CONTROL OF THE NON-GEOMETRICALLY INTEGRAL REDUCTIONS

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Abstract. — In this paper, for a geometrically integral projective scheme, we will give an upper bound of the product of the norms of its non-geometrically integral reductions over an arbitrary number field. For this aim, we take the adelic viewpoint on this subject.

Résumé **(Contrôle des réductions non-géométriquement intègres)**
Dans cet article, pour un schéma projectif géométriquement intègre, on donnera une majoration du produit des norms de ses réductions non-géométriquement intègre sur un corps de nombres arbitraire. Pour le but, on prend le point de vue adélique autour de ce sujet.

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Contents

1. Introduction

Let $X \hookrightarrow \mathbb{P}^n_K \to \operatorname{Spec} K$ be a geometrically integral closed sub-scheme over a number field K, $\mathscr{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K} \to \operatorname{Spec} \mathcal{O}_K$ be its Zariski closure, and $\mathscr{X}_{\mathbb{F}_\mathfrak{p}} =$

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$\mathscr{X}\times_{\operatorname{Spec}\mathcal{O}_K}\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}\to\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}$ be its fiber at $\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K$. By [**8**, Théorème 9.7.7], the set

$$\mathcal{Q}(\mathscr{X})=\left\{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K \mid \mathscr{X}_{\mathbb{F}_{\mathfrak{p}}}\to\operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \text{ is not geometrically integral}\right\} \tag{1}$$

is finite.

If we replace the geometrically integral property of $X\to\operatorname{Spec}K$ above by the integral property, the set

$$\mathcal{Q}'(\mathscr{X})=\left\{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K \mid \mathscr{X}_{\mathbb{F}_{\mathfrak{p}}}\to\operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \text{ is not integral}\right\}$$

is not finite in general. For example, we consider

$$X=\operatorname{Proj}\left(\mathbb{Q}[T_0,T_1,T_2]/\left(T_0^2+T_1^2\right)\right)\to\operatorname{Spec}\mathbb{Q}.$$

Then we have

$$\mathcal{Q}'(\mathscr{X})=\{p \text{ prime} \mid p\equiv 1 (\text{mod } 4) \text{ or } p=2\}$$

by properties on the quadric residue, which is a infinite set.

By the above reasons, it is reasonable to give a numerical description of the set of non-geometrically integral reductions $\mathcal{Q}(\mathscr{X})$. Actually, it is easy to construct examples such that $\mathcal{Q}(\mathscr{X})=\emptyset$, for example, the scheme X is a hyperplane in $\mathbb{P}^n_K$. In addition, we can construct examples whose non-geometrically integral reductions are any products of prime ideals. Let

$$X=\operatorname{Proj}\left(\mathbb{Q}[T_0,T_1,T_2]/\left(T_0^2+aT_1T_2\right)\right)\to\operatorname{Spec}\mathbb{Q},$$

where $a\in\mathbb{Z}$. In this case, we have

$$\mathcal{Q}(\mathscr{X})=\{p \text{ prime} \mid p\mid a\}.$$

Hence, we are interested in the upper bound of $\sum\limits_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p})$.

1.1. Brief history. — Traditionally, we only focus on the case of hypersurfaces when $K=\mathbb{Q}$ and $\mathcal{O}_K=\mathbb{Z}$, and there are fruitful results reported on this subject. By [**13**, Exercise 2.4.1], we only need to study whether the polynomial defining this hypersurface is absolutely irreducible over the residue field. Usually, lots of former works considered the case of plane curves only.

Up to the author's knowledge, this subject was first considered by A. Ostrowski in [**14**] implicitly. In [**18**], W. M. Schmidt gave an explicit estimate, which is refined by E. Kaltofen in [**10**] (see also [**11**]).

In [**15**], W. M. Ruppert transferred the criterion of absolute irreducibility of polynomials into the existence of certain polynomial solutions to a certain system of partial differential equations, where he considered the de Rham cohomology of some particular complexes. By this result, he gave an upper bound of non-geometrically reductions for the case of arbitrary hypersurfaces, and a sharper upper bound for the case of plane curves. This result improved some previous results, and was generalized by [**19**] and [**5**] to different directions.

In [**22**], U. Zannier gave an upper bound depending on the multi-degree of a polynomial $f(x,y)$ over $\mathbb{Z}$. This result is improved by W. M. Ruppert in [**17**] by refining his method in [**15**]. In [**16**], he considered a special kind of plane curves and gave a better upper bound.

In [**6**], Shuhong Gao and V. M. Rodrigues applied Newton polytopes to refine the estimate in [**17**], where they involved the number of integral points of Newton polytopes into the estimate.

1.2. Relation to the arithmetic Hodge index theorem. — In [**3**, §5], G. Faltings proved an arithmetic analogue of the Hodge index theorem. Let $\mathscr{X} \to \operatorname{Spec}\mathcal{O}_K$, be an arithmetic surface, which means it is flat and projective with the relative dimension 1, whose generic fiber is $X \to \operatorname{Spec} K$. In [**3**, Theorem 4 d)], the author gave an estimate of

$$\sum_{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K}\left(\left(\text{number of components of } \mathscr{X}_{\mathbb{F}_\mathfrak{p}}\right)-1\right),$$

which is related the rank of jacobian of X over K. In fact, the above estimate is able to provide an estimate of the number of non-geometrically integral reductions for the case of curves, while the invariant $\sum\limits_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p})$ can provide the same estimate. But their methods and fundamental ideas are quite different.

1.3. Adelic viewpoint. — In this paper, we will give such an upper bound for the case of an arbitrary number field. Let $X \hookrightarrow \mathbb{P}^n_K$ be a hypersurface, and we consider the Zariski closure $\mathscr{X}$ of X in $\mathbb{P}^n_{\mathcal{O}_K}$. In this case, only when $\mathcal{O}_K$ is a principal ideal domain, $\mathscr{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ can always be defined by a primitive equation with coefficients in $\mathcal{O}_K$.

Similar to the method in [**12**] to study the non-reduced reductions over an arbitrary number field, we introduce the adelic viewpoint to overcome this obstruction. We consider the polynomial with coefficients in K defining $X \hookrightarrow \mathbb{P}^n_K$ as coefficients in the adelic ring $\mathbb{A}_K$ with respect to the diagonal embedding, and then we can obtain a primitive $\mathbb{A}_{\mathcal{O}_K}$-coefficient polynomial by multiplying an element in $\mathbb{A}_K$ which does not change the height of polynomial in the adelic sense. Then for each $\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K$, the $\mathfrak{p}$-part of this primitive polynomial of $\mathbb{A}_{\mathcal{O}_K}$-coefficients is primitive over $\mathcal{O}_{K,\mathfrak{p}}$, which defines $\mathscr{X}_\mathfrak{p} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_{K,\mathfrak{p}}}$ from $\mathscr{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ via the base change $\operatorname{Spec}\mathcal{O}_{K,\mathfrak{p}} \to \operatorname{Spec}\mathcal{O}_K$. Then we can consider the reduction type of each $\mathscr{X}_\mathfrak{p}$ modulo $\mathfrak{p}$.

In order to judge whether a projective hypersurface is geometrically integral, we use a numerical criterion of Ruppert [**15**, Satz 3, Satz 4]. For the general case, we use the theory of Chow varieties and Cayley varieties to reduce it to the case of hypersurfaces, which is similar to that in [**12**, §6]. In fact, we have the following estimate in Theorem 4.5.

Theorem. — *Let X be a geometrically integral closed sub-scheme of $\mathbb{P}^n_K$ of pure dimension d and degree δ, and $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}^n_{\mathcal{O}_K}$. Then we have*

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p}) \leqslant (\delta^2-1)h(X)+C(n,d,\delta),$$

where $h(X)$ is a height of X and $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$. We will give the above constant $C(n,d,\delta)$ explicitly in Theorem 4.5, and we have $C(n,d,\delta) \ll_n \delta^3$.

If we consider the case of plane curves ($d = 1$ and $n = 2$) and use the naive height (see Definition 2.2) in the above theorem, we are able to obtain $C(n,d,\delta) \ll_n \delta^2 \log \delta$ in the above estimate, see Proposition 4.3.

1.4. Structure of the article. — This paper is organized as follows. In §2, we introduce the useful notions on Diophantine geometry and Arakelov geometry. In §3, we recall some results of Ruppert on the criterion of the geometrically integral property. In §4, we give an upper bound of the non-geometrically reductions for the case of hypersurfaces by the above results of Ruppert under the adelic viewpoint, and such an upper bound for the general case by applying the theory of Chow varieties and Cayley varieties. In Appendix A, we provide the solution to an exercise in [**13**], which is useful in this work.

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2. Height functions

The height of arithmetic varieties is an invariant which evaluates the arithmetic complexity of varieties. In order to study it, we introduce some preliminaries of Arakelov geometry and Diophantine geometry.

2.1. Normed vector bundles. — Normed vector bundles are one of the main research objects in Arakelov geometry. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. We denote by M_K the set of places of K, by $M_{K,f}$ the set of its finite places and by $M_{K,\infty}$ the set of its infinite places. A *normed vector bundle* on $\operatorname{Spec} \mathcal{O}_K$ is a pair $\overline{E} = \left(E, (\|\cdot\|_v)_{v \in M_{K,\infty}}\right)$, where:

— E is a projective $\mathcal{O}_K$-module of finite rank;

— $(\|\cdot\|_v)_{v \in M_{K,\infty}}$ is a family of norms, where $\|\cdot\|_v$ is a norm over $E \otimes_{\mathcal{O}_K,v} \mathbb{C}$ which is invariant under the action of $\operatorname{Gal}(\mathbb{C}/K_v)$.

If all the norms $(\|\cdot\|_v)_{v \in M_{K,\infty}}$ are Hermitian, we call $\overline{E}$ a *Hermitian vector bundle* on $\operatorname{Spec} \mathcal{O}_K$. In particular, if $\operatorname{rk}_{\mathcal{O}_K}(E) = 1$, we say that $\overline{E}$ is a *Hermitian line bundle* since all Archimedean norms are Hermitian in this case.

2.2. Height of arithmetic varieties. — In this part, we introduce a kind of height functions of arithmetic varieties defined via the arithmetic intersection theory developed by Gillet and Soulé in [**7**], which is first introduced by Faltings in [**4**, Definition 2.5], see also [**20**, III.6].

***Definition 2.1* (Arakelov height).** — Let K be a number field, $\mathcal{O}_K$ be its ring of integers, $\overline{\mathcal{E}}$ be a Hermitian vector bundle of rank $n+1$ on $\operatorname{Spec}\mathcal{O}_K$, and $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathbb{P}(\mathcal{E})$. Let X be a pure dimensional closed sub-scheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension d, and $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. The *Arakelov height* of X is defined as the arithmetic intersection number

$$\frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}\cdot[\mathscr{X}]\right),$$

where $\widehat{c}_1(\overline{\mathcal{L}})$ is the arithmetic first Chern class of $\overline{\mathcal{L}}$ (see [**20**, Chap. III.4, Proposition 1] for its definition), and $\widehat{\deg}(\cdot)$ is the *Arakelov degree* of arithmetic cycles. This height is denoted by $h_{\overline{\mathcal{L}}}(X)$ or $h_{\overline{\mathcal{L}}}(\mathscr{X})$.

2.3. Height of hypersurfaces. — Let X be a hypersurface in $\mathbb{P}^n_K$. By [**9**, Proposition 7.6, Chap. I], X is defined by a homogeneous polynomial. We define a height function of hypersurfaces by considering its defining polynomial.

***Definition 2.2* (Naive height).** — Let $f(T_0,\ldots,T_n)=\sum\limits_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}}a_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}$ be a polynomial. We define the *naive height* of $f(T_0,\ldots,T_n)$ as

$$H_K(f)=\prod_{v\in M_K}\max_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}}\left\{|a_{i_0,\ldots,i_n}|_v\right\}^{[K_v:\mathbb{Q}_v]},$$

and $h(f)=\frac{1}{[K:\mathbb{Q}]}\log H_K(f)$. In addition, if $f(T_0,\ldots,T_n)$ is the defining polynomial of the hypersurface $X\hookrightarrow\mathbb{P}^n_K$, we define the *naive height* of X as

$$h(X)=h(f).$$

2.4. Adelic height. — In order to consider the reductions over an arbitrary number field, we will introduce the so-called *adelic height* of a polynomial, which has been applied to study the non-reduced reductions in [**12**].

Let K be a number field, $\mathcal{O}_K$ be its ring of integers. In addition, we denote by

$$\mathbb{A}_K=\left\{(a_v)_v\in\prod_{v\in M_K}K_v\ \middle|\ a_v\in\mathcal{O}_{K,v}\text{ except a finite number of }v\in M_{K,f}\right\}$$

the adelic ring of K, by

$$\mathbb{A}_{\mathcal{O}_K}=\{(a_v)_v\in\mathbb{A}_K|\ a_v\in\mathcal{O}_{K,v}\text{ for all }v\in M_{K,f}\}$$

the integral adelic ring of K, and by $\Delta:K\hookrightarrow\mathbb{A}_K$ the diagonal embedding. Let $a=(a_v)_{v\in M_K}\in\mathbb{A}_K$, we define

$$|a|_{\mathbb{A}_K}=\prod_{v\in M_K}|a_v|_v^{[K_v:\mathbb{Q}_v]}.$$

***Definition 2.3* (Local part).** — Let $\{a_{i_0,\ldots,i_n}\}=\{(a^v_{i_0,\ldots,i_n})_{v\in M_K}\}$ be a finite family of elements in $\mathbb{A}_K$ with the indices $(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}$, and

$$f(T_0,\ldots,T_n)=\sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}}a_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}$$

be a non-zero polynomial in $\mathbb{A}_K[T_0, \ldots, T_n]$. For each $v \in M_K$, we denote by

$$f^{(v)}(T_0, \ldots, T_n) = \sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} a^v_{i_0,\ldots,i_n} T_0^{i_0} \cdots T_n^{i_n}$$

the *v-part* of $f(T_0, \ldots, T_n)$, or by $f^{(\mathfrak{p})}(T_0, \ldots, T_n)$ for $\mathfrak{p} \in \operatorname{Spm} \mathcal{O}_K$ corresponding to the place $v \in M_{K,f}$, which is called the *$\mathfrak{p}$-part* of $f(T_0, \ldots, T_n)$.

Definition 2.4 **(Adelic height).** — Let $f(T_0, \ldots, T_n) = \sum\limits_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} a_{i_0,\ldots,i_n} T_0^{i_0} \cdots T_n^{i_n}$ be a polynomial with coefficients in $\mathbb{A}_K$, where we denote $a_{i_0,\ldots,i_n} = (a^v_{i_0,\ldots,i_n})_{v\in M_K} \in \mathbb{A}_K$ for every index $(i_0, \ldots, i_n)$ in the above polynomial. We define

$$H_{\mathbb{A}_K}(f) = \prod_{v\in M_K} \max_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} \{|a^v_{i_0,\ldots,i_n}|_v\}^{[K_v:\mathbb{Q}_v]}$$

as the *adelic height* of f. In addition, we denote $h(f) = \frac{1}{[K:\mathbb{Q}]} \log H_{\mathbb{A}_K}(f)$.

Let $f(T_0, \ldots, T_n) = \sum\limits_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} a_{i_0,\ldots,i_n} T_0^{i_0} \cdots T_n^{i_n}$ be a polynomial with coefficients in K, and $c \in \mathbb{A}_K$ with $|c|_{\mathbb{A}_K} = 1$. Let

$$g(T_0, \ldots, T_n) = \sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} c\Delta(a_{i_0,\ldots,i_n}) T_0^{i_0} \cdots T_n^{i_n}$$

be the polynomial with coefficients in $\mathbb{A}_K$. Then by definition, we have

$$H_{\mathbb{A}_K}(g) = H_K(f), \tag{2}$$

where $H_K(f)$ is defined in Definition 2.2.

Let

$$f(T_0, \ldots, T_n) = \sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} a_{i_0,\ldots,i_n} T_0^{i_0} \cdots T_n^{i_n}$$

be a polynomial with coefficients in K. By [**12**, Lemme 3.8], there exists an element $c \in \mathbb{A}_K$ with $|c|_{\mathbb{A}_K} = 1$, such that for each $v \in M_{K,f}$, we have

$$\max_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} \{|c\Delta(a_{i_0,\ldots,i_n})|_v\} = 1.$$

Let $b_{i_0,\ldots,i_n} = c\Delta(a_{i_0,\ldots,i_n})$, then

$$F(T_0, \ldots, T_n) = \sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} b_{i_0,\ldots,i_n} T_0^{i_0} \cdots T_n^{i_n} \in \mathbb{A}_{\mathcal{O}_K}[T_0, \ldots, T_n], \tag{3}$$

which is called an *adelicly primitive polynomial* of f.

3. A criterion of non-geometrically property

Let X be a geometrically integral hypersurface of $\mathbb{P}^n_K$ defined by the homogeneous polynomial $f(T_0,\ldots,T_n)$, and $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}^n_{\mathcal{O}_K}$. For all $\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K$, in order to study the reduction of $\mathscr{X}\hookrightarrow\mathbb{P}^n_{\mathcal{O}_K}\to\operatorname{Spec}\mathcal{O}_K$ at $\mathfrak{p}$, we factor the reduction through the localization at $\mathfrak{p}$. More precisely, we consider the Cartesian diagram

$$\begin{array}{ccccc}
\mathscr{X}_{\mathbb{F}_\mathfrak{p}} & \longrightarrow & \mathscr{X}_{\mathcal{O}_{K,\mathfrak{p}}} & \longrightarrow & \mathscr{X} \\
\big\downarrow & \square & \big\downarrow & \square & \big\downarrow \\
\mathbb{P}^n_{\mathbb{F}_\mathfrak{p}} & \longrightarrow & \mathbb{P}^n_{\mathcal{O}_{K,\mathfrak{p}}} & \longrightarrow & \mathbb{P}^n_{\mathcal{O}_K} \\
\big\downarrow & \square & \big\downarrow & \square & \big\downarrow \\
\operatorname{Spec}\mathbb{F}_\mathfrak{p} & \longrightarrow & \operatorname{Spec}\mathcal{O}_{K,\mathfrak{p}} & \longrightarrow & \operatorname{Spec}\mathcal{O}_K.
\end{array}$$

By definition, $\mathscr{X}_{\mathcal{O}_{K,\mathfrak{p}}}\hookrightarrow\mathbb{P}^n_{\mathcal{O}_{K,\mathfrak{p}}}$ is defined by the $\mathfrak{p}$-part $F^{(\mathfrak{p})}(T_0\ldots,T_n)$ of $F(T_0,\ldots,T_n)$ (see Definition 2.3 for its definition, which is primitive over $\mathcal{O}_{K,\mathfrak{p}}$ by the construction of $F(T_0,\ldots,T_n)$ in (3).

By [**13**, Exercise 2.4.1] (see [**12**, Remarque 5.2] for a projective version), for an arbitrary $\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K$, the fact that the polynomial $F^{(\mathfrak{p})}(T_0\ldots,T_n)$ modulo $\mathfrak{p}[T_0,\ldots,T_n]$ is not absolutely irreducible over $\mathbb{F}_\mathfrak{p}$ is verified if and only if $\mathscr{X}_{\mathbb{F}_\mathfrak{p}}$ is not geometrically integral over $\operatorname{Spec}\mathbb{F}_\mathfrak{p}$. So in order to control the set $\mathcal{Q}(\mathscr{X})$ introduced in (1), we need to study the absolute irreducibility of $F^{(\mathfrak{p})}(T_0,\ldots,T_n) \mod \mathfrak{p}[T_0,\ldots,T_n]$ for all $\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K$.

The first result is for the case of plane curves.

Proposition 3.1 **([15], Satz 3).** — *Let*

$$g(T_0,T_1,T_2)=\sum_{\substack{(i_0,i_1,i_2)\in\mathbb{N}^3\\ i_0+i_1+i_2=\delta}} b_{i_0,i_1,i_2}T_0^{i_0}T_1^{i_1}T_2^{i_2}$$

be a homogeneous polynomial of degree δ over an algebraically closed field k. Then there exists a family of homogeneous polynomial $\{\phi_j\}_{j\in J}\in\mathbb{Z}[b_{i_0,i_1,i_2}]$ with the index set J and variables $\{b_{i_0,i_1,i_2}|\ (i_0,i_1,i_2)\in\mathbb{N}^3,\ i_0+i_1+i_2=\delta\}$, which are of degree δ^2-1 and length smaller than $\delta^{3\delta^2-3}$, such that

1. *If F is reducible, then $\phi_j(b_{i_0,i_1,i_2})=0$ for every $j\in J$;*
2. *If F is irreducible and k is of characteristic 0, then there exists at least one $j\in J$, such that $\phi_j(b_{i_0,i_1,i_2})\neq 0$.*

The second one is for the case of general hypersurfaces.

Proposition 3.2 **([15], Satz 4).** — *Let*

$$g(T_0,\ldots,T_n)=\sum_{\substack{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}\\ i_0+\cdots+i_n=\delta}} b_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}$$

be a homogeneous polynomial of degree δ over an algebraically closed field k. Then there exists a family of homogeneous polynomial $\{\phi_j\}_{j\in J} \in \mathbb{Z}[b_{i_0,\ldots,i_n}]$ with the index set J and variables $\{b_{i_0,\ldots,i_n} |\ (i_0,\ldots,i_n) \in \mathbb{N}^{n+1},\ i_0+\cdots+i_n=\delta\}$, which are of degree δ^2-1 and length smaller than $\delta^{3\delta^2-3}\left[\binom{n+\delta}{\delta}3^\delta\right]^{\delta^2-1}$, such that

1. *If F is reducible, then $\phi_j(b_{i_0,\ldots,i_n})=0$ for every $j\in J$;*
2. *If F is irreducible and k is of characteristic 0, then there exists at least one $j\in J$, such that $\phi_j(b_{i_0,\ldots,i_n})\neq 0$.*

4. Control of the non-geometrically integral reductions

By Proposition 3.1 and 3.2, Ruppert gives a control of non-geometrically integral reductions of hypersurfaces in $\mathbb{P}^n_{\mathbb{Z}}$ in [**15**, Korollar 1, Korollar 2]. In this part, we will give such a control over an arbitrary number field K for general projective schemes.

4.1. Non-geometrically integral reductions of hypersurfaces. — For the case of hypersurfaces, by applying Proposition 3.1 and 3.2 to an adelicly primitive polynomial introduced at (3), we have the following two results. Since their proofs are quite similar, we only provide the detailed proof for the case of general hypersurfaces.

Proposition 4.1. — *Let X be a geometrically integral hypersurface in $\mathbb{P}^n_K$ of degree δ, $\mathscr{X}$ be its Zariski closure in $\mathbb{P}^n_{\mathcal{O}_K}$, $\mathscr{X}_{\mathbb{F}_\mathfrak{p}} = \mathscr{X}\times_{\operatorname{Spec}\mathcal{O}_K}\operatorname{Spec}\mathbb{F}_\mathfrak{p}$, and*

$$\mathcal{Q}(\mathscr{X}) = \left\{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K |\ \mathscr{X}_{\mathbb{F}_\mathfrak{p}}\to\operatorname{Spec}\mathbb{F}_\mathfrak{p} \text{ is not geometrically integral}\right\}.$$

Then we have

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p}) \leqslant (\delta^2-1)h(X)+C(n,\delta),$$

where $N(\mathfrak{p})=\#(\mathcal{O}_K/\mathfrak{p})$, $h(X)$ is the classic height of X in $\mathbb{P}^n_K$ defined in Definition 2.2, and the constant

$$C(n,\delta)=(\delta^2-1)\left(3\log\delta+\delta\log 3+\log\binom{n+\delta}{\delta}\right).$$

Proof. — Suppose X is defined by the homogeneous polynomial

$$f(T_0,\ldots,T_n)=\sum_{\substack{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}\\ i_0+\cdots+i_n=\delta}} a_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}$$

with coefficients in K, and

$$F(T_0,\ldots,T_n)=\sum_{\substack{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}\\ i_0+\cdots+i_n=\delta}} b_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}$$

be an adelicly primitive polynomial of $f(T_0,\ldots,T_n)$ constructed in (3). We use the notations in Proposition 3.2, and choose an index $j\in J$ of the polynomial $\phi_j(b_{i_0,\ldots,i_n})$ with variables $b_{i_0,\ldots,i_n}$, such that $\phi_j(a_{i_0,\ldots,i_n})\neq 0$ for the coefficients of $f(T_0,\ldots,T_n)$.

For each $\mathfrak{p} \in \operatorname{Spm} \mathcal{O}_K$, since $b^{(\mathfrak{p})}_{i_0,\ldots,i_n} \in \mathcal{O}_{K,\mathfrak{p}}$, we have $\left|\phi_j\left(b^{(\mathfrak{p})}_{i_0,\ldots,i_n}\right)\right|_{\mathfrak{p}} \leqslant 1$ if $\phi_j\left(b^{(\mathfrak{p})}_{i_0,\ldots,i_n}\right) \neq 0$. By definition, if the maximal ideal $\mathfrak{p} \in \mathcal{Q}(\mathscr{X})$, we have $\left|\phi_j\left(b^{(\mathfrak{p})}_{i_0,\ldots,i_n}\right)\right|_{\mathfrak{p}} < 1$. Then we obtain

$$\begin{aligned}\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p}) &\leqslant -\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\frac{[K_{\mathfrak{p}}:\mathbb{Q}_{\mathfrak{p}}]}{[K:\mathbb{Q}]}\log\left(\left|\phi_j\left(b^{(\mathfrak{p})}_{i_0,\ldots,i_n}\right)\right|_{\mathfrak{p}}\right)\\ &\leqslant -\sum_{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K}\frac{[K_{\mathfrak{p}}:\mathbb{Q}_{\mathfrak{p}}]}{[K:\mathbb{Q}]}\log\left(\left|\phi_j\left(b^{(\mathfrak{p})}_{i_0,\ldots,i_n}\right)\right|_{\mathfrak{p}}\right)\\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_{K,\infty}}\log\left(\left|\phi_j\left(b^{(v)}_{i_0,\ldots,i_n}\right)\right|_{v}\right).\end{aligned}$$

In order to estimate $\log\left(\left|\phi_j\left(b^{(v)}_{i_0,\ldots,i_n}\right)\right|_{v}\right)$ for a fixed $v \in M_{K,\infty}$, from the properties of ϕ_j given in Proposition 3.2, we have

$$\begin{aligned}\log\left(\left|\phi_j\left(b^{(v)}_{i_0,\ldots,i_n}\right)\right|_{v}\right) &\leqslant (\delta^2-1)\log\left(\max_{\substack{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}\\ i_0+\cdots+i_n=\delta}}\left\{|b^{(v)}_{i_0,\ldots,i_n}|_v\right\}\right)\\ &\quad +(\delta^2-1)\left(3\log\delta+\delta\log 3+\log\binom{n+\delta}{\delta}\right).\end{aligned} \tag{4}$$

Then from (4), we obtain

$$\begin{aligned}&\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_{K,\infty}}\log\left(\left|\phi_j\left(b^{(v)}_{i_0,\ldots,i_n}\right)\right|_{v}\right)\\ \leqslant\ &\frac{\delta^2-1}{[K:\mathbb{Q}]}\sum_{v\in M_{K,\infty}}\log\left(\max_{\substack{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}\\ i_0+\cdots+i_n=\delta}}\left\{|b^{(v)}_{i_0,\ldots,i_n}|_v\right\}\right)\\ &+(\delta^2-1)\left(3\log\delta+\delta\log 3+\log\binom{n+\delta}{\delta}\right)\\ =\ &(\delta^2-1)h(X)+(\delta^2-1)\left(3\log\delta+\delta\log 3+\log\binom{n+\delta}{\delta}\right),\end{aligned}$$

where the last equality is from (2) and (3). Then we have the assertion. □

Remark 4.2. — With all the notations in Proposition 4.1, we have $C(n,\delta) \ll_n \delta^3$.

By applying Proposition 3.1 to the proof of Proposition 4.1, we have the following estimate for plane curves, where we will only point out the key difference in the proofs.

Proposition 4.3. — *Let X be a geometrically integral plane curve in $\mathbb{P}^2_K$ of degree δ, $\mathscr{X}$ be its Zariski closure in $\mathbb{P}^2_{\mathcal{O}_K}$, $\mathscr{X}_{\mathbb{F}_{\mathfrak{p}}} = \mathscr{X}\times_{\operatorname{Spec}\mathcal{O}_K}\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}$, and*

$$\mathcal{Q}(\mathscr{X}) = \left\{\mathfrak{p}\in\operatorname{Spm}\mathcal{O}_K \mid \mathscr{X}_{\mathbb{F}_{\mathfrak{p}}}\to\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}\ \textit{is not geometrically integral}\right\}.$$

Then we have

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})}\log N(\mathfrak{p})\leqslant(\delta^2-1)h(X)+C(\delta),$$

where $N(\mathfrak{p})=\#(\mathcal{O}_K/\mathfrak{p})$, $h(X)$ *is the classic height of* X *in* $\mathbb{P}^n_K$ *defined in Definition 2.2, and the constant* $C(\delta)=(3\delta^2-3)\log\delta$.

Sketch of the proof. — We replace (4) by the upper bound of the length in Proposition 3.1 in the proof of Proposition 4.1, then we prove the assertion. □

Remark 4.4. — With all the notations in Proposition 4.3, we have $C(\delta)\ll\delta^2\log\delta$, which has a better dependence on the degree than the case of general hypersurfaces provided in Proposition 4.1. If we only consider the dependence on the degree of plane curves, this estimate has the same as the later improvements.

4.2. Non-geometrically reductions of general projective schemes.

— In order to study the non-geometrically reductions of general schemes, it is significant to understand the reductions over their Chow varieties or Cayley varieties. Then we will reduce the general case to that of hypersurfaces. In this paper, will only use Cayley varieties, and Chow varieties are only mentioned for a historical reason.

4.2.1. *Cayley variety.* — First, we briefly recall the construction of Cayley varieties. For more details applied in the quantitative arithmetics, we refer the readers to [**2**, §3], see also [**12**, §2] for the application to the study of the non-reduced reductions.

Let A be a Dedekind domain or a field, $\mathcal{E}$ be a vector bundle of rank $n+1$ over $\operatorname{Spec}A$, and $d\in\mathbb{N}$ satisfying $1\leqslant d\leqslant n-1$. We denote

$$\theta:\mathcal{E}^\vee\otimes_A\left(\wedge^{d+1}\mathcal{E}\right)\to\wedge^d\mathcal{E}$$

the homomorphism which maps $\xi\otimes(x_0\wedge\cdots\wedge x_n)$ to

$$\sum_{i=0}^{d}(-1)^i\xi(x_i)x_0\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_d.$$

Let Γ be the sub-variety of $\mathbb{P}(\mathcal{E})\times_{\operatorname{Spec}A}\mathbb{P}(\wedge^{d+1}\mathcal{E}^\vee)$ which classifies the all the points (ξ,α) such that $\theta(\xi\otimes\alpha)=0$. Let $p:\mathbb{P}(\mathcal{E})\times_{\operatorname{Spec}A}\mathbb{P}\left(\wedge^{d+1}\mathcal{E}^\vee\right)\to\mathbb{P}(\mathcal{E})$ and $q:\mathbb{P}(\mathcal{E})\times_{\operatorname{Spec}A}\mathbb{P}\left(\wedge^{d+1}\mathcal{E}^\vee\right)\to\mathbb{P}\left(\wedge^{d+1}\mathcal{E}^\vee\right)$ be the two canonical projections.

Next, let $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$, X be a pure dimensional closed sub-scheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension d and degree δ, and $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. By [**2**, Proposition 3.4] or [**12**, Proposition 2.2, Proposition 2.7], the scheme $q(\Gamma\cap p^{-1}(X))$ (*resp.* $q(\Gamma\cap p^{-1}(\mathscr{X}))$) is a geometrically integral hypersurface in $\mathbb{P}\left(\wedge^{d+1}\mathcal{E}_K^\vee\right)$ (*resp.* $\mathbb{P}\left(\wedge^{d+1}\mathcal{E}^\vee\right)$), and $q(\Gamma\cap p^{-1}(X))$ is of degree δ. We call these hypersurfaces the *Cayley varieties* of X and $\mathscr{X}$. We denote by $\Psi_X\hookrightarrow\mathbb{P}\left(\wedge^{d+1}\mathcal{E}_K^\vee\right)$ and $\Psi_{\mathscr{X}}\hookrightarrow\mathbb{P}\left(\wedge^{d+1}\mathcal{E}^\vee\right)$ the Cayley varieties of X and $\mathscr{X}$ respectively.

By [**1**, §4.3.2 (i), (iv)], the construction of Cayley varieties commutes with the extension from $X\hookrightarrow\mathbb{P}(\mathcal{E}_K)$ to $\mathscr{X}\hookrightarrow\mathbb{P}(\mathcal{E})$, and commutes with the base change from $\mathcal{O}_K$ to its residue field, see [**1**, §4.3.1] or [**12**, Proposition 2.7] for more details of the above argument. Then in order to control the non-geometrically integral reductions of $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_K$, we are able to consider the non-geometrically reductions of its Cayley varieites.

4.2.2. *Control of the non-geometrically integral reductions.* — With the above constructions, we consider the non-geometrically reductions of general projective schemes below. We pick $\overline{\mathcal{E}} = \left(\mathcal{O}_K^{\oplus(n+1)}, (\|\cdot\|_v)_{v\in M_{K,\infty}}\right)$, where for each $v \in M_{K,\infty}$, the norm $\|\cdot\|_v$ maps $(x_0, \ldots, x_n)$ to $\sqrt{|x_0|_v^2 + \cdots + |x_n|_v^2}$. In this case, we denote $\mathbb{P}(\mathcal{E}_K)$ and $\mathbb{P}(\mathcal{E})$ by $\mathbb{P}^n_K$ and $\mathbb{P}^n_{\mathcal{O}_K}$ respectively for simplicity.

Theorem 4.5. — *With all the above notations and conditions in §4.2.2. Let X be a geometrically integral closed sub-scheme of $\mathbb{P}^n_K$ of pure dimension d and degree δ, $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}^n_{\mathcal{O}_K}$, $\mathscr{X}_{\mathbb{F}_\mathfrak{p}} = \mathscr{X} \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}_\mathfrak{p}$, and*

$$\mathcal{Q}(\mathscr{X}) = \left\{\mathfrak{p} \in \operatorname{Spm}\mathcal{O}_K \mid \mathscr{X}_{\mathbb{F}_\mathfrak{p}} \to \operatorname{Spec}\mathbb{F}_\mathfrak{p} \text{ is not geometrically integral}\right\}.$$

We denote $N(n,d) = \binom{n+1}{d+1} - 1$, $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$, and $\mathcal{H}_m = 1 + \cdots + \frac{1}{m}$. Then we have

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})} \log N(\mathfrak{p}) \leqslant (\delta^2-1)h_{\overline{\mathcal{O}(1)}}(X) + C'(n,d,\delta),$$

where $\overline{\mathcal{O}(1)}$ is equipped with the corresponding Fubini-Study metrics for all $v \in M_{K,\infty}$, $h_{\overline{\mathcal{O}(1)}}(X)$ is the Arakelov height of X in $\mathbb{P}^n_K$ defined in Definition 2.1, and the constant

$$\begin{aligned} C'(n,d,\delta) \quad = \quad & (\delta^2-1)\Bigg(3\log\delta + \log\binom{N(n,d)+\delta}{\delta} + \\ & \left(\left(N(n,d)+1\right)\log 2 + 4\log(N(n,d)+1) + \log 3 - \frac{1}{2}\mathcal{H}_{N(n,d)}\right)\delta\Bigg). \end{aligned}$$

Proof. — Let $\Psi_\mathscr{X}$ be the Cayley variety of $\mathscr{X}$, and

$$\mathcal{Q}(\Psi_\mathscr{X}) = \{\mathfrak{p} \in \operatorname{Spm}\mathcal{O}_K | \Psi_\mathscr{X} \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}_\mathfrak{p} \text{ is not geometrically integral}\}.$$

Then by [**12**, Proposition 2.2, 2.7], the fact $\mathfrak{p} \in \mathcal{Q}(\Psi_\mathscr{X})$ is verified if and only if $\mathscr{X} \times_{\operatorname{Spec}\mathcal{O}_K} \operatorname{Spec}\mathbb{F}_\mathfrak{p} \to \operatorname{Spec}\mathbb{F}_\mathfrak{p}$ is not geometrically integral. So we obtain

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\mathscr{X})} \log N(\mathfrak{p}) = \frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\Psi_\mathscr{X})} \log N(\mathfrak{p}).$$

By Proposition 4.1, we have

$$\begin{aligned} & \frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\in\mathcal{Q}(\Psi_\mathscr{X})} \log N(\mathfrak{p}) \\ \leqslant \quad & (\delta^2-1)h(\Psi_X) + (\delta^2-1)\left(3\log\delta + \delta\log 3 + \log\binom{N(n,d)+\delta}{\delta}\right), \end{aligned}$$

where $h(\Psi_X)$ is defined in Definition 2.2.

By [**12**, Proposition 3.7], we have

$$h(\Psi_X) - h_{\overline{\mathcal{O}(1)}}(X) \leqslant (N(n,d)+1)\delta\log 2 + 4\delta\log(N(n,d)+1) - \frac{1}{2}\delta\mathcal{H}_{N(n,d)}.$$

So we obtain the assertion by combining the above estimates. □

Remark 4.6. — We consider the constant $C'(n,d,\delta)$ in Theorem 4.5. Then we have $C'(n,d,\delta) \ll_n \delta^3$. Due to the comparison of heights, we have the same estimate of this constant to the case of curves and of general dimensions if we choose the Arakelov height.

Appendix A. A criterion of the reduced and irreducible properties

In this appendix, we will give a criterion of the reduced and irreducible hypersurfaces, which is exactly a solution to the exercise [**13**, Exercise 2.4.1]. This result is useful to judge whether a scheme is geometrically integral.

Proposition A.1. — *Let k be a field and $P \in k[T_1,\ldots,T_n]$. Then the scheme* $\mathrm{Spec}\,(k[T_1,\ldots,T_n]/(P))$ *is reduced (resp. irreducible; resp. integral) if and only if P has no square factor (resp. admits only one irreducible factor; resp. is irreducible).*

Proof. — First, we consider the reduced property. By [**13**, Definition 2.4.1, Proposition 2.4.2 (b)], $\mathrm{Spec}\,(k[T_1,\ldots,T_n]/(P))$ is reduced if and only if the ring $k[T_1,\ldots,T_n]/(P)$ is reduced. In fact, the nilradical of $k[T_1,\ldots,T_n]/(P)$ is not zero if and only if there exists a non-zero element $f \in k[T_1,\ldots,T_n]$, such that $f \notin (P)$ but $f^m \in (P)$ for some $m \in \mathbb{N}_+$, which is verified if and only if (P) has at least one square factor.

Next, we consider the irreducible property. Let $I = (P)$ be an ideal of $k[T_1,\ldots,T_n]$, and then we have $\mathrm{Spec}\,(k[T_1,\ldots,T_n]/(P)) = V(I)$. By [**13**, Proposition 2.4.7 (a)], $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime, which is verified if and only if P admits only one square factor.

By [**13**, Definition 2.4.16], a scheme is integral if and only it is both reduced and irreducible. Then we obtain the result about the integral property. □

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