 8.1]).

Our next result shows that our decomposition in Theorem 3.1 for  $p = 1$  is stronger than the noncommutative Lépingle-Yor inequality.

**Corollary 3.8.** *Let  $(\xi_n)_{n \geq 1}$  be an adapted sequence in  $L_1(\mathcal{M})$ . Then we have*

$$\left\| \left( \sum_{n \geq 1} |\mathcal{E}_{n-1}(\xi_n)|^2 \right)^{1/2} \right\|_1 \leq 2\sqrt{2} \left\| \left( \sum_{n \geq 1} |\xi_n|^2 \right)^{1/2} \right\|_1.$$

For classical martingales, the inequality above is known as the Lépingle-Yor inequality ([30]). Its noncommutative analogue as stated in Corollary 3.8 was proved in [38] with constant 2. Unfortunately, our alternative proof below yields only the constant  $2\sqrt{2}$ . We should note that the optimal constant for the classical situation is  $\sqrt{2}$  (see [34]).

*Proof of Corollary 3.8.* Let  $\xi = (\xi_n)$  be an adapted sequence in  $L_1(\mathcal{M}; \ell_2^c)$ . Apply Theorem 3.1 to get a decomposition into two adapted sequences  $y$  and  $z$  such that:

$$\|y\|_{\ell_1(L_1(\mathcal{M}))} + \|z\|_{L_1^{\text{cond}}(\mathcal{M}; \ell_2^c)} \leq 2\sqrt{2} \|\xi\|_{L_1(\mathcal{M}; \ell_2^c)}.$$

Note that since we are not using item (iii) of Theorem 3.1, the assumption that the adapted sequence  $\xi$  belongs to  $L_\infty(\mathcal{M}; \ell_2^c)$  is not needed. Since  $|\mathcal{E}_{n-1}(b)|^2 \leq \mathcal{E}_{n-1}(|b|^2)$  for every  $b \in L_2(\mathcal{M})$ , it follows that

$$(3.9) \quad \left\| \left( \sum_{n \geq 1} |\mathcal{E}_{n-1}(z_n)|^2 \right)^{1/2} \right\|_1 \leq \|z\|_{L_1^{\text{cond}}(\mathcal{M}; \ell_2^c)}.$$

Moreover, as  $\ell_1(L_1(\mathcal{M}))$  embeds contractively into  $L_1(\mathcal{M}; \ell_2^c)$  and the expectations  $\mathcal{E}_n$ 's are contraction in  $L_1(\mathcal{M})$ , we also have

$$(3.10) \quad \left\| \left( \sum_{n \geq 1} |\mathcal{E}_{n-1}(y_n)|^2 \right)^{1/2} \right\|_1 \leq \sum_{n \geq 1} \|\mathcal{E}_{n-1}(y_n)\|_1 \leq \sum_{n \geq 1} \|y_n\|_1.$$

Combining (3.9) and (3.10), we deduce that

$$\left\| \left( \sum_{n \geq 1} |\mathcal{E}_{n-1}(\xi_n)|^2 \right)^{1/2} \right\|_1 \leq \|y\|_{\ell_1(L_1(\mathcal{M}))} + \|z\|_{L_1^{\text{cond}}(\mathcal{M})}.$$

This proves the desired inequality.  $\square$

In the following, we extend the Davis decomposition to the case of martingales in a certain class of noncommutative symmetric spaces. This is one of the main tools that we use in the next section.

**Theorem 3.9.** *Let  $1 < p < q < \infty$  and  $E$  be a symmetric Banach function space with the Fatou property and  $E \in \text{Int}(L_p, L_q)$ . There exist two positive constants  $\alpha_E$  and  $\beta_E$  such that:*

(i) *for every  $x \in \mathcal{H}_E^c(\mathcal{M})$ , the following inequality holds:*

$$\alpha_E^{-1} \inf \left\{ \|x^d\|_{\mathfrak{h}_E^d} + \|x^c\|_{\mathfrak{h}_E^c} \right\} \leq \|x\|_{\mathcal{H}_E^c},$$

*where the infimum is taken over all  $x^d \in \mathfrak{h}_E^d(\mathcal{M})$  and  $x^c \in \mathfrak{h}_E^c(\mathcal{M})$  such that  $x = x^d + x^c$ .*

(ii) *for every  $x \in \mathfrak{h}_E^d \cap \mathfrak{h}_E^c(\mathcal{M})$ , the following inequality holds:*

$$\|x\|_{\mathcal{H}_E^c} \leq \beta_E \max \left\{ \|x\|_{\mathfrak{h}_E^d}, \|x\|_{\mathfrak{h}_E^c} \right\}.$$

*Proof.* Throughout the proof we make use of the following notations for the compatible couples

$$\overline{A} := (L_p(\mathcal{M} \overline{\otimes} \ell_\infty), L_q(\mathcal{M} \overline{\otimes} \ell_\infty)) \quad \text{and} \quad \overline{B} := (L_p(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))), L_q(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))).$$

Also we may fix a symmetric function space  $F$  with nontrivial Boyd indices such that for any semifinite von Neumann algebra  $\mathcal{N}$ , we have  $E(\mathcal{N}) = [L_p(\mathcal{N}), L_q(\mathcal{N})]_{F, j}$  as stated in Lemma 2.2. We note that when

$1 < p < q < \infty$ ,  $\mathcal{H}_p^c(\mathcal{M})$  and  $\mathcal{H}_q^c(\mathcal{M})$  embed complementedly into  $L_p(\mathcal{M} \overline{\otimes} B(\ell_2))$  and  $L_q(\mathcal{M} \overline{\otimes} B(\ell_2))$ , respectively. This implies that for every  $x \in \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_q^c(\mathcal{M})$  and  $t > 0$ ,

$$K(t, x; \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M})) \leq C_{p,q} K(t, \mathcal{D}_c(x); L_p(\mathcal{M} \overline{\otimes} B(\ell_2)), L_q(\mathcal{M} \overline{\otimes} B(\ell_2))).$$

As a consequence, we have that for every  $x \in \mathcal{H}_E^c(\mathcal{M})$ ,

$$\|x\|_{\mathcal{H}_E^c} = \|\mathcal{D}_c(x)\|_{E(\mathcal{M} \overline{\otimes} B(\ell_2))} \approx_E \|x\|_{F, \underline{j}},$$

where the interpolation on the last norm is taken with respect to the couple  $(\mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))$ .

We are now ready to present the proof of (i). Let  $x \in \mathcal{H}_E^c(\mathcal{M})$  and fix a representation  $x = \sum_{\nu \in \mathbb{Z}} u_\nu$  in the compatible couple  $(\mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))$  such that:

$$(3.11) \quad \left\| \underline{j}(\cdot, \{u_\nu\}_\nu) \right\|_F \leq 2 \|x\|_{F, \underline{j}}.$$

We recall that for every  $\nu \in \mathbb{Z}$ ,  $u_\nu \in \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_q^c(\mathcal{M})$ . By Corollary 3.4, there exist  $a_\nu \in \mathfrak{h}_p^d(\mathcal{M}) \cap \mathfrak{h}_q^d(\mathcal{M})$  and  $b_\nu \in \mathfrak{h}_p^c(\mathcal{M}) \cap \mathfrak{h}_q^c(\mathcal{M})$ , satisfying:

$$(3.12) \quad u_\nu = a_\nu + b_\nu$$

and if  $s \in \{p, q\}$ , then

$$(3.13) \quad \|a_\nu\|_{\mathfrak{h}_s^d} + \|b_\nu\|_{\mathfrak{h}_s^c} \leq C(p, q) \|u_\nu\|_{\mathcal{H}_s^c}.$$

For each  $\nu \in \mathbb{Z}$ , we consider  $\mathcal{D}_d(a_\nu) \in \Delta(\overline{A})$  and  $U\mathcal{D}_c(b_\nu) \in \Delta(\overline{B})$ . As in [40], inequality (3.13) can be reinterpreted by using the  $J$ -functionals as follows:

$$(3.14) \quad \begin{cases} J(t, \mathcal{D}_d(a_\nu); \overline{A}) \leq C(p, q) J(t, u_\nu), & t > 0, \\ J(t, U\mathcal{D}_c(b_\nu); \overline{B}) \leq C(p, q) J(t, u_\nu), & t > 0. \end{cases}$$

One can show as in [40, Sublemma 3.3] that the series  $\sum_{\nu \in \mathbb{Z}} \mathcal{D}_d(a_\nu)$  is weakly unconditionally Cauchy in  $E(\mathcal{M} \overline{\otimes} \ell_\infty)$ . A fortiori, it is convergent in  $\Sigma(\overline{A})$ . Similarly, we can also get that  $\sum_{\nu \in \mathbb{Z}} U\mathcal{D}_c(b_\nu)$  is convergent in  $\Sigma(\overline{B})$ . Set

$$\alpha := \sum_{\nu \in \mathbb{Z}} \mathcal{D}_d(a_\nu) \in \Sigma(\overline{A}) \quad \text{and} \quad \beta := \sum_{\nu \in \mathbb{Z}} U\mathcal{D}_c(b_\nu) \in \Sigma(\overline{B}).$$

The series  $\sum_{\nu \in \mathbb{Z}} \mathcal{D}_d(a_\nu)$  may be viewed as a representation of  $\alpha$  in the interpolation couple  $\overline{A}$ . Similarly,  $\sum_{\nu \in \mathbb{Z}} U\mathcal{D}_c(b_\nu)$  is a representation of  $\beta$  in the interpolation couple  $\overline{B}$ . We claim that:

$$(3.15) \quad \|\alpha\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} + \|\beta\|_{E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))} \leq \kappa_E \|x\|_{\mathcal{H}_E^c}.$$

To see this claim, we observe from (3.14) that for every  $s > 0$ ,

$$\underline{j}(s, \{\mathcal{D}_d(a_\nu)\}_\nu; \overline{A}) + \underline{j}(s, \{U\mathcal{D}_c(b_\nu)\}_\nu; \overline{B}) \leq 2C(p, q) \underline{j}(s, \{u_\nu\}_\nu).$$

Taking the norms on the function space  $F$  together with (3.11) gives

$$\|\alpha\|_{F, \underline{j}} + \|\beta\|_{F, \underline{j}} \leq 4C(p, q) \|x\|_{F, \underline{j}}.$$

This proves (3.15). To conclude the proof, it is plain that there exist  $a \in \mathfrak{h}_E^d(\mathcal{M})$  and  $b \in \mathfrak{h}_E^c(\mathcal{M})$  such that  $\alpha = \mathcal{D}_d(a)$  and  $\beta = U\mathcal{D}_c(b)$ . Moreover, it is clear from the construction that  $x = a + b$ . Indeed, the fact that  $\alpha$  is a martingale difference sequence follows from the convergence of the series  $\sum_{\nu \in \mathbb{Z}} \mathcal{D}_d(a_\nu)$  in  $\Sigma(\overline{A})$ . Similarly, the representation above also gives that  $\beta$  is in the range of  $U\mathcal{D}_c$  in  $\Sigma(\overline{B})$ . We may now conclude from (3.15) that

$$\|a\|_{\mathfrak{h}_E^d(\mathcal{M})} + \|b\|_{\mathfrak{h}_E^c(\mathcal{M})} \leq \kappa_E \|x\|_{\mathcal{H}_E^c}.$$

The proof of (i) is complete. Item (ii) can be obtained by duality in the same manner as in Part III of the proof of [40, Theorem 3.1]. The details are left to the reader.  $\square$

As an immediate application of Theorem 3.9, we have the following result:

**Corollary 3.10.** *Let  $E$  be a symmetric Banach function space with the Fatou property.*

- (i) If  $E \in \text{Int}(L_p, L_2)$  for some  $1 < p < 2$ , then  $\mathcal{H}_E^c(\mathcal{M}) = \mathfrak{h}_E^d(\mathcal{M}) + \mathfrak{h}_E^c(\mathcal{M})$  with equivalent norms.
- (ii) If  $E \in \text{Int}(L_2, L_q)$  for some  $2 < q < \infty$ , then  $\mathcal{H}_E^c(\mathcal{M}) = \mathfrak{h}_E^d(\mathcal{M}) \cap \mathfrak{h}_E^c(\mathcal{M})$  with equivalent norms.

*Proof.* We already have from Theorem 3.9 that  $\mathcal{H}_E^c(\mathcal{M}) \subseteq \mathfrak{h}_E^d(\mathcal{M}) + \mathfrak{h}_E^c(\mathcal{M})$ . The reverse inclusion follows from the fact that for  $1 < v \leq 2$ ,  $\mathfrak{h}_v^d(\mathcal{M}) + \mathfrak{h}_v^c(\mathcal{M}) = \mathcal{H}_v^c(\mathcal{M})$ . Indeed, this identification implies in particular that  $\mathfrak{h}_v^d(\mathcal{M}) \subset \mathcal{H}_v^c(\mathcal{M})$  and  $\mathfrak{h}_v^c(\mathcal{M}) \subseteq \mathcal{H}_v^c(\mathcal{M})$ . A standard use of interpolation then gives  $\mathfrak{h}_E^d(\mathcal{M}) \subseteq \mathcal{H}_E^c(\mathcal{M})$  and  $\mathfrak{h}_E^c(\mathcal{M}) \subseteq \mathcal{H}_E^c(\mathcal{M})$ . We should recall here that since  $E \in \text{Int}(L_p, L_2)$  and  $1 < p < 2$ , by complementation, we have  $\mathfrak{h}_E^d(\mathcal{M}) \in \text{Int}(\mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_2^d(\mathcal{M}))$ ,  $\mathfrak{h}_E^c(\mathcal{M}) \in \text{Int}(\mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_2^c(\mathcal{M}))$ , and  $\mathcal{H}_E^c(\mathcal{M}) \in \text{Int}(\mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_2^c(\mathcal{M}))$ . The argument for (ii) is identical.  $\square$

*Remark 3.11.* We do not know if the statement in the first item of Theorem 3.9 (respectively of Corollary 3.10) remains valid if one only assumes that  $E \in \text{Int}(L_1, L_q)$  for  $1 < q < \infty$  (respectively  $E \in \text{Int}(L_1, L_2)$ ). Since both assertions hold for  $E = L_1$ , it is reasonable to conjecture that this should be the case in general. We leave these as open problems.

*Remark 3.12.* We conclude this section by calling to the reader's attention that all statements above admit corresponding row versions.

#### 4. APPLICATIONS TO NONCOMMUTATIVE BURKHOLDER/ROSENTHAL INEQUALITIES

**4.1. The case of noncommutative symmetric spaces.** The main objective of this subsection is to provide versions of the Burkholder/Rosenthal inequality for martingales in noncommutative symmetric spaces. Recall that the noncommutative Burkholder/Rosenthal inequalities were proved by Junge and the third named author for martingales in noncommutative  $L_p$ -spaces for  $1 < p < \infty$  ([25]). We should emphasize here that it is essential to separate the case  $1 < p \leq 2$  from the one  $2 \leq p < \infty$ . In fact, the original classical case was only proved for  $2 \leq p < \infty$  and it was in [25] that the corresponding case  $1 < p < 2$  was discovered. Our main result in this subsection strengthens the versions of Burkholder/Rosenthal inequalities for noncommutative symmetric spaces from [11, 40]. It reads as follows:

**Theorem 4.1.** *Let  $E$  be a symmetric Banach function space with the Fatou property.*

- (i) *If  $E \in \text{Int}(L_p, L_2)$  for some  $1 < p < 2$ , then  $E(\mathcal{M}) = \mathfrak{h}_E^d(\mathcal{M}) + \mathfrak{h}_E^c(\mathcal{M}) + \mathfrak{h}_E^r(\mathcal{M})$  with equivalent norms.*
- (ii) *If  $E \in \text{Int}(L_2, L_q)$  for some  $2 < q < \infty$ , then  $E(\mathcal{M}) = \mathfrak{h}_E^d(\mathcal{M}) \cap \mathfrak{h}_E^c(\mathcal{M}) \cap \mathfrak{h}_E^r(\mathcal{M})$  with equivalent norms.*

We remark that (ii) was also obtained recently by Jiao *et al.* in [22, Theorem 1.5] under the more restrictive assumption that  $E \in \text{Int}(L_2, L_4)$ . It is important to note that through the use of interpolation, it is not difficult to deduce that if  $1 < p < 2$  and  $E \in \text{Int}(L_p, L_2)$  then  $\mathfrak{h}_E^w(\mathcal{M}) \subseteq E(\mathcal{M})$  for  $w \in \{d, c, r\}$ . Therefore, we always have in this case that  $\mathfrak{h}_E^d(\mathcal{M}) + \mathfrak{h}_E^c(\mathcal{M}) + \mathfrak{h}_E^r(\mathcal{M}) \subseteq E(\mathcal{M})$ . Similarly, when  $2 < q < \infty$  and  $E \in \text{Int}(L_2, L_q)$  then  $E(\mathcal{M}) \subseteq \mathfrak{h}_E^d(\mathcal{M}) \cap \mathfrak{h}_E^c(\mathcal{M}) \cap \mathfrak{h}_E^r(\mathcal{M})$ . Thus, it is only necessary to prove the respective reverse inclusions. To this end, we will prove the following more general result:

**Theorem 4.2.** *Let  $1 < p < q < \infty$  and  $E$  be a symmetric Banach function space with the Fatou property and such that  $E \in \text{Int}(L_p, L_q)$ . Then there exist positive constants  $\delta_E$  and  $\eta_E$  such that:*

- (i) *for every  $x \in E(\mathcal{M})$ , the following inequality holds:*

$$\delta_E^{-1} \inf \left\{ \|x^d\|_{\mathfrak{h}_E^d} + \|x^c\|_{\mathfrak{h}_E^c} + \|x^r\|_{\mathfrak{h}_E^r} \right\} \leq \|x\|_{E(\mathcal{M})},$$

*where the infimum is taken over all  $x^d \in \mathfrak{h}_E^d(\mathcal{M})$ ,  $x^c \in \mathfrak{h}_E^c(\mathcal{M})$ , and  $x^r \in \mathfrak{h}_E^r(\mathcal{M})$  such that  $x = x^d + x^c + x^r$ ;*

- (ii) *for every  $x \in \mathfrak{h}_E^d(\mathcal{M}) \cap \mathfrak{h}_E^c(\mathcal{M}) \cap \mathfrak{h}_E^r(\mathcal{M})$ , the following inequality holds:*

$$\|x\|_{E(\mathcal{M})} \leq \eta_E \max \left\{ \|x\|_{\mathfrak{h}_E^d}, \|x\|_{\mathfrak{h}_E^c}, \|x\|_{\mathfrak{h}_E^r} \right\}.$$

Clearly, Theorem 4.2 and the preceding discussion imply Theorem 4.1. Our strategy for the proof of Theorem 4.2 is to use the corresponding Burkholder-Gundy along side our Davis decomposition stated in Theorem 3.9. The following version of the Burkholder-Gundy inequalities is implicit in [13]:

**Proposition 4.3.** *Let  $1 < p < q < \infty$  and  $E$  be a symmetric Banach function space with the Fatou property such that  $E \in \text{Int}(L_p, L_q)$ . Then there exist positive constants  $C_E$  and  $c_E$  such that:*

(i) *for every  $x \in E(\mathcal{M})$ , the following inequality holds:*

$$\inf \left\{ \|x^c\|_{\mathcal{H}_E^c} + \|x^r\|_{\mathcal{H}_E^r} \right\} \leq C_E \|x\|_{E(\mathcal{M})},$$

*where the infimum is taken over all  $x^c \in \mathcal{H}_E^c(\mathcal{M})$ , and  $x^r \in \mathcal{H}_E^r(\mathcal{M})$  such that  $x = x^c + x^r$ ;*

(ii) *for every  $x \in \mathcal{H}_E^c(\mathcal{M}) \cap \mathcal{H}_E^r(\mathcal{M})$ , the following inequality holds:*

$$\|x\|_{E(\mathcal{M})} \leq c_E \max \left\{ \|x\|_{\mathcal{H}_E^c}, \|x\|_{\mathcal{H}_E^r} \right\}.$$

*Proof.* Assume that  $E \in \text{Int}(L_p, L_q)$  with  $1 < p < q < \infty$ . By the boundedness of martingale transforms on  $E(\mathcal{M})$  ([39, Proposition 4.9]), there exists a constant  $\kappa_E$  such that for any given finite martingale  $x$  in  $E(\mathcal{M})$ ,

$$\kappa_E^{-1} \mathbb{E} \left\| \sum_{n \geq 1} \varepsilon_n dx_n \right\|_{E(\mathcal{M})} \leq \|x\|_{E(\mathcal{M})} \leq \kappa_E \mathbb{E} \left\| \sum_{n \geq 1} \varepsilon_n dx_n \right\|_{E(\mathcal{M})}$$

where  $(\varepsilon_n)$  denotes a Rademacher sequence on a given probability space. According to [13, Theorem 4.3], we then have,

$$(\kappa'_E)^{-1} \|(dx_n)\|_{E(\mathcal{M}; \ell_2^c) + E(\mathcal{M}; \ell_2^r)} \leq \|x\|_{E(\mathcal{M})} \leq c_E \|(dx_n)\|_{E(\mathcal{M}; \ell_2^c) \cap E(\mathcal{M}; \ell_2^r)}.$$

Using the the noncommutative Stein inequality on the first inequality ([20]), we deduce that

$$C_E^{-1} \|x\|_{\mathcal{H}_E^c + \mathcal{H}_E^r} \leq \|x\|_{E(\mathcal{M})} \leq c_E \|x\|_{\mathcal{H}_E^c \cap \mathcal{H}_E^r}.$$

This proves both items.  $\square$

*Proof of Theorem 4.2.* Let  $x \in E(\mathcal{M})$  and  $\varepsilon > 0$ . By Proposition 4.3(i), there exists a decomposition  $x = a^c + a^r$  so that

$$(4.1) \quad \|a^c\|_{\mathcal{H}_E^c} + \|a^r\|_{\mathcal{H}_E^r} \leq C_E \|x\|_{E(\mathcal{M})} + \varepsilon.$$

Applying Theorem 3.9 separately on  $a^c$  and  $(a^r)^*$ , there exist further decompositions  $a^c = y^d + y^c$  and  $a^r = z^d + z^r$  with  $y^d, z^d \in \mathfrak{h}_E^d(\mathcal{M})$ ,  $y^c \in \mathfrak{h}_E^c(\mathcal{M})$ , and  $z^r \in \mathfrak{h}_E^r(\mathcal{M})$  satisfying:

$$\|y^d\|_{\mathfrak{h}_E^d} + \|y^c\|_{\mathfrak{h}_E^c} \leq \delta_E \|a^c\|_{\mathcal{H}_E^c} + \varepsilon \quad \text{and} \quad \|z^d\|_{\mathfrak{h}_E^d} + \|z^r\|_{\mathfrak{h}_E^r} \leq \delta_E \|a^r\|_{\mathcal{H}_E^r} + \varepsilon.$$

Set  $x^d := y^d + z^d$ ,  $x^c := y^c$ , and  $x^r := z^r$ . Clearly,  $x = x^d + x^c + x^r$  and the previous two inequalities lead to:

$$\|x^d\|_{\mathfrak{h}_E^d} + \|x^c\|_{\mathfrak{h}_E^c} + \|x^r\|_{\mathfrak{h}_E^r} \leq \delta_E C_E \|x\|_{E(\mathcal{M})} + (\delta_E + 2)\varepsilon.$$

This proves (i). Item (ii) is similar. Indeed, from combining Theorem 3.9(ii) and Proposition 4.3(ii), we deduce that for every finite martingale  $x$ , one has:

$$\begin{aligned} \|x\|_{E(\mathcal{M})} &\leq c_E \max \{ \|x\|_{\mathcal{H}_E^c(\mathcal{M})}, \|x\|_{\mathcal{H}_E^r(\mathcal{M})} \} \\ &\leq c_E \beta_E \max \{ \|x\|_{\mathfrak{h}_E^d(\mathcal{M})}, \|x\|_{\mathfrak{h}_E^c(\mathcal{M})}, \|x\|_{\mathfrak{h}_E^r(\mathcal{M})} \}. \end{aligned}$$

The proof is complete.  $\square$

*Remark 4.4.* In order to have equivalences of norms as stated in Theorem 4.1, the assumptions used there are in general necessary. Indeed, if  $E$  is a symmetric Banach function space that satisfies the equivalences of norms as stated in Theorem 4.1 then a fortiori, martingale difference sequences are unconditional in  $E(\mathcal{M})$ . From [31], it follows that there exist  $1 < p \leq q < \infty$  so that  $E \in \text{Int}(L_p, L_q)$ . On the other hand, it was noted in [29] that if  $1 < p < 2 < q < \infty$  then  $L_p(\mathcal{M}) \cap L_q(\mathcal{M})$  fails to satisfy the noncommutative Khintchine inequalities. In particular, it must fail the equivalences of norms stated in Theorem 4.1. This shows that separating the two cases  $E \in \text{Int}(L_p, L_2)$  for  $1 < p < 2$  and  $E \in \text{Int}(L_2, L_q)$  for  $2 < q < \infty$  are necessary. On the other hand, there are symmetric function spaces with Boyd indices equal to 2 but do not appear to belong to either of the two classes of functions considered in Theorem 4.1. For instance, we do

not know if either of the versions of the noncommutative Burkholder/Rosenthal inequalities in Theorem 4.1 apply to martingales in  $L_{2,\infty}(\mathcal{M}, \tau)$ , or more generally in  $L_{2,q}(\mathcal{M}, \tau)$  for any  $1 \leq q \neq 2 \leq \infty$ .

**4.2. Modular inequalities.** In this subsection, we focus on noncommutative moment inequalities associated with Orlicz functions, which were considered in [1, 14, 41]. We will assume throughout that  $\Phi$  is an Orlicz function satisfying the  $\Delta_2$ -condition, that is, for some constant  $C > 0$ ,

$$(4.2) \quad \Phi(2t) \leq C\Phi(t) \quad t \geq 0.$$

We denote by  $L_\Phi$  the Orlicz function space associated to  $\Phi$ . Below, we write  $\mathcal{H}_\Phi^c(\mathcal{M})$ ,  $\mathfrak{h}_\Phi^c(\mathcal{M})$ , *etc.* for martingale Hardy spaces  $\mathcal{H}_{L_\Phi}^c(\mathcal{M})$ ,  $\mathfrak{h}_{L_\Phi}^c(\mathcal{M})$ , *etc.* We make the observation that if  $L_\Phi \not\subseteq L_2 + L_\infty$ , then for an  $x \in \mathfrak{h}_\Phi^c(\mathcal{M})$ , the  $\Phi$ -moment  $\tau[\Phi(s_c(x))]$  is understood to be the quantity  $\tau \otimes \text{tr}[\Phi(|UD_c(x)|)]$  as fully detailed in [41].

Given  $1 \leq p \leq q < \infty$ , we recall that an Orlicz function  $\Phi$  is said to be  $p$ -convex if the function  $t \mapsto \Phi(t^{1/p})$  is convex, and to be  $q$ -concave if the function  $t \mapsto \Phi(t^{1/q})$  is concave. The function  $\Phi$  satisfies the  $\Delta_2$ -condition if and only if it is  $q$ -concave for some  $q < \infty$ . Recall the so-called Matuzewska-Orlicz indices  $p_\Phi$  and  $q_\Phi$  of  $\Phi$ :

$$p_\Phi = \lim_{t \rightarrow 0^+} \frac{\log M_\Phi(t)}{\log t} \quad \text{and} \quad q_\Phi = \lim_{t \rightarrow \infty} \frac{\log M_\Phi(t)}{\log t},$$

where

$$M_\Phi(t) = \sup_{s > 0} \frac{\Phi(ts)}{\Phi(s)}.$$

The indices  $p_\Phi$  and  $q_\Phi$  are used in the previous papers [1, 2, 14, 41] instead of the convexity and concavity indices in the present one. It is easy to see that  $p \leq p_\Phi \leq q_\Phi \leq q$  if  $\Phi$  is  $p$ -convex and  $q$ -concave. We refer to [32] for backgrounds on Orlicz functions and spaces.

As part of our motivation, we state the following  $\Phi$ -moment version of the noncommutative Burkholder-Gundy inequality:

**Theorem 4.5** ([1, 14]). *Let  $1 < p < q < \infty$  and  $\Phi$  be a  $p$ -convex and  $q$ -concave Orlicz function. Then there exists a positive constant  $c_\Phi$  such that for every  $x \in L_\Phi(\mathcal{M})$ ,*

$$c_\Phi^{-1} \inf \left\{ \tau[\Phi(S_c(y))] + \tau[\Phi(S_r(z))] \right\} \leq \tau[\Phi(|x|)] \leq c_\Phi \max \left\{ \tau[\Phi(S_c(x))], \tau[\Phi(S_r(x))] \right\},$$

where the infimum on the first inequality is taken over all  $y \in \mathcal{H}_\Phi^c(\mathcal{M})$  and  $z \in \mathcal{H}_\Phi^r(\mathcal{M})$  such that  $x = y + z$ .

*Proof.* The second inequality is from [14, Corollary 3.3]. The first one follows from a  $\Phi$ -moment Khintchine inequality proved in [1] which states that for any given finite sequence  $(a_k)$  in  $L_\Phi(\mathcal{M})$ ,

$$(4.3) \quad \inf \left\{ \tau \left[ \Phi \left( \left( \sum_k |b_k|^2 \right)^{1/2} \right) \right] + \tau \left[ \Phi \left( \left( \sum_k |c_k^*|^2 \right)^{1/2} \right) \right] \right\} \leq C\mathbb{E} \left[ \tau \left[ \Phi \left( \left| \sum_k \varepsilon_k a_k \right| \right) \right] \right],$$

where  $(\varepsilon_k)_{k \geq 1}$  is a Rademacher sequence and the infimum runs over all decompositions  $a_k = b_k + c_k$  with  $b_k$  and  $c_k$  in  $L_\Phi(\mathcal{M})$ . We should note that (4.3) was stated in [1] under the assumption that  $1 < p_\Phi \leq q_\Phi < 2$  but the proof given there apply verbatim to the present situation. It is now standard to deduce the first inequality from (4.3) using the  $\Phi$ -moment versions of the noncommutative Stein inequality and martingale transforms. Both of these results were proved in [1].  $\square$

It is a natural question if the Burkholder/Rosenthal version of the above theorem holds. A first attempt in this direction was done in [41] but the results obtained there require far more restrictive assumption than the one in Theorem 4.5. As in the case of noncommutative symmetric spaces, our approach is based on the consideration of our Davis decomposition. The following is one of our main results in this subsection. It is the  $\Phi$ -moment analogue of the Davis decomposition stated in Theorem 3.9.

**Theorem 4.6.** *Let  $1 < p < q < \infty$  and  $\Phi$  be a  $p$ -convex and  $q$ -concave Orlicz function. Then there exist positive constants  $\alpha_\Phi$  and  $\beta_\Phi$  such that:*

(i) for every martingale  $x \in \mathcal{H}_\Phi^c(\mathcal{M})$ , the following inequality holds:

$$\alpha_\Phi^{-1} \inf \left\{ \tau[\Phi(s_c(x^c))] + \sum_{n \geq 1} \tau[\Phi(|dx_n^d|)] \right\} \leq \tau[\Phi(S_c(x))],$$

where the infimum is taken over all  $x^c \in \mathfrak{h}_\Phi^c(\mathcal{M})$  and  $x^d \in \mathfrak{h}_\Phi^d(\mathcal{M})$  such that  $x = x^c + x^d$ ;

(ii) for every  $x \in \mathfrak{h}_\Phi^d(\mathcal{M}) \cap \mathfrak{h}_\Phi^c(\mathcal{M})$ , the following inequality holds:

$$\tau[\Phi(S_c(x))] \leq \beta_\Phi \max \left\{ \sum_{n \geq 1} \tau[\Phi(|dx_n|)], \tau[\Phi(s_c(x))] \right\}.$$

Before we present the proof, we need some preparations. We first record few technical facts from interpolation and duality that we will need in the sequel.

**Lemma 4.7** ([21, Lemma 6.2]). *Let  $(\mathcal{M}_1, \tau_1)$  and  $(\mathcal{M}_2, \tau_2)$  be semifinite von Neumann algebras and  $\Phi$  be a  $p$ -convex and  $q$ -concave Orlicz function for  $1 \leq p \leq q < \infty$ . If  $W : L_p(\mathcal{M}_1) \rightarrow L_p(\mathcal{M}_2)$  and  $\tilde{W} : L_q(\mathcal{M}_1) \rightarrow L_q(\mathcal{M}_2)$  are bounded linear operators, then there exists a constant  $C_\Phi$  satisfying:*

$$\tau_2[\Phi(|Wx|)] \leq C_\Phi \tau_1[\Phi(|x|)], \quad x \in L_\Phi(\mathcal{M}_1).$$

Lemma 4.7 shows in particular that if  $\Phi$  is  $p$ -convex and  $q$ -concave then the Orlicz function space  $L_\Phi$  belongs to  $\text{Int}(L_p, L_q)$ .

**Lemma 4.8** ([41]). *Let  $\mathcal{N}$  be a semifinite von Neumann algebra and  $\Phi$  be an Orlicz function such that  $1 < p < p_\Phi \leq q_\Phi < q < \infty$ . The following inequalities hold:*

(i) For every  $y \in L_p(\mathcal{N}) + L_q(\mathcal{N})$ ,

$$\int_0^\infty \Phi[t^{-1}K(t, y; L_1(\mathcal{N}), \mathcal{N})] dt \leq C_{\Phi, p, q} \int_0^\infty \Phi[t^{-1/p}K(t^{1/p-1/q}, y; L_p(\mathcal{N}), L_q(\mathcal{N}))] dt.$$

(ii) If  $y \in L_p(\mathcal{N}) \cap L_q(\mathcal{N})$  and  $u(\cdot)$  is a representation of  $y$  in the couples  $(L_p(\mathcal{N}), L_q(\mathcal{N}))$  and  $(L_1(\mathcal{N}), \mathcal{N})$  then,

$$\int_0^\infty \Phi[t^{-1/p}J(t^{1/p-1/q}, u(t); L_p(\mathcal{N}), L_q(\mathcal{N}))] dt \leq C_{\Phi, p, q} \int_0^\infty \Phi[t^{-1}J(t, u(t); L_1(\mathcal{N}), \mathcal{N})] dt.$$

(iii) For every  $y \in L_p(\mathcal{N}) + L_q(\mathcal{N})$ ,

$$\int_0^\infty \Phi[t^{-1/p}K(t^{1/p-1/q}, y; L_p(\mathcal{N}), L_q(\mathcal{N}))] dt \leq C_{\Phi, p, q} \inf \left\{ \int_0^\infty \Phi[t^{-1/p}J(t^{1/p-1/q}, u(t); L_p(\mathcal{N}), L_q(\mathcal{N}))] dt \right\},$$

where the infimum is taken over all representations  $u(\cdot)$  of  $y$ .

Below  $\Phi^*$  denotes the Orlicz complementary function to  $\Phi$ . The next lemma will be used for duality purposes.

**Lemma 4.9** ([41, Proposition 2.3]). *Let  $\Phi$  be an Orlicz function which is  $p$ -convex and  $q$ -concave for some  $1 < p \leq q < \infty$  and  $\mathcal{N}$  be a semifinite von Neumann algebra. For every  $0 \leq x \in L_\Phi(\mathcal{N})$  there exists  $0 \leq y \in L_{\Phi^*}(\mathcal{N})$  such that  $y$  commutes with  $x$  and satisfies:  $xy = \Phi(x) + \Phi^*(y)$ .*

*Proof of Theorem 4.6.* The proof is an adaptation of the argument used in [41] so we will only highlight the main points. We begin with the proof of (i). Since  $\Phi$  is  $p$ -convex and  $q$ -concave, we have  $p \leq p_\Phi \leq q_\Phi \leq q$ . Let  $1 < p_0 < p$  and  $q < q_0 < \infty$ . It is clear that  $\Phi$  is  $p_0$ -convex and  $q_0$ -concave. Replacing  $p$  by  $p_0$  and  $q$  by  $q_0$  if necessary, we may assume without loss of generality that  $1 < p < p_\Phi \leq q_\Phi < q < \infty$ . Under this assumption, Lemma 4.8 applies to  $\Phi$ . Below,  $C_{\Phi, p, q}$  denotes a constant whose value may change from line to line.

Fix a martingale  $x$  such that  $\xi = (dx_n)_{n \geq 1} \in L_1(\mathcal{M}; \ell_2^c) \cap L_\infty(\mathcal{M}; \ell_2^c)$ . We make the observation that by complementation,

$$K(t, \xi) = K(t, \xi; L_1(\mathcal{M}; \ell_2^c), L_\infty(\mathcal{M}; \ell_2^c)) = \int_0^t \mu_s(S_c(x)) ds, \quad t > 0.$$



This leads to

$$(4.4) \quad \tau[\Phi(S_c(x))] = \int_0^\infty \Phi(\mu_t(S_c(x))) dt \approx \int_0^\infty \Phi[t^{-1}K(t, \xi)] dt$$

where the equivalence comes from the boundedness of the Hilbert operator on  $L_r$  for  $1 < r < \infty$  and Lemma 4.7.

Choose  $u(\cdot)$  a representation of  $\xi$  in the compatible couple  $(L_1(\mathcal{M}; \ell_2^c), L_\infty(\mathcal{M}; \ell_2^c))$  such that:

$$(4.5) \quad J(t, u(t)) \leq CK(t, \xi), \quad t > 0,$$

where  $C$  is an absolute constant. Thus, since  $\Phi$  has the  $\Delta_2$ -condition, we have from (4.4) and (4.5) that

$$(4.6) \quad \int_0^\infty \Phi[t^{-1}J(t, u(t))] dt \leq C_\Phi \tau[\Phi(S_c(x))].$$

It is important to note that  $u(\cdot)$  is also a representation of  $\xi$  for the couple  $(L_p(\mathcal{M}; \ell_2^c), L_q(\mathcal{M}; \ell_2^c))$ . Putting (4.6) together with Proposition 4.8(ii) yields:

$$\int_0^\infty \Phi[t^{-1/p}J(t^{1/p-1/q}, u(t); L_p(\mathcal{M}; \ell_2^c), L_q(\mathcal{M}; \ell_2^c))] dt \leq C_{\Phi, p, q} \tau[\Phi(S_c(x))].$$

Consider  $\Theta : L_p(\mathcal{M}; \ell_2^c) + L_q(\mathcal{M}; \ell_2^c) \rightarrow \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_q^c(\mathcal{M})$  defined by:

$$\Theta((a_n)_{n \geq 1}) = \sum_{n \geq 1} [\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)].$$

By the noncommutative Stein inequality,  $\Theta$  is bounded and one can easily verify that  $\Theta(u(\cdot))$  is a representation of  $x$  for the couple  $(\mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))$ . Moreover, we have:

$$\int_0^\infty \Phi[t^{-1/p}J(t^{1/p-1/q}, \Theta(u(t)); \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))] dt \leq C_{\Phi, p, q} \tau[\Phi(S_c(x))].$$

As in [41], we need to modify the representation as follows: set  $\theta = 1/p - 1/q$  and define:

$$v(t) = \frac{1}{\theta} \Theta(u(t^{1/\theta})).$$

Then  $v(\cdot)$  is a representation of  $x$  in the couple  $(\mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))$  and the preceding inequality becomes:

$$(4.7) \quad \int_0^\infty \Phi[t^{-1/p}J(t^{1/p-1/q}, v(t^{1/p-1/q}); \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))] dt \leq C_{\Phi, p, q} \tau[\Phi(S_c(x))].$$

Next, we discretize the integral in (4.7). If we set  $v_\nu = \int_{2^\nu}^{2^{\nu+1}} v(t) dt/t$  for every  $\nu \in \mathbb{Z}$ , then

$$(4.8) \quad x = \sum_{\nu \in \mathbb{Z}} v_\nu, \quad \text{convergence in } \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_q^c(\mathcal{M}).$$

By [41, Lemma 3.12(i)], we deduce from (4.7) that

$$(4.9) \quad \sum_{\nu \in \mathbb{Z}} 2^{\nu/\theta} \Phi[2^{-\nu/(\theta p)} J(2^\nu, v_\nu; \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M}))] \leq C_{\Phi, p, q} \tau[\Phi(S_c(x))].$$

The next step is to apply the simultaneous Davis decomposition (Corollary 3.4 and Remark 3.5). For each  $\nu \in \mathbb{Z}$ , there exist  $a_\nu \in \mathfrak{h}_p^d(\mathcal{M}) \cap \mathfrak{h}_q^d(\mathcal{M})$  and  $b_\nu \in \mathfrak{h}_p^c(\mathcal{M}) \cap \mathfrak{h}_q^c(\mathcal{M})$  such that:

$$(4.10) \quad v_\nu = a_\nu + b_\nu$$

and if  $s$  is equal to either  $p$  or  $q$ , then

$$(4.11) \quad \|a_\nu\|_{\mathfrak{h}_s^d} + \|b_\nu\|_{\mathfrak{h}_s^c} \leq C(p, q) \|v_\nu\|_{\mathcal{H}_s^c}.$$

As above, the inequalities in (4.11) can be reinterpreted using the  $J$ -functionals as follows:

$$(4.12) \quad \begin{aligned} J(t, a_\nu; \mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_q^d(\mathcal{M})) &\leq C(p, q) J(t, v_\nu; \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M})), \quad t > 0, \\ J(t, b_\nu; \mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_q^c(\mathcal{M})) &\leq C(p, q) J(t, v_\nu; \mathcal{H}_p^c(\mathcal{M}), \mathcal{H}_q^c(\mathcal{M})), \quad t > 0. \end{aligned}$$

Following similar argument used in the proof of [40, Sublemma 3.3], we have that the series  $\sum_{\nu \in \mathbb{Z}} a_\nu$  and  $\sum_{\nu \in \mathbb{Z}} b_\nu$  are convergent in  $\mathfrak{h}_\Phi^d(\mathcal{M})$  and  $\mathfrak{h}_\Phi^c(\mathcal{M})$  respectively. Set

$$(4.13) \quad a := \sum_{\nu \in \mathbb{Z}} a_\nu \in \mathfrak{h}_\Phi^d(\mathcal{M}) \text{ and } b := \sum_{\nu \in \mathbb{Z}} b_\nu \in \mathfrak{h}_\Phi^c(\mathcal{M}).$$

Combining (4.9) with (4.12) we further get:

$$(4.14) \quad \begin{aligned} \sum_{\nu \in \mathbb{Z}} 2^{\nu/\theta} \Phi \left[ 2^{-\nu/(\theta p)} J(2^\nu, a_\nu; \mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_q^d(\mathcal{M})) \right] &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))], \\ \sum_{\nu \in \mathbb{Z}} 2^{\nu/\theta} \Phi \left[ 2^{-\nu/(\theta p)} J(2^\nu, b_\nu; \mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_q^c(\mathcal{M})) \right] &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))]. \end{aligned}$$

Next, we convert the above inequalities into their corresponding continuous forms. By setting for  $t \in [2^\nu, 2^{\nu+1})$ ,

$$a(t) = \frac{a_\nu}{\log 2} \in \mathfrak{h}_p^d(\mathcal{M}) \cap \mathfrak{h}_q^d(\mathcal{M}) \text{ and } b(t) = \frac{b_\nu}{\log 2} \in \mathfrak{h}_p^c(\mathcal{M}) \cap \mathfrak{h}_q^c(\mathcal{M}),$$

we get that  $a(\cdot)$  is a representation of  $a$  for the couple  $(\mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_q^d(\mathcal{M}))$  and  $b(\cdot)$  is a representation of  $b$  for the couple  $(\mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_q^c(\mathcal{M}))$ . Moreover, [41, Lemma 3.12(ii)] and (4.14) provide integral estimates involving the  $J$ -functionals:

$$(4.15) \quad \begin{aligned} \int_0^\infty \Phi \left[ t^{-1/p} J(t^{1/p-1/q}, a(t^{1/p-1/q}); \mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_q^d(\mathcal{M})) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))], \\ \int_0^\infty \Phi \left[ t^{-1/p} J(t^{1/p-1/q}, b(t^{1/p-1/q}); \mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_q^c(\mathcal{M})) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))]. \end{aligned}$$

By Lemma 4.8(iii), these further yield:

$$(4.16) \quad \begin{aligned} \int_0^\infty \Phi \left[ t^{-1/p} K(t^{1/p-1/q}, a; \mathfrak{h}_p^d(\mathcal{M}), \mathfrak{h}_q^d(\mathcal{M})) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))], \\ \int_0^\infty \Phi \left[ t^{-1/p} K(t^{1/p-1/q}, b; \mathfrak{h}_p^c(\mathcal{M}), \mathfrak{h}_q^c(\mathcal{M})) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))]. \end{aligned}$$

Let  $\mathcal{N}_1 := \mathcal{M} \overline{\otimes} \ell_\infty$  and  $\mathcal{N}_2 := \mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))$ . Since for every  $1 \leq r < \infty$ ,  $\mathcal{D}_d : \mathfrak{h}_r^d(\mathcal{M}) \rightarrow L_r(\mathcal{N}_1)$  and  $U\mathcal{D}_c : \mathfrak{h}_r^c(\mathcal{M}) \rightarrow L_r(\mathcal{N}_2)$  are isometries, inequalities (4.16) implies:

$$(4.17) \quad \begin{aligned} \int_0^\infty \Phi \left[ t^{-1/p} K(t^{1/p-1/q}, \mathcal{D}_d(a); L_p(\mathcal{N}_1), L_q(\mathcal{N}_1)) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))], \\ \int_0^\infty \Phi \left[ t^{-1/p} K(t^{1/p-1/q}, U\mathcal{D}_c(b); L_p(\mathcal{N}_2), L_q(\mathcal{N}_2)) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))]. \end{aligned}$$

We apply Lemma 4.8(i) to see that the next two inequalities follow from (4.17):

$$(4.18) \quad \begin{aligned} \int_0^\infty \Phi \left[ t^{-1} K(t, \mathcal{D}_d(a); L_1(\mathcal{N}_1), \mathcal{N}_1) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))], \\ \int_0^\infty \Phi \left[ t^{-1} K(t, U\mathcal{D}_c(b); L_1(\mathcal{N}_2), \mathcal{N}_2) \right] dt &\leq C_{\Phi, p, q} \tau [\Phi(S_c(x))]. \end{aligned}$$

To conclude the proof, we observe that  $\int_0^\infty \Phi \left[ t^{-1} K(t, \mathcal{D}_d(a); L_1(\mathcal{N}_1), \mathcal{N}_1) \right] dt \approx_\Phi \tau_1 [\Phi(|\mathcal{D}_d(a)|)]$  and  $\int_0^\infty \Phi \left[ t^{-1} K(t, U\mathcal{D}_c(b); L_1(\mathcal{N}_2), \mathcal{N}_2) \right] dt \approx_\Phi \tau_2 [\Phi(|U\mathcal{D}_c(b)|)]$  where  $\tau_1$  and  $\tau_2$  are the natural traces on  $\mathcal{N}_1$  and  $\mathcal{N}_2$ , respectively. It is now straightforward to verify that  $\tau_1 [\Phi(|\mathcal{D}_d(a)|)] = \sum_{n \geq 1} \tau [\Phi(|da_n|)]$  and

$\tau_2[\Phi(|U\mathcal{D}_c(b)|)] = \tau[\Phi(s_c(b))]$ , that is, we obtain that  $x = a + b$  and  $\sum_{n \geq 1} \tau[\Phi(|da_n|)] + \tau[\Phi(s_c(b))] \leq C_{\Phi} \tau[\Phi(S_c(x))]$ . The proof of (i) is complete.

Now we provide the argument for (ii). We adapt the duality technique used in [41]. Assume that  $\Phi$  is  $p$ -convex and  $q$ -concave. If  $1/p + 1/p' = 1$  and  $1/q + 1/q' = 1$ , then  $1 < q' < p' < \infty$ . We observe that  $\Phi^*$  is  $q'$ -convex and  $p'$ -concave. Therefore the inequality stated in (i) applies to  $\Phi^*$ . By approximation, it is enough to verify the inequality for  $x \in L_1(\mathcal{M}) \cap \mathcal{M}$ . Let  $\mathcal{N} = \mathcal{M} \otimes B(\ell_2)$  equipped with its natural trace which we will denote by  $\tau_{\mathcal{N}}$ .

For  $1 < r < \infty$ , consider  $\Pi : L_r(\mathcal{N}) \rightarrow L_r(\mathcal{N})$  defined by setting  $\Pi((a_{ij})) = \sum_{n \geq 1} a_{1n} \otimes e_{n,1}$ . Clearly,  $\Pi$  is a contraction. Using the noncommutative Stein inequality,  $\tilde{\Pi} : L_r(\mathcal{N}) \rightarrow L_r(\mathcal{N})$  given by  $\tilde{\Pi}((a_{ij})) = \sum_{n \geq 1} [\mathcal{E}_n(a_{1n}) - \mathcal{E}_{n-1}(a_{1n})] \otimes e_{n,1}$  is also bounded for all  $1 < r < \infty$ . By Lemma 4.7, there exists a constant  $C_{\Phi^*}$  so that

$$(4.19) \quad \tau_{\mathcal{N}}[\Phi^*(|\tilde{\Pi}((a_{ij}))|)] \leq C_{\Phi^*} \tau_{\mathcal{N}}[\Phi^*(|(a_{ij})|)].$$

As in [41], we may fix  $t_{\Phi} > 0$  so that for every operator  $0 \leq z \in L_{\Phi^*}(\mathcal{N})$ ,

$$(4.20) \quad \Phi^*(t_{\Phi} z) \leq (2C_{\Phi^*} \alpha_{\Phi^*})^{-1} \Phi^*(z)$$

where  $\alpha_{\Phi^*}$  is the constant from (i) applied to  $\Phi^*$ .

Set  $w := \sum_{n \geq 1} dx_n \otimes e_{n,1} \in L_1(\mathcal{N}) \cap \mathcal{N}$ . It is clear that  $|w| = S_c(x) \otimes e_{1,1}$ . By Lemma 4.9, we may choose  $0 \leq y \in L_{\Phi^*}(\mathcal{N})$  such that  $y$  commutes with  $|w|$  and

$$(4.21) \quad \Phi(|w|) + \Phi^*(y) = y|w|.$$

If  $w = u|w|$  is the polar decomposition of  $w$ , we set  $z := yu^* \in L_1(\mathcal{N}) \cap \mathcal{N}$ . Write  $z = (z_{ij})$  where  $z_{ij} \in L_1(\mathcal{M}) \cap \mathcal{M}$  and set

$$v = \sum_{n \geq 1} \mathcal{E}_n(z_{1n}) - \mathcal{E}_{n-1}(z_{1n}).$$

It is clear that  $\tau_{\mathcal{N}}(y|w|) = \tau_{\mathcal{N}}(zw) = \tau(vx)$  and  $v \in L_1(\mathcal{M}) \cap \mathcal{M}$ . In particular, we may view  $v$  as a martingale in  $\mathcal{H}_{\Phi^*}^c(\mathcal{M})$ . We now apply (i) to  $v$ . There exists a decomposition  $v = v^d + v^c$  with  $v^c \in \mathfrak{h}_{\Phi^*}^c(\mathcal{M})$  and  $v^d \in \mathfrak{h}_{\Phi^*}^d(\mathcal{M})$  satisfying:

$$(4.22) \quad \tau[\Phi^*(s_c(v^c))] + \sum_{n \geq 1} \tau[\Phi^*(|dv_n^d|)] \leq 2\alpha_{\Phi^*} \tau[\Phi^*(S_c(v))].$$

Taking traces on (4.21) together with the decomposition of  $v$ , we have

$$\begin{aligned} \tau_{\mathcal{N}}[\Phi(|w|)] + \tau_{\mathcal{N}}[\Phi^*(y)] &= \tau(xv) \\ &= \tau(xv^d) + \tau(xv^c). \end{aligned}$$

One can verify that the following estimates hold (see [41] for details):

$$\tau(xv^d) \leq \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)] + (2C_{\Phi^*} \alpha_{\Phi^*})^{-1} \sum_{n \geq 1} \tau[\Phi^*(|dv_n^d|)]$$

and

$$\tau(xv^c) \leq \tau[\Phi(t_{\Phi}^{-1}s_c(x))] + (2C_{\Phi^*} \alpha_{\Phi^*})^{-1} \tau[\Phi^*(s_c(v^c))].$$

Applying (4.19), (4.22), and taking the sum of the previous two estimates, we arrive at:

$$\begin{aligned} \tau_{\mathcal{N}}[\Phi(|w|)] + \tau_{\mathcal{N}}[\Phi^*(y)] &\leq \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)] + \tau[\Phi(t_{\Phi}^{-1}s_c(x))] + C_{\Phi^*}^{-1} \tau[\Phi^*(S_c(v))] \\ &\leq \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)] + \tau[\Phi(t_{\Phi}^{-1}s_c(x))] + C_{\Phi^*}^{-1} \tau_{\mathcal{N}}[\Phi^*(|\tilde{\Pi}(z)|)] \\ &\leq \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)] + \tau[\Phi(t_{\Phi}^{-1}s_c(x))] + \tau_{\mathcal{N}}[\Phi^*(|z|)]. \end{aligned}$$

As we clearly have  $\tau_{\mathcal{N}}[\Phi^*(|z|)] \leq \tau_{\mathcal{N}}[\Phi^*(y)]$ , we deduce that

$$\begin{aligned} \tau[\Phi(S_c(x))] &= \tau_{\mathcal{N}}[\Phi(|w|)] \leq \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)] + \tau[\Phi(t_{\Phi}^{-1}s_c(x))] \\ &\leq 2 \max \left\{ \sum_{n \geq 1} \tau[\Phi(t_{\Phi}^{-1}|dx_n|)], \tau[\Phi(t_{\Phi}^{-1}s_c(x))] \right\}. \end{aligned}$$

The existence of the constant  $\beta_{\Phi}$  in the statement (ii) now follows from the  $\Delta_2$ -condition.  $\square$

The next result solves [41, Problem 6.3]. It can be deduced at once from combining both the row and column versions of Theorem 4.6 with Theorem 4.5.

**Theorem 4.10.** *Let  $1 < p < q < \infty$  and  $\Phi$  be an Orlicz function that is  $p$ -convex and  $q$ -concave. Then there exist positive constants  $\delta_{\Phi}$  and  $\eta_{\Phi}$  such that:*

(i) *for every martingale  $x \in L_{\Phi}(\mathcal{M})$ , the following inequality holds:*

$$\delta_{\Phi}^{-1} \inf \left\{ \tau[\Phi(s_c(x^c))] + \tau[\Phi(s_r(x^r))] + \sum_{n \geq 1} \tau[\Phi(|dx_n^d|)] \right\} \leq \tau[\Phi(|x|)],$$

*with the infimum being taken over all  $x^c \in \mathfrak{h}_{\Phi}^c(\mathcal{M})$ ,  $x^r \in \mathfrak{h}_{\Phi}^r(\mathcal{M})$ , and  $x^d \in \mathfrak{h}_{\Phi}^d(\mathcal{M})$  such that  $x = x^d + x^c + x^r$ ;*

(ii) *for every  $x \in \mathfrak{h}_{\Phi}^d(\mathcal{M}) \cap \mathfrak{h}_{\Phi}^c(\mathcal{M}) \cap \mathfrak{h}_{\Phi}^r(\mathcal{M})$ , the following inequality holds:*

$$\tau[\Phi(|x|)] \leq \eta_{\Phi} \max \left\{ \sum_{n \geq 1} \tau[\Phi(|dx_n|)], \tau[\Phi(s_c(x))], \tau[\Phi(s_r(x))] \right\}.$$

We are now in a position of stating our  $\Phi$ -moment version of Burkholder/Rosenthal inequality. It should be compared with a recent version of the Burkholder-Gundy inequality from [22, Theorem 7.2]. Our result is much more general than the version obtained in [41]. In fact, it solves [41, Problems 6.4].

**Theorem 4.11.** *Let  $\Phi$  be an Orlicz function.*

(i) *If  $\Phi$  is  $p$ -convex for some  $1 < p < 2$  and 2-concave, then there exists a positive constant  $C_{\Phi}$  so that the following holds for every martingale  $x \in L_{\Phi}(\mathcal{M})$ :*

$$C_{\Phi}^{-1} \tau[\Phi(|x|)] \leq \inf \left\{ \tau[\Phi(s_c(x^c))] + \tau[\Phi(s_r(x^r))] + \sum_{n \geq 1} \tau[\Phi(|dx_n^d|)] \right\} \leq C_{\Phi} \tau[\Phi(|x|)],$$

*where the infimum is taken over all  $x^c \in \mathfrak{h}_{\Phi}^c(\mathcal{M})$ ,  $x^r \in \mathfrak{h}_{\Phi}^r(\mathcal{M})$ , and  $x^d \in \mathfrak{h}_{\Phi}^d(\mathcal{M})$  such that  $x = x^d + x^c + x^r$ ;*

(ii) *If  $\Phi$  is 2-convex and  $q$ -concave for some  $q > 2$ , then there exists a positive constant  $c_{\Phi}$  so that the following holds for every martingale  $x \in L_{\Phi}(\mathcal{M})$ :*

$$c_{\Phi}^{-1} \tau[\Phi(|x|)] \leq c_{\Phi} \max \left\{ \sum_{n \geq 1} \tau[\Phi(|dx_n|)], \tau[\Phi(s_c(x))], \tau[\Phi(s_r(x))] \right\} \leq c_{\Phi} \tau[\Phi(|x|)].$$

*Proof.* To prove (i), it is enough to verify that there exists a constant  $C_{\Phi}$  so that for every decomposition  $x = x^d + x^c + x^r$ ,

$$\tau[\Phi(|x|)] \leq C_{\Phi} \left\{ \tau[\Phi(s_c(x^c))] + \tau[\Phi(s_r(x^r))] + \sum_{n \geq 1} \tau[\Phi(|dx_n^d|)] \right\},$$

as the reverse inequality is already contained in Theorem 4.10. This follows from the facts that  $\mathfrak{h}_2^s(\mathcal{M}) = L_2(\mathcal{M})$ ,  $\mathfrak{h}_p^s(\mathcal{M}) \subset L_p(\mathcal{M})$  for  $s \in \{d, c, r\}$ , and that the Hardy spaces are complemented subspaces of noncommutative  $L_p$ -spaces. Indeed, for  $w \in \{p, 2\}$ , let  $\Pi : L_w(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \rightarrow \mathfrak{h}_w^c(\mathcal{M})$  be the bounded projection (see [23] for the fact that the projections are simultaneously bounded) and  $\Theta : \mathfrak{h}_w^c(\mathcal{M}) \rightarrow L_w(\mathcal{M})$  the formal inclusion. By the noncommutative Burkholder inequality,  $\Theta \circ \Pi : L_w(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \rightarrow L_w(\mathcal{M})$  is bounded. By Lemma 4.7, we have for every  $a \in \mathfrak{h}_{\Phi}^c(\mathcal{M})$ ,

$$\tau[\Phi(|a|)] = \tau[\Phi(|\Theta \Pi(UD_c(a))|)] \leq C_{\Phi} \tau \otimes \text{tr}[\Phi(|UD_c(a)|)] = C_{\Phi} \tau[\Phi(s_c(a))].$$

Similar arguments can be applied to the diagonal and the row parts.

(ii) Assume now that  $\Phi$  is 2-convex and  $q$ -concave for some  $q > 2$ . By the noncommutative Burkholder inequalities, the formal inclusion is bounded from  $L_w(\mathcal{M})$  into  $\mathfrak{h}_w^c(\mathcal{M})$  for all  $w \geq 2$ . Denote this by  $I$ . We have  $UD_cI : L_w(\mathcal{M}) \rightarrow L_w(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2)))$  is bounded for all  $w \geq 2$ . We deduce from Lemma 4.7 that for every  $b \in L_\Phi(\mathcal{M})$ ,

$$\tau[\Phi(s_c(b))] = \tau \otimes \text{tr}[\Phi(|UD_cI(b)|)] \leq C_\Phi \tau[\Phi(|b|)].$$

Applying the same argument for the diagonal and the row parts, we have for every  $x \in L_\Phi(\mathcal{M})$ ,

$$\max \left\{ \sum_{n \geq 1} \tau[\Phi(|dx_n|)], \tau[\Phi(s_c(x))], \tau[\Phi(s_r(x))] \right\} \leq C_\Phi \tau[\Phi(|x|)].$$

The reverse inequality is already contained in Theorem 4.10(ii).  $\square$

We conclude by exhibiting examples of Orlicz functions for which the  $\Phi$ -moment versions of the noncommutative Burkholder inequalities apply but not covered by the results from [41].

*Example 4.12.* Let  $\Phi = t^p \log(1 + t^q)$  with  $p > 1$  and  $q > 0$ . One can check that  $p_\Phi = p$  and  $q_\Phi = p + q$ . Also, since  $\Phi(t)/t^p$  is increasing and  $\Phi(t)/t^{p+q}$  is decreasing,  $\Phi$  is  $p$ -convex and  $p + q$ -concave.

- (i) If  $p + q = 2$  then the equivalence in Theorem 4.11(i) holds for  $\Phi$ .
- (ii) If  $p = 2$  then the equivalence in Theorem 4.11(ii) holds for  $\Phi$ .

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DEPARTMENT OF MATHEMATICS, MIAMI UNIVERSITY, OXFORD, OHIO 45056, USA  
*E-mail address:* `randrin@miamioh.edu`

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA 410075, CHINA  
*E-mail address:* `wulian@cnu.edu.cn`

INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN 150001, CHINA; AND  
LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ, 25030 BESANÇON CEDEX, FRANCE; AND  
INSTITUT UNIVERSITAIRE DE FRANCE  
*E-mail address:* `qxu@univ-fcomte.fr`