Forced waves of the Fisher-KPP equation in a shifting environment*

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July 5, 2019

Abstract

This paper concerns the equation

$$u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}, \tag{0.1}$$

where $c \geq 0$ is a forcing speed and $f : (s, u) \in \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is asymptotically of KPP type as $s \to -\infty$. We are interested in the questions of whether such a forced moving KPP nonlinearity from behind can give rise to traveling waves with the same speed and how they attract solutions of initial value problems when they exist. Under a sublinearity condition on $f(s, u)$, we obtain the complete existence and multiplicity of forced traveling waves as well as their attractivity except for some critical cases. In these cases, we provide examples to show that there is no definite answer unless one imposes further conditions depending on the heterogeneity of f in $s \in \mathbb{R}$.

2010 AMS MSC: 35C07, 35B40, 35K57, 92D25

Keywords: Shifting environment, Fisher-KPP equation, traveling waves, long time behavior.

*The research leading to these results was supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi -Reaction-Diffusion Equations, Propagation and Modelling held by Henri Berestycki. This work was also partially supported by the NSF of Heilongjiang province (LC2017002) and the French National Research Agency (ANR), within the project NONLOCAL ANR-14-CE25-0013. We thank the anonymous referee for the comments that improve the paper.

1 Introduction

This paper deals with the equation

$$u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}, \tag{1.1}$$

where $c \geq 0$ and $f \in C^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ is assumed to have the following properties:

$$f(s, 0) = 0 \text{ for all } s \in \mathbb{R}; \tag{1.2}$$
$$\text{the limits } f(\pm\infty, u) \text{ and } \partial_u f(\pm\infty, u) \text{ exist and are continuous for } u \geq 0; \tag{1.3}$$
$$f(-\infty, u) = 0 \text{ has a unique positive solution } \alpha; \tag{1.4}$$
$$f(s, u)/u \text{ is non-increasing in } u > 0 \text{ for any } s \in \mathbb{R}; \tag{1.5}$$
$$\text{there exists } M > 0 \text{ such that } f(s, u) < 0 \text{ for } u \geq M, \text{ for all } s \in \mathbb{R}. \tag{1.6}$$

A typical example of such a nonlinearity is $f(s, u) = u(a(s) - u)$, where a is a smooth function and has limits at $\pm\infty$ with $a(-\infty) > 0$. Here $a(s)$ may have negative limit at $+\infty$ and may also change sign when s is away from $\pm\infty$. Another example is $f(s, u) = b(s)u(1 - u)$, where $b > 0$ has positive limits at $\pm\infty$.

A forced wave solution of (1.1) has the form $u(t, x) = U_c(x - ct)$, where c is the forced speed and U_c is the profile satisfying

$$U_c''(x) + cU_c'(x) + f(x, U_c(x)) = 0, \quad x \in \mathbb{R}. \tag{1.7}$$

The main purpose of this paper is to study under what conditions a forward shifting KPP nonlinearity gives rise to this kind of forced wave solutions. Let S be the set of all positive and bounded solutions of (1.7). Our goal here is to draw a complete picture of S and to study the attractivity of forced waves for the initial value problem of (1.1). Let us note that in our proofs to establish the main results about (1.7) we employ both ODE and PDE arguments, even though we treat an ODE.

We first recall some related developments and unsolved questions about this problem.

If $f(s, u)$ does not depend on $s \in \mathbb{R}$, that is, $f(s, u) \equiv g(u)$ is *homogeneous*, then under the assumptions (1.2)-(1.6) the nonlinearity $g(u)$ is of KPP type, that is,

$$g \text{ has a unique positive zero } \alpha \text{ and } g(u) \leq g'(0)u \text{ for } u \geq 0. \tag{1.8}$$

This equation $u_t = u_{xx} + g(u)$ in such a case has been extensively studied, since the classical works of Fisher [**?**] and Kolmogorov, Petrovsky and Piskunov (KPP) [**?**]. It is well-known that $c^* := 2\sqrt{g'(0)}$ is the minimal speed for traveling waves solution $U_c(x - ct)$, which satisfies $U_c'' + cU_c' + g(U_c) = 0$ with $U_c(+\infty) = 0$ and $U_c(-\infty) = \alpha$. Such a U_c is unique up to translations. Further, $\lim_{x \to +\infty} U_c(x)x^{1-m_c}e^{-\lambda_c x}$ is a positive number, where λ_c is the largest negative solution of $\lambda^2 + c\lambda + g'(0) = 0$ and m_c is its multiplicity. Aronson and Weinberger [**?**] showed that c^* is also the spreading speed of solutions of (1.1) having compactly supported initial data. For further studies in the long time behavior of solutions of monostable reaction-diffusion equations with different kinds of initial values, including the convergence to traveling waves, we refer to Hamel and Roques [?] and references therein.

If $f(s,u)$ is non-increasing in $s \in \mathbb{R}$ and both $f(\pm\infty, u)$ are of KPP type, then (1.7) becomes a special case of the higher dimensional cylinder problem investigated by Hamel [?]. From this work it follows that (1.7) admits at least one solution U_c with $U_c(-\infty) = \alpha$ and $U_c(+\infty) = 0$ if $c \geq c_+^* := 2\sqrt{\partial_u f(-\infty, 0)}$ but no such solutions if $c < c_-^* := 2\sqrt{\partial_u f(+\infty, 0)}$. By the monotonicity assumption of $f(s,u)$ in $s \in \mathbb{R}$, we always have $c_+^* \geq c_-^*$. In some cases, for instance, $f(s,u) = u(1 - d(s)u)$ where $d(s)$ decreasingly connects 1 to 2, one has $c_+^* = c_-^*$. Otherwise, one may have $c_+^* > c_-^*$, and then it remains open whether (1.7) admits such solutions when $c \in [c_-^*, c_+^*)$ (see the remark in page 574 of [?]). In a companion paper to [?], Hamel [?] showed that the solution set of (1.7) is one dimensional if $f(s,u)$ is a small perturbation of a heterogeneous bistable nonlinearity. This naturally suggests the same question for the monostable situation.

If $f(\pm\infty, u) < 0$ for $u > 0$, it was shown by Berestycki, Diekmann, Nagelkerke and Zegeling [?] that (1.7) admits a (unique) positive solution if and only if the generalized eigenvalue λ_1 of the linearized problem at 0 satisfies $\lambda_1 < 0$. Here, $\lambda_1 = \lambda_1(c)$ is defined by

$$\lambda_1 := \sup\{\lambda \mid \exists \phi \in C^2(\mathbb{R}), \phi > 0, s.t., \phi'' + c\phi' + \partial_u f(x,0)\phi + \lambda\phi \leq 0\}. \tag{1.9}$$

For the definition and properties of generalized eigenvalues, we refer the reader to [?]. By a Louville transform, it is clear that $\lambda_1(c) = \frac{c^2}{4} + \lambda_1(0)$. Thus, the condition $\lambda_1(c) < 0$ gives rise to a threshold value for $|c|$ so that (1.7) is solvable. Similar results in spirit were then established for more general equations in [?].

However, the sign of λ_1 cannot always determine the solvability of (1.7) if the condition $f(\pm\infty, u) < 0$ is removed. In [?] and [?], the first author of this paper and his collaborators introduced another notion of generalized eigenvalue λ_1', for a general elliptic operator including the linear part of (1.7). This generalized eigenvalue is defined by:

$$\lambda_1' := \inf\{\lambda \mid \exists \phi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}), s.t., \phi'' + c\phi' + \partial_u f(x,0)\phi + \lambda\phi \geq 0\}. \tag{1.10}$$

Applying their results to (1.7), we can infer that (1.7) admits at least one solution if $\lambda_1' < 0$ and no solution if $\lambda_1' > 0$. The critical case $\lambda_1' = 0$ remains an open question (see the problem 4.6 in [?]). Furthermore, there is a unique solution if $c^2 < 4\partial_u f(\pm\infty, 0)$.

From the aforementioned works, we see that the signs of λ_1 and λ_1' play essential roles for the solvability of (1.7), but one still needs to relate these eigenvalues to the coefficients of the operator. Recently, the second author, Lou and Wu [?] derived explicit formulas for λ_1 and λ_1' defined in (1.10) and (1.9), respectively, for the special case where $\frac{d}{ds}\partial_u f(s,0) < 0$. These formulas read

$$\lambda_1 = -\partial_u f(-\infty,0) + \frac{c^2}{4}, \quad \lambda_1' = \begin{cases} -\partial_u f(-\infty,0), & c \le 0 \\ \lambda_1, & c \in (0,\bar{c}) \\ -\partial_u f(+\infty,0), & c \ge \bar{c}, \end{cases} \tag{1.11}$$

where $\bar{c} := 2\sqrt{\partial_u f(-\infty,0) - \partial_u f(+\infty,0)}$. Moreover, when $\partial_u f(-\infty,0) > 0 > \partial_u f(+\infty,0)$ and $c = 2\sqrt{\partial_u f(-\infty,0)}$, we see $\lambda_1' = \lambda_1 = 0$ thanks to the explicit formulas. In such a case, it was shown in [**?**] that (1.7) does not have a solution. Then it is natural to ask whether this special case is sufficient to suggest that the problem 4.6 in [**?**] has a definite answer. However, it is not. We will come back to this critical case later.

Consider the special case when $f(s,u) = u\,(\,a(s) - u\,)$ and $a(s)$ is non-decreasing with respect to s and connects (in the sense of limits at $\pm\infty$) a negative constant to a positive constant. For this case, the article [**?**] showed on the one hand that the solutions having compactly supported initial values converge to zero in the moving frame if $c > 2\sqrt{\partial_u f(+\infty,0)}$. On the other hand, in the case $c < 2\sqrt{\partial_u f(+\infty,0)}$, the solutions starting from compactly supported initial values do propagate along the shifting environment at the asymptotic speed $2\sqrt{\partial_u f(+\infty,0)}$.

More recently, Bouhours and Giletti [**?**] considered the following discontinuous nonlinearity

$$f(s,u) = \begin{cases} -u, & s < 0 \\ g(u), & s \ge 0, \end{cases} \tag{1.12}$$

where g is of KPP type. They show that there are three possible phenomenas depending on the forcing speed c: extinction, grounding and spreading. More precisely, if $c \in [0, 2\sqrt{g'(0)})$, then spreading occurs; if c is greater than some $c^* \ge 2\sqrt{g'(0)}$, then extinction occurs; if $c \in [2\sqrt{g'(0)}, c^*)$ (provided such c exists), then both spreading and extinction may happen depending on the choice of initial value. Furthermore, there is a sharp selection if considering a continuous family of increasing initial values.

Unlike [**?**, **?**], in this paper we consider the optimistic scenario that the favorable environment moves forward. As we will see, there are some interesting and novel dynamics different from the other cases.

We now state our main results. The first one shows that there is an ordering structure for the solution set S of (1.7). In the following, we will denote the solutions U_c of the traveling wave equation (1.7) simply by u when there is no confusion.

Theorem 1.1. *If the set S has more than one element, then it is a totally ordered continuum of dimension one. Moreover,* $\sup\{u : u \in S\} \in S$.

With this structure, we see that either there is no solution, or a unique solution, or there are infinitely many solutions belonging to a one-dimensional manifold and the maximal solution exists.

The next result gives a necessary and sufficient condition for the minimal positive solution to exist.

Theorem 1.2. *The following statements are equivalent:*

(i) The minimal positive solution exists.

(ii) For any $x_0 \in \mathbb{R}$, the boundary value problem

$$\begin{cases} u'' + cu' + f(x,u) = 0, x < x_0, \\ u(-\infty) = \alpha, u(x_0) = 0. \end{cases} \tag{1.13}$$

has a unique positive solution.

(iii) $\lambda_1 < 0$ (where λ_1 is the generalized principal eigenvalue defined in (1.9)*).*

Thus, the natural question now is to determine the conditions under which the solution set S is empty, or a singleton, or a continuum. Define

$$\beta := \partial_u f(+\infty, 0). \tag{1.14}$$

We use $\tilde{\beta}$ to denote the unique positive solution of $f(+\infty, u) = 0$ whenever $\beta > 0$ (compare assumptions (1.5) and (1.6)):

$$f(+\infty, \tilde{\beta}) = 0 = f(+\infty, 0) \quad \text{and} \quad f(+\infty, u) > 0, u \in (0, \tilde{\beta}).$$

In the following two theorems, we describe the set S as determined by the various parameters β, c as well as the eigenvalue λ_1 (except for some critical cases). The structure of S is summarized in Figure 1.

Theorem 1.3 ($\lambda_1 < 0$)**.** *Assume that $\lambda_1 < 0$. Then the following statements hold:*

(i) If $\beta < 0$ then S is a singleton that has limit zero at $+\infty$.

(ii) If $\beta > 0$ and $c < 2\sqrt{\beta}$ then, S is a singleton that has limit $\tilde{\beta}$ at $+\infty$.

(iii) If $\beta > 0$ and $c > 2\sqrt{\beta}$ then, S is a continuum. Furthermore, only the maximal solution has limit $\tilde{\beta}$ at $+\infty$ whereas all other solutions have limit zero. The minimal solution decays to 0 with the exponential rate $\frac{-c-\sqrt{c^2-4\beta}}{2}$, while others decay with the rate $\frac{-c+\sqrt{c^2-4\beta}}{2}$.

Part (iii) of Theorem 1.3 and the eigenvalue formulas (1.11) answer the open question proposed in page 574 of [**?**] in the non-critical case when dimension is one.

Theorem 1.4 ($\lambda_1 \geq 0$)**.** *Assume that $\lambda_1 \geq 0$. If $\beta < 0$, then $S = \emptyset$. If $\beta > 0$, then S is a continuum. Furthermore, the maximal solution has limit $\tilde{\beta}$ at $+\infty$ and all others decays to zero at $+\infty$ with the exponential rate $\frac{-c+\sqrt{c^2-4\beta}}{2}$.*

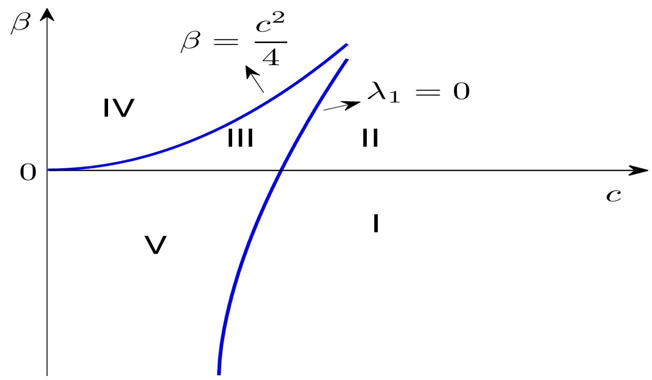

Figure 1: The curve $\lambda_1 = 0$ is not explicit but it always lies below the curve $\beta = \frac{c^2}{4}$. These two curves and the axis separate the half plane into five regions. In region I, $S = \emptyset$. In region II, S is a continuum without the minimal element. In region III, S is a continuum with the minimal element. In region IV, S is a singleton $\{u^*\}$ and $u^*(+\infty) > 0$. In region V, S is a singleton $\{u^*\}$ and $u^*(+\infty) = 0$.

If $f(s, u)$ is homogeneous in s, say $f(s, u) \equiv g(u)$, then (1.7) becomes the wave profile equation of the classical KPP equation $u_t = u_{xx} + g(u)$, for which it is known that all traveling waves with speed greater than $2\sqrt{g'(0)}$ decay exponentially at $+\infty$ and the exponent is the largest negative solution of the characteristic equation $\lambda^2 + c\lambda + g'(0) = 0$ although it has two negative solutions. But if $f(s, u)$ is asymptotic to a homogeneous KPP nonlinearity as $s \to +\infty$, then it is interesting to see from Theorem 1.3[iii] that both candidates of the exponential decay rate can be selected by different waves in the case where c is moderate. As a consequence, if $f(s, u)$ is a local perturbation of a KPP nonlinearity, that is, $f(s, u) = g(u)(1 + b(s))$, where b has compact support, then such a local heterogeneity may affect the asymptotic behavior of solutions of (1.7) at $+\infty$.

Having these information, we are ready to state the long time behavior of solutions of (1.1) in the moving frame. A "bistable" type dynamics may appear in some range of parameters due to the heterogeneity of $f(s, u)$.

Theorem 1.5. *Let $u(t, x; \psi)$ be the solution of* (1.1) *with initial value ψ. Then in the sense of locally uniform convergence we have the following limits as $t \to +\infty$.*

(i) $u(t, x + ct; \psi)$ converges to 0 provided that $\lambda_1 \geq 0$ and ψ has compact support.

(ii) $u(t, x + ct; \psi)$ converges to the element in S provided that $\lambda_1 < 0$ and S is a singleton.

(iii) In the case where $\lambda_1 < 0, \beta > 0$ and $c > 2\sqrt{\beta}$, consider the family of initial values

$$\psi_\gamma(x) = \min\{1, e^{-\gamma x}\}, \quad \gamma \in (0, +\infty).$$

Then $u(t, x+ct; \psi_\gamma)$ converges to the minimal solution of (1.7) *if* $\gamma > \frac{c-\sqrt{c^2-4\beta}}{2}$ *and to the maximal solution if* $\gamma < \frac{c-\sqrt{c^2-4\beta}}{2}$.

Part (iii) of this statement leaves several questions open : a) for the case where $\gamma = \frac{c-\sqrt{c^2-4\beta}}{2}$, does the solution converge to some element in S in the moving frame? b) if the solution were convergent, which element in S would be selected for the limit? c) in the original frame, what is the spreading speed when $\lambda_1 \geq 0$ and $c > 2\sqrt{\alpha}$? .

For the critical cases $\beta = 0$ or $\beta = \frac{c^2}{4}$, we give a specific example of nonlinearity to illustrate the fact that more information is needed to determine the structure of S.

Proposition 1.1. *Assume that $f(s,u) = u(a(s) - u)$. If $\beta = 0$ or $c = 2\sqrt{\beta}$, then the structure of S depends on the rate of convergence of $a(s)$ to β as $s \to +\infty$.*

If, further, $a'(s) \leq 0$, then by [**?**] we have $\lambda_1 = -\alpha + \frac{c^2}{4}$, hence, Figure 1 reduces to Figure 2. By Proposition 1.1, we see that $\lambda_1' = 0$ cannot determine the existence or non-existence of solutions for $u'' + cu' + a(x)u = u^2$. Moreover, Proposition 1.1 shows that the problem 4.6 in [**?**] and the open question in page 574 of [**?**] in the critical case does not have a definite answer unless further conditions on the heterogeneity are imposed.

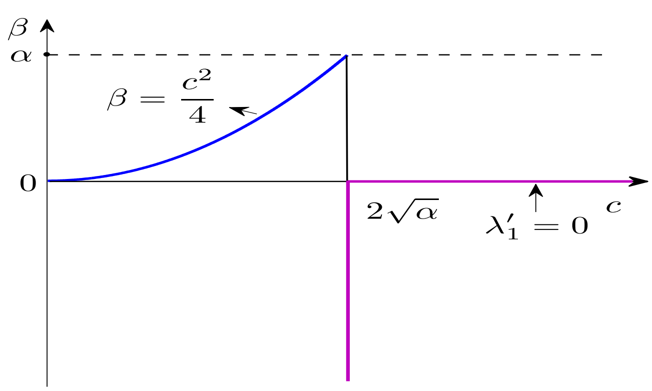

Figure 2: Since $a' \leq 0$, $\beta \leq \alpha$. The dashed line $\beta = \alpha$ corresponds to the classical Fisher-KPP case. The semi-line $\{(c,\beta) : c = 2\sqrt{\alpha}, \beta \leq \alpha\}$ corresponds to $\lambda_1 = 0$. The two semi-lines $\{(c,\beta) : c = 2\sqrt{\alpha}, \beta \leq 0\}$ and $\{(c,\beta) : c \geq 2\sqrt{\alpha}, \beta = 0\}$ correspond to $\lambda_1' = 0$.

The following theorem states the existence of a time global solution of (1.1) that connects 0 as $t \to -\infty$ and the minimal of maximal forced wave as $t \to +\infty$.

Theorem 1.6 (Heteroclinic orbits)**.** *Assume that $f(s,u) = u(a(s) - u)$, $a' \leq 0$ and $c \in (2\sqrt{\beta}, 2\sqrt{\alpha})$. Then there is one heteroclinc orbit from 0 to the maximal forced wave and another one from 0 to the minimal forced wave.*

We mention that the spatio-temporal heterogeneity $x - ct$ in the reaction term of (1.1) typically may arise in the modeling of a shifting environment. Patapov and Lewis [?], and Berestycki, Diekmann, Nagelkerke and Zegeling [?] proposed some reaction-diffusion systems with such a heterogeneity to study the persistence of species under the effect of climate change due to global warming. We refer to [?] for a nonlocal reaction diffusion model by combining a mutation of phenotype. Zhou and Kot [?] incorporate such a factor into an integro-difference equation. We refer to [?] by combining age or stage structure and [?] by combining a stochastic factor. Du, Wei and Zhou [?] proposed a free boundary problem in such a shifting environment, see also [?, ?]. Hu and Li [?] formulated such a problem in a discrete media. Moreover,"shifting environment" can also arise indirectly. Holzer and Scheel [?] consider a partially decoupled reaction diffusion system of two equations, where a wave solution for the first equation provides a shifting environment for the second one. Fang, Lou and Wu [?] also derived such a system by considering pathogen spread when their hosts are invading new environment. Cosner [?] pointed out several challenging topics on the modeling and analysis using reaction-diffusion equations in changing environment. Vo [?] investigated the persistence of species facing a forced time periodic and locally favorable environment in a cylindrical or partially periodic domain and established various existence and uniqueness of the forced waves.

The paper is organized as follows. In the preliminary section, we present some inequalities which will be used later. In section 3, we establish an ordering lemma, from which we derive Theorems 1.1 and 1.2. We investigate in section 4, the tail behavior of the elements in S. From this and Theorems 1.1 and 1.2, we obtain in section 5, a complete characterization of S, except for some critical cases. With the characterization of S, we prove Theorem 1.5. In section 7, we consider the critical cases and show they are indeterminate unless one imposes further conditions. In the last section, we construct two heteroclinic orbits.

2 Preliminaries

From the assumption $f \in C^1$ and assumptions (1.2)-(1.5), we derive some inequalities, which will be used hereafter on several occasions. By (1.5), we have $f(s,u)/u \le f(s,v)/v$ for $u \ge v > 0$. Letting $v \to 0$ yields

$$f(s,u) \le \partial_u f(s,0)u, \quad s \in \mathbb{R}, u \ge 0 \tag{2.1}$$

by using (1.2). Letting $s \to -\infty$ in (2.1) we obtain $f(-\infty,u) \le \partial_u f(-\infty,0)u$, which, together with (1.3), implies that

$$\partial_u f(-\infty,0) > 0. \tag{2.2}$$

Note that $f(s,u) = \partial_u f(s,\xi)u$ for some $\xi \in (0,u)$. It then follows that for $\epsilon > 0$ there exists $\delta > 0$ and $s_0 < 0$ such that

$$f(s,u) \ge (1-\epsilon)\partial_u f(-\infty,0)u, \quad \text{for} \quad s \le s_0, u \in (0,\delta]. \tag{2.3}$$

Since $\partial_u f(\pm\infty, u)$ exists and $\partial_u f(s, u)$ is continuous, we see that for $M > 0$ there exists L such that

$$|f(s,u) - f(s,v)| \leq L|u - v|, \quad u, v \in [0, M]. \tag{2.4}$$

By (1.5), we also have

$$f(s, \rho u) \leq \rho f(s, u), \quad \rho > 1, s \in \mathbb{R}, u \geq 0. \tag{2.5}$$

Letting $s \to +\infty$ in (2.1), (2.5) and (1.6), we infer that $f(+\infty, u)$ is of KPP type (as defined in (1.8)) if $\partial_u f(+\infty, 0) > 0$ and $f(+\infty, u) \leq 0$ if $\partial_u f(+\infty, 0) \leq 0$. We use $\tilde{\beta}$ to denote the unique positive solution of $f(+\infty, u) = 0$ whenever $\partial_u f(+\infty, 0) > 0$. Using the limiting version of (2.5) as $s \to +\infty$, we have

$$\frac{f(+\infty, u) - f(+\infty, u - \eta)}{\eta} \leq \frac{f(+\infty, u - \eta)}{u - \eta}, \quad 0 < \eta < u,$$

which implies that

$$\partial_u f(+\infty, u) < 0, \quad u > \tilde{\beta}, \tag{2.6}$$

where the fact $f(+\infty, u) < 0$ for $u > \tilde{\beta}$ is used. Similarly,

$$\partial_u f(-\infty, u) < 0, \quad u > \alpha. \tag{2.7}$$

3 An ordering lemma

In order to describe the set of bounded solutions of (1.7) in the whole line, we first investigate the solutions on semi-infinite intervals $(-\infty, \bar{x}]$. We begin with establishing the limiting behavior of these solutions at $-\infty$.

Lemma 3.1. *Assume that $u \not\equiv 0$ is a bounded functions of class $C^2((-\infty, \bar{x}), \mathbb{R}_+) \cap C^1((-\infty, \bar{x}], \mathbb{R}_+)$ that solves*

$$u'' + cu' + f(x, u) = 0, \quad \textit{for} \quad x \leq \bar{x}$$

for some $\bar{x} \in \mathbb{R}$. Then, $\lim_{x \to -\infty} u(x) = \alpha$ *and* $\lim_{x \to -\infty} u'(x) = 0$.

Proof. As a first step, we prove that either $u \to \alpha$ or $u \to 0$ as $x \to -\infty$. We argue by contradiction and suppose that u does not converge neither to 0 nor to α. We have the following alternative:

- (i) either for some sufficiently large A, $u'(x) \neq 0$ for all $x \leq -A$,
- (ii) or there exists a sequence $\{x_i\}$ with $x_i \to -\infty$ such that $u'(x_i) = 0$ for all i.

Let us start with case (i). Then, $u(x)$ has a finite limit as $x \to -\infty$ and from the equation it is easy to see that this limit must satisfy $f(-\infty, u(-\infty)) = 0$ whence $u(-\infty) = 0$ or $u(-\infty) = \alpha$.

Turn to case (ii). For any $\delta > 0$ fixed, for A sufficiently large, if $x \le -A$ and $0 \le u(x) \le \alpha - \delta$, then $f(x, u(x)) > 0$ while if $u(x) \ge \alpha + \delta$ then $f(x, u(x)) < 0$. From this, using the equation, we infer that u' can have at most one zero in $(-\infty, -A]$, where u achieves a value not in $(\alpha - \delta, \alpha + \delta)$. Since u is monotone between two consecutive zeroes of u', this implies that $\alpha - \delta \le u(x) \le \alpha + \delta$ for x close to $-\infty$. We have thus shown in this case that $u(-\infty) = \alpha$.

The second step is to show that $u(-\infty) = 0$ is impossible. Indeed, if this were the case, then, from the preceding proof we see that we would have $u'(x) > 0$ for x sufficiently close to $-\infty$. From the equation, we then infer that $u''(x) < 0$ for $x \le x_0$ form some x_0. And we have thus reached a contradiction. $\square$

Next, the following lemma reveals an ordering structure, which plays an important role in the rest of the paper.

Lemma 3.2. *Fix $\bar{x} \in \mathbb{R}$. Assume that $u_i \not\equiv 0, i = 1, 2$ are two bounded non-negative functions in $C^2((-\infty, \bar{x}), \mathbb{R}_+) \cap C^1((-\infty, \bar{x}], \mathbb{R}_+)$ satisfiying*

$$u_i'' + cu_i' + f(x, u_i) = 0, \quad x < \bar{x}.$$

Then either of the following holds: (i) $u_1 > u_2$, (ii) $u_1 < u_2$, (iii) $u_1 \equiv u_2$.

Before moving to the proof of the lemma, we notice that this ordering property is determined by the sign of $u_1(\bar{x}) - u_2(\bar{x})$.

Proof. From the previous Lemma, we know that $u_1(-\infty) = u_2(-\infty) = \alpha$. We claim that $u_1 \ge u_2$ provided that $u_1(\bar{x}) \ge u_2(\bar{x})$. Indeed, by the strong maximum principle and the Hopf lemma, we have $u_i > 0$ on $(-\infty, \bar{x})$ and $u_i'(\bar{x}) < 0$ in the case $u_i(\bar{x}) = 0, i = 1, 2$. By L'Hôpital's rule, this implies that $\lim_{x \uparrow \bar{x}} u_2(x)/u_1(x)$ is finite. Thus, the quotient $q(x) := u_2(x)/u_1(x)$ is uniformly bounded for $x \in (-\infty, \bar{x})$ because $q(-\infty) = 1$. Let k^* be the least upper bound of $q(x)$. Clearly, $k^* \ge q(-\infty) = 1$. If $k^* = 1$, then $u_1 \ge u_2$ and the claim is proved. Otherwise, $k^* > 1$ and then there exists $x^* \in (-\infty, \bar{x}]$ such that $k^* = \lim_{x \uparrow x^*} q(x)$. Define the difference $w := k^* u_1 - u_2$, which has the following properties: $w \ge 0, w(x^*) = 0$ and

$$\begin{aligned} w'' + cw' \quad &= -k^* f(x, u_1) + f(x, u_2) \\ &\le -f(x, k^* u_1) + f(x, u_2), \quad \text{by (2.5)} \\ &\le Lw, \quad x < \bar{x}, \quad \text{by (2.4)} \end{aligned} \tag{3.1}$$

where $L \in (0, +\infty)$ depends on $\sup_{x \le \bar{x}, i=1,2}\{u_i(x)\}$. Then the strong maximum principle implies that either $w \equiv 0$ or $x^* = \bar{x}$. Further, $w \equiv 0$ is impossible due to $w(-\infty) = (k^* - 1)u_1(-\infty) > 0$. Hence, $x^* = \bar{x}$. Consequently,

$$k^* = \lim_{x \uparrow \bar{x}} q(x) = \begin{cases} u_2(\bar{x})/u_1(\bar{x}), & u_1(\bar{x}) > 0, \\ u_2'(\bar{x})/u_1'(\bar{x}), & u_1(\bar{x}) = 0, \end{cases} \tag{3.2}$$

where the fact that $u_2(\bar{x}) = 0$ when $u_1(\bar{x}) = 0$ is used. Since $w(\bar{x}) = 0$ and $w(x) > 0$ for $x < \bar{x}$, by the Hopf lemma we have $w'(\bar{x}) < 0$, that is, $k^* u_1'(\bar{x}) < u_2'(\bar{x})$, which implies that $k^* = u_2(\bar{x})/u_1(\bar{x}) > 1$ in (3.2). As such, $u_2(\bar{x}) > u_1(\bar{x})$, a contradiction. The claim is proved.

Finally, exchanging the role of u_1 and u_2 in the above arguments, we see that $u_1 \le u_2$ provided that $u_1(\bar{x}) \le u_2(\bar{x})$. Further, $u_1 \equiv u_2$ if $u_2(\bar{x}) = u_1(\bar{x})$. By using the strong maximum principle again, we have that u_1 and u_2 cannot touch at any $x \in (-\infty, \bar{x})$ if they are not identical. □

Next we investigate the existence of half line solutions which we have just shown to be ordered. By (2.2), one may choose x_0 sufficiently large in the negative direction such that

$$\partial_u f(x, 0) > 0, \quad x \le x_0. \tag{3.3}$$

Consider the following boundary value problem:

$$\begin{cases} u'' + cu' + f(x, u) = 0, x < x_0, \\ u(-\infty) = \alpha, u(x_0) = \theta. \end{cases} \tag{3.4}$$

Then by the method of sub and super solution, we see that for each $\theta > 0$ there is a positive solution u_θ^-. By the ordering lemma, we infer that such a solution is unique. Based on u_θ^-, we then can shoot it to $+\infty$. More precisely, we consider the following problem:

$$\begin{cases} u'' + cu' + f(x, u) = 0, x > x_0 \\ u(x_0) = \theta, u'(x_0) = \lim_{x \uparrow x_0} (u_\theta^-)'(x), \end{cases} \tag{3.5}$$

for which, by the standard theory of ordinary differential equations, one has the maximal interval $I_\theta^{\max}$ for the existence of solution, where $I_\theta^{\max} = (x_0, y)$ for some $y \in (x_0, +\infty]$, and either $u(x)$ or $u'(x)$ blows up as $x \uparrow y$ if $y < +\infty$. Furhter, integrating $u'' + cu' + f(x, u) = 0$ from x_0 to $x < y$ gives

$$u'(x) = u'(x_0) - c(u(x) - u(x_0)) - \int_{x_0}^{x} f(\xi, u(\xi)) d\xi, \tag{3.6}$$

which implies that $\limsup_{x \uparrow y} |u(x)| = +\infty$ if $y < +\infty$. We use u_θ^+ to denote the solution of (3.5). Clearly,

$$u_\theta := \begin{cases} u_\theta^-, & x \le x_0 \\ u_\theta^+, & x \in I_\theta^{\max} \end{cases} \tag{3.7}$$

is a solution of (1.7) whenever $I_\theta^{\max} = (x_0, +\infty)$. We notice that u is a solution of (1.7) if and only if there exists $\theta > 0$ such that $u_\theta \equiv u$.

Proof of Theorem 1.1. Assume that (1.7) admits two solutions. That is, there are $\theta_1 < \theta_2$ such that $u_{\theta_1} < u_{\theta_2}$ are two ordered solutions owing to the ordering lemma, by which again we infer that u_θ is a solution for any $\theta \in (\theta_1, \theta_2)$. Indeed, note that it follows from Lemma 3.2 that once the solution is "sandwiched" to the left of some real $\bar{x}$ all the way to $-\infty$ by two solutions u_{θ_1} and u_{θ_2}, it will remain sandwiched by them to the right as well. That is, $u = u_\theta$ and $u_{\theta_1} > u > u_{\theta_2}$ on $\mathbb{R}$. In particular, $I_\theta^{\max} = (x_0, +\infty)$, that is, u_θ is a solution.

Define

$$\theta_{\sup} := \sup\{\theta > 0 : u_\theta \in S\}, \tag{3.8}$$

where S is the bounded positive solution set of (1.7). Next we show $\theta_{\sup} < +\infty$. For $u_\theta \in S$, one has $u_\theta(-\infty) = \alpha$ and that $u_\theta(+\infty)$ is a non-negative zero of $f(+\infty, u)$. If we assume, by contraction, that $\theta_{\sup} = +\infty$, then $\max_{x\in\mathbb{R}} u_\theta(x)$ is attained at some finite point for any large θ. Meanwhile, all large constants are super solutions of (1.7) thanks to (1.6). By the strong maximum principle one can see a contradiction. Therefore, $\theta_{\sup} < +\infty$. Finally we use a limiting argument to show $u_{\theta_{\sup}} \in S$. Choose $\theta_n \in (\theta_1, \theta_{\sup})$ with $\theta_n \uparrow \theta_{\sup}$ as $n \to \infty$. Then the uniform boundedness of u_{θ_n} implies that of u'_{θ_n}. This, together with the monotonicity of u_{θ_n} in n, further implies that u_{θ_n} converges to some u^* locally uniformly. Moreover, $u^*(x_0) = \theta_{\sup}$ and $u^* \in S$. By the ordering lemma again, we have $u_{\theta_{\sup}} = u^* \in S$. The proof is complete.

Proof of Theorem 1.2. We first prove that (i) $\Rightarrow$ (ii). Since the minimal positive solution exists, there is $\theta_{\inf} > 0$ such that u_θ attains 0 at some point $x_\theta > x_0$ for any $\theta \in (0, \theta_{\inf})$, thanks to the ordering lemma. Meanwhile, one can infer that $x_\theta : (0, \theta_{\inf}) \to (x_0, +\infty)$ is a homeomorphism. In particular,

$$\lim_{\theta\uparrow\theta_{\inf}} x_\theta = +\infty, \quad \lim_{\theta\downarrow 0} x_\theta = x_0.$$

Since x_0 is arbitrary, we obtain statement (ii).

Next we prove that (ii) $\Rightarrow$ (iii). Suppose to the contrary that $\lambda_1 \geq 0$. Let $\phi > 0$ be an eigenfunction associated with λ_1, that is,

$$\phi'' + c\phi' + \partial_u f(x, 0)\phi + \lambda_1\phi = 0.$$

By our assumptions and in view of (2.2), there exists $m > 0$ such that $\partial_u f(x, 0) \geq m > 0$ for $x \ll -1$. Thus we have

$$\phi'' + c\phi' + m\phi \leq 0, \quad \text{for} \quad x \ll -1. \tag{3.9}$$

We claim that this implies that $\phi(-\infty) = +\infty$. To prove this, suppose first that for some x_0 very negative, $\phi'(x_0) > 0$. Then $\phi''(x) < 0$ for all $x \leq x_0$, as long as $\phi(x)$ remains positive to the left of x_0. Hence, ϕ would have to change sign which is a contraditcion. Thus, $\phi'(x) < 0$ for all $x \leq x_1$ (for some x_1). Therefore, ϕ has a limit, possibly infinite but positive. If this limite is finite, then, using the inequality (3.9), we find that $\phi'' + c\phi' \leq -\rho < 0$ for all $x \leq x_1'$ (for some x_1'). Integrating this equation, we get that $\phi'(-\infty) = -\infty$, an obvious contradiction. Therefore, $\phi(-\infty) = +\infty$.

Let $\tilde{u}(x), x \leq x_0$ be a semi-wave ending at $x = 0$. Then $k\phi(x)$ with any $k > 0$ is a super solution thanks to (2.1), and $k\phi(x) \geq \tilde{u}(x), x < x_0$ as long as k is large enough. By letting k decrease to its minimum value, and using the strong maximum principle we obtain a contradiction. The proof for statement (iii) is complete.

Finally, we show (iii) $\Rightarrow$ (i). Since $\lambda_1 < 0$, by [**?**], there exists a family of small compactly supported sub-solutions and $S \neq \emptyset$. Assume, by contradiction, the minimal positive solution does not exist. Then from the ordering lemma we see that $u_\theta, \theta \in (0, \theta_{\sup})$ is a solution and $\lim_{\theta \to 0} u_\theta(x) = 0$ locally uniformly. By the strong maximum principle again we reach a contradiction to the existence of small compactly supported sub-solutions. The proof is complete.

4 A priori estimates on the tail at $+\infty$

In this section, we investigate the decay rates to zero at $+\infty$ for the solutions of (1.7). For this purpose, we first recall an example given in [**?**] to gain some motivations. The function $\frac{1}{1+x}$ is a solution of

$$u'' + u' + u^2 - 2u^3 = 0, \quad x \geq 1,$$

while e^{-x} is a solution of the linearized equation $u'' + u' = 0$. This then suggests that (even in homogeneous case) the decay rate of solutions for the linearized equation may not determine that of solutions for the nonlinear equation. Note that there is a degeneracy near zero for the nonlinearity $u^2 - 2u^3$.

Next we proceed with three different situations distinguished by the sign of $\beta := \partial_u f(+\infty, 0)$ for (1.7). Indeed, $\beta \neq 0$ will exclude the aforementioned degeneracy. Then as a special case in [**?**, Theorem 1, Appendix], one may immediately obtain the following exact asymptotic behavior of $u(x)$ as $x \to +\infty$.

Lemma 4.1. *Assume, in addition to* (1.2)-(1.6)*, that* $f(x, s)$ *is non increasing in* $x \in \mathbb{R}$ *and there exists* $\delta > 0$ *such that*

$$\left| \frac{f(x,s) - f(+\infty, s)}{s} \right| = o(e^{-\delta x}) \text{ as } x \to +\infty \text{ uniformly for } s \in (0, \delta). \tag{4.1}$$

If $u \in S$ *with* $u(+\infty) = 0$*, then*

(i) $c \geq 2\sqrt{\beta}$ *when* $\beta > 0$*. Further,*

$$u(x) = C_1 e^{\lambda x} + o(e^{\lambda x}) \text{ as } x \to +\infty \text{ if } c > 2\sqrt{\beta}$$

or

$$u(x) = C_2 x e^{\lambda x} + o(e^{\lambda x}) \text{ as } x \to +\infty \text{ if } c = 2\sqrt{\beta}$$

with $C_1, C_2 > 0$ *and* λ *being a negative solution of* $\lambda^2 + c\lambda + \beta = 0$.

(ii) $u(x) = C_3 e^{\lambda x} + o(e^{\lambda x})$ *for some* $C_3 > 0$ *when* $\beta < 0$*, where* $\lambda < 0$ *is the unique negative solution of* $\lambda^2 + c\lambda + \beta = 0$.

Recall that in [**?**] condition (4.1) was used to show that the decay rate of u is dominated by a solution of linearized equation of (1.7) and it is crucial in obtaining the exact behavior. In statement (i) with $c > 2\sqrt{\beta}$, $\lambda^2+c\lambda+\beta = 0$ have two negative solutions. In such a case, only the bigger one can be selected for the exponent of decay if $f(s,u)$ is homogeneous in $s \in \mathbb{R}$, while it remains unclear for inhomogeneous f. We will come back to this question in the next section.

Now we want to modify the proof of [**?**, Theorem 1, Appendix] to drop the conditions of monotonicity and (4.1) in Lemma 4.1. The results are then not expected as exact as above, but they are sufficient for the purpose of this paper.

For a bounded function $\phi \in C(\mathbb{R}, \mathbb{R}^+)$ which decays exponentially to 0 at $+\infty$, we define $\gamma_\phi \in [-\infty, 0)$ as follows

$$\gamma_\phi := \inf \left\{ \gamma < 0 : \exists C_\gamma > 0 \, s.t. \, \phi(x) \le C_\gamma e^{\gamma x}, \, x \in \mathbb{R} \right\}. \tag{4.2}$$

Lemma 4.2. *If $u \in S$ with $u(+\infty) = 0$ and $\beta > 0$, then $c \ge 2\sqrt{\beta}$ and $\gamma_u \in \left\{ \frac{-c\pm\sqrt{c^2-4\beta}}{2} \right\}$, that is, γ_u is a solution of $\gamma^2 + c\gamma + \beta = 0$.*

When $c > 2\sqrt{\beta}$, there are two different candidates for the decay rate. In the next section, we will show that both of them may exist.

Proof. From the cases 1 and 2 of [**?**, Lemma 2, Appendix], we see that the decay rate of $u(x)$ to 0 as $x \to +\infty$ is in between two exponentially decay functions. Consequently, $\gamma_u \in (-\infty, 0)$ is well defined. It then remains to show that γ_u is a zero of $\gamma^2 + c\gamma + \beta = 0$.

Next, we claim that if $u(x) \le C_1 e^{\gamma x}, x \in \mathbb{R}$ with $\gamma^2 + c\gamma + \beta \ge \delta$ for some $\delta > 0$, then there exists $\sigma = \sigma(\delta) < 0$ (independent of γ) such that $u(x) \le C_2 e^{(\gamma+\sigma)x}$. Indeed, set $\zeta(x) := u(x)e^{-\gamma x}$. Then

$$\begin{aligned} 0 \qquad &= \zeta'' + (c+2\gamma)\zeta' + (\gamma^2 + c\gamma + \frac{f(x,u)}{u})\zeta \\ &\ge \zeta'' + (c+2\gamma)\zeta' + \frac{\delta}{2}\zeta \quad \text{for all large } x, \end{aligned} \tag{4.3}$$

which further implies that $\zeta(x) = O(e^{-\sigma x})$ for some $\sigma = \sigma(\delta) > 0$ according to [**?**, Lemma 5.2]. The claim is proved. Recall that for all sufficiently large x, $u(x)$ is in between two exponentially decay functions. It then follows that $\gamma_u \in [\frac{-c-\sqrt{c^2-4\beta}}{2}, \frac{-c+\sqrt{c^2-4\beta}}{2}]$, outside which $\gamma^2 + c\gamma + \beta > 0$.

Now we are ready to show that γ_u has to be a zero of $\gamma + c\gamma + \beta$. Assume for the sake of contradiction that $\gamma_u \in (\frac{-c-\sqrt{c^2-4\beta}}{2}, \frac{-c+\sqrt{c^2-4\beta}}{2})$. Then $\gamma_u^2 + c\gamma_u + \beta < 0$, and hence, there exists $\delta_0 > 0$ and $\epsilon_0 > 0$ such that $(\gamma_u+\epsilon)^2 + c(\gamma_u+\epsilon) + \beta < -\delta_0, \forall \epsilon \in [0, \epsilon_0]$. Set the bounded function $\zeta_\epsilon = ue^{-(\gamma_u+\epsilon)x}$. By direct computations we have

$$\begin{aligned} \zeta_\epsilon'' + (c + 2(\gamma_u+\epsilon))\zeta_\epsilon' \qquad &= e^{-(\gamma_u+\epsilon)x}\{u'' + cu' - [(\gamma_u+\epsilon)^2 + c(\gamma_u+\epsilon)]u\} \\ &= -\zeta_\epsilon[\frac{f(x,u)}{u} + (\gamma_u+\epsilon)^2 + c(\gamma_u+\epsilon)] \end{aligned}$$

Since $u(+\infty) = 0$ and $\lim_{x\to+\infty} \frac{f(x,u(x))}{u(x)} = \beta > 0$, it then follows that there exists $R > 0$ such that $\frac{f(x,u(x))}{u(x)} \le \beta + \frac{\delta_0}{2}$ for $x \ge R$, and hence,

$$\begin{cases} \zeta_\epsilon'' + (c + 2(\gamma_u + \epsilon))\zeta_\epsilon' - \frac{\delta_0}{2}\zeta_\epsilon \ge 0, & x \ge R \\ \zeta_\epsilon \text{ is bounded and } \zeta_\epsilon(+\infty) = 0. \end{cases} \tag{4.4}$$

By the proof of [**?**, Proposition 4.1] we conclude that as $x \to +\infty$, $\zeta_\epsilon(x) = o(e^{\eta x}), \forall \eta < \eta_0(\epsilon)$, where $\eta_0(\epsilon)$ is the negative solution of $\eta^2 + (c + 2(\gamma_u + \epsilon))\eta - \frac{\delta_0}{2} = 0$. Since $\lim_{\epsilon\to0} \eta_0(\epsilon) < 0$, we have $\frac{\eta_0(\epsilon)}{2} + \epsilon < 0$ when ϵ is small enough. Meanwhile, $u(x) = \zeta_\epsilon(x)e^{(\gamma_u+\epsilon)x} = o(e^{(\gamma_u+\epsilon+\frac{\eta_0(\epsilon)}{2})x})$ as $x \to +\infty$, which contradicts the definition of γ_u. $\square$

Lemma 4.3. *If $\beta < 0$, then $\gamma_u = \frac{-c-\sqrt{c^2-4\beta}}{2}$ and $c < 2\sqrt{\sup_{x\in\mathbb{R}} \partial_u f(x,0)}$.*

Proof. By [**?**, Lemma 5.2], we obtain that $\gamma_u = \frac{-c-\sqrt{c^2-4\beta}}{2}$. Next we show that $c < 2\sqrt{\sup_{x\in\mathbb{R}} \partial_u f(x,0)}$. Otherwise, set $c_1 := 2\sqrt{\sup_{x\in\mathbb{R}} \partial_u f(x,0)}$ and $\mu := \frac{-c+\sqrt{c^2-c_1^2}}{2}$. Then any translation of the exponential function $e^{\mu x}$ is a super solution of (1.7) thanks to (2.1). Since $0 > \mu > \gamma_u$ due to $\beta < 0 < c_1$, there exists $l \in \mathbb{R}$ such that $e^{\mu(x+l)} - u(x) \ge 0$ and vanishes at some $x \in \mathbb{R}$. By the strong maximum principle, we reach $e^{\mu(x+l)} = u(x)$ for all $x \in \mathbb{R}$, a contradiction. The proof is complete. $\square$

In the end of this section, we give a rough bound on the decay rate of solutions of (1.7) with the degeneracy $\beta = 0$.

Lemma 4.4. *If $\beta = 0$, then $\liminf_{x\to+\infty} u(x)e^{\gamma x} = +\infty$ for $\gamma > c$.*

Proof. Fix $\gamma > c$ and set $w(x) := u(x)e^{\gamma x}$. Then one has

$$w'' + (c - 2\gamma)w' + [\gamma^2 - c\gamma + \frac{f(x,u)}{u}]w = 0, \tag{4.5}$$

which implies that there exists $\delta > 0$ and $\bar{x} \in \mathbb{R}$ such that $w'' + (c-2\gamma)w' + \delta w \le 0$ for $x \ge \bar{x}$ due to $u(+\infty) = 0 = \beta$. Therefore, $w(+\infty) = +\infty$. $\square$

5 Uniqueness of waves and unique characteristics for the minimal and maximal waves

By Theorem 1.1, we have seen that either there exists at most one forced wave or there are infinitely many waves that form an ordered continuum of dimension one. Further, in the later case the maximum solution always exists and the minimal solution exists when the generalized eigenvalue $\lambda_1 < 0$ (see Theorem 1.2). The purpose of this section is to find when there exists at most one solution and whether the maximal and minimal solutions have certain unique characteristics comparing with all other possible solutions.

Recall that $\beta := \partial_u f(+\infty, 0)$. We first consider the case where $\beta \le 0$.

Lemma 5.1. *If either of the following conditions holds, then there exists at most one forced wave.*

(i) $\beta < 0$;

(ii) There exist $\sigma > c, \delta > 0$ *and* $\eta > 0$ *such that* $\partial_u f(x,0) = o(e^{-\sigma x})$ *as* $x \to +\infty$ *and* $f(x,u) \leq \partial_u f(x,0)u - \eta u^2$ *for* $u \in [0,\delta]$.

The second statement asserts the uniqueness when $\partial_u f(x,0)$ tends to zero with an exponential decay rate. In the next section we will give an example illustrating the non-uniqueness when $\partial_u f(x,0)$ tends to zero with an algebraic decay rate. The proof of Lemma 5.1 will be given together with that of the next lemma, which deals with the case where $\beta > 0$ and gives some characteristics about the minimal and maximal positive solutions.

Lemma 5.2. *Assume that* $\beta > 0$. *Then there exists at most one forced wave of the following two types.*

(i) $u(+\infty) > 0$;

(ii) $u(x) = o(e^{\sigma x})$ *as* $x \to +\infty$ *for some* $\sigma < 0$ *with* $\sigma^2 + c\sigma + \beta < 0$.

By the similar arguments as in Lemma 3.1, we know that $u(+\infty)$ always exists and it is either 0 or the unique positive zero of $f(+\infty,\cdot)$. Further, by Lemma 5.2(i) and the ordering structure we can infer that only the maximum wave may have a positive limit at $+\infty$. In Lemma 4.2, we have shown that there are two candidates for the exponential decay rates of u and they are the solutions of $\gamma^2 + c\gamma + \beta = 0$. By the assumption $\sigma^2 + c\sigma + \beta < 0$, we know σ is in between these two candidates, and hence, any solution belonging to type (ii) must select the faster decay rate. Combining with the ordering structure and Lemma 5.2(ii), we can infer that only the minimal wave has the possibility to select the faster decay rate among two candidates. In the next section, we will show that the minimal wave does select this decay rate.

Proof of Lemma 5.1 and and Lemma 5.2(i). The idea of the proof is highly motivated by [**?**, Theorem 3.3]. Assume that $u_i, i = 1,2$ are two different wave solutions, then $u_i(-\infty) = \alpha$, while $u_i(+\infty) = 0$ or $\tilde{\beta}$, where $\tilde{\beta}$ is the unique positive zero, if exists, of $f(+\infty,\cdot)$. In any case, $u_1(\pm\infty) = u_2(\pm\infty)$. We may assume $u_1 < u_2$ thanks to the ordering lemma.

For $\epsilon > 0$ define the set

$$K_\epsilon := \{k > 0 : ku_1 > u_2 - \epsilon \text{ in } \mathbb{R}\}, \tag{5.1}$$

which is not empty since

$$\sup \left\{0, \frac{u_2(x) - \epsilon}{u_1(x)}\right\} \text{ is bounded uniformly in } x \in \mathbb{R}. \tag{5.2}$$

Set $k_\epsilon = \inf K_\epsilon$. Clearly, k_ϵ is nonincreasing in ϵ. Define $k^* = \lim_{\epsilon\to 0^+} k(\epsilon)$ and $w_\epsilon = k_\epsilon u_1 - u_2 + \epsilon$. Clearly, $k^* \in [1, +\infty), w_\epsilon \geq 0$ in $\mathbb{R}$ and

$$w := \lim_{\epsilon\to 0} w_\epsilon = k^* u_1 - u_2. \tag{5.3}$$

We intend to prove that $k^* = 1$. Assume, for the sake of contraction, that $k^* > 1$. Then $k_\epsilon > 1$ for all small ϵ. Consequently,

$$w_\epsilon(-\infty) = (k_\epsilon - 1)\alpha + \epsilon > 0, \quad w_\epsilon(+\infty) = \epsilon > 0, \tag{5.4}$$

and hence, there exists x_ϵ such that $w_\epsilon(x_\epsilon) = 0$ owing to the definition of k_ϵ. We claim that x_ϵ is uniformly bounded in ϵ. Let us postpone the proof of the claim and reach a contradiction with $k^* > 1$. Now that x_ϵ is bounded, there exists a convergent subsequence, $(\epsilon_n, x_{\epsilon_n}) \to (0, \bar{x})$ for some $\bar{x} \in \mathbb{R}$. Note that $w \geq 0, w(\bar{x}) = 0$ and w satisfies inequality (3.1) in the whole line. It then follows from the strong maximum principle that $w \equiv 0$, which is equivalent to $k^* u_1 \equiv u_2$. In particular,

$$\alpha < k^*\alpha = k^* u_1(-\infty) = u_2(-\infty) = \alpha, \tag{5.5}$$

a contradiction. We have proved $k^* = 1$ and can infer that $u_1 \geq u_2$, a contradiction.

To prove the claim, we assume, for the sake of contradiction, that there exist ϵ_n and x_{ϵ_n} such that

$$(\epsilon_n, |x_{\epsilon_n}|) \to (0, +\infty), \quad w_{\epsilon_n}(x_{\epsilon_n}) = 0.$$

We may choose x_{ϵ_n} appropriately such that $w_{\epsilon_n}(x)$ is not identically zero in some neighborhood of x_{ϵ_n} due to that fact that $w_{\epsilon_n}(\pm\infty) \geq \epsilon_n > 0$. Recall that here we still have the assumption $k^* > 1$, and hence, $k_{\epsilon_n} > 1$ for large n. Further, by (2.5) we obtain

$$\begin{aligned} w''_{\epsilon_n}(x) + c w'_{\epsilon_n}(x) \quad &= -k_{\epsilon_n} f(x, u_1(x)) + f(x, u_2(x)) \\ &\leq -f(x, k_{\epsilon_n} u_1(x)) + f(x, u_2(x)), \quad x \in \mathbb{R}. \end{aligned} \tag{5.6}$$

In the following we proceed with four cases.

Case 1. $x_{\epsilon_n} \to -\infty$ (up to subsequence). By the mean value theorem,

$$-f(x, k_{\epsilon_n} u_1(x)) + f(x, u_2(x)) = -\partial_u f(x, \xi_n(x))(k_{\epsilon_n} u_1(x) - u_2(x)). \tag{5.7}$$

where $\xi_n(x)$ is chosen to be the biggest value in between $k_{\epsilon_n} u_1(x)$ and $u_2(x)$ such that the equality holds. Passing $x \to -\infty$ and $n \to \infty$ yields that

$$A^\infty := \lim_{n\to\infty} \lim_{x\to-\infty} \partial_u f(x, \xi_n(x))$$

exists. And $f(-\infty, k^*\alpha) - f(-\infty, \alpha) = A^\infty (k^* - 1)\alpha$. Therefore, by the mean value theorem again, we know that there exists $s^\infty \in (\alpha, k^*\alpha)$ such that $A^\infty = \partial_u f(-\infty, s^\infty) < 0$ owing to (2.7). Consequently, for $-x$ and n large enough one has

$\partial_u f(x, \xi_n(x)) < 0$. Meanwhile, by the continuity of $w_{\epsilon_n}(x)$ and the fact $w_{\epsilon_n}(x_{\epsilon_n}) = 0$, for any n there exists a ball $B_{\rho_n}(x_{\epsilon_n})$ centered at x_{ϵ_n} such that

$$k_{\epsilon_n} u_1(x) - u_2(x)) < 0, \quad x \in B_{\rho_n}(x_{\epsilon_n}).$$

Therefore, for sufficiently large n,

$$w_{\epsilon_n}(x) \geq 0, \quad w_{\epsilon_n}(x_{\epsilon_n}) = 0, \quad w''_{\epsilon_n}(x) + c w'_{\epsilon_n}(x) \leq 0, \quad x \in B_{\rho_n}(x_{\epsilon_n}).$$

By the strong maximum principle, we infer that $w_{\epsilon_n}(x) \equiv 0$ for $x \in B_{\rho_n}(x_{\epsilon_n})$. This contradicts the choice of x_{ϵ_n}.

Case 2. $x_{\epsilon_n} \to +\infty$ (up to subsequence), $\beta > 0$ and $u_i(+\infty) > 0$. Then we reach a similar contradiction to Case 1. Thus, Lemma 5.2(i) is proved.

Case 3. $x_{\epsilon_n} \to +\infty$ (up to subsequence) and $\beta < 0$. Then we have $u_i(+\infty) = 0$ due to $\beta < 0$. We also have (5.7), where $\lim_{x \to +\infty} \partial_u f(x, \xi_n(x)) = \beta < 0$. Thus, for n large enough we reach a similar contradiction to Case 1. Thus, Lemma 5.1(i) is proved.

Case 4. $x_{\epsilon_n} \to +\infty$ (up to subsequence) and the condition in the second statement of Lemma 5.1 holds. In this case, $u_i(+\infty) = 0$ and $\lim_{x \to +\infty} u_i(x) e^{\gamma x} \to +\infty$ for $\gamma > c$ (see Lemma 4.4). Moreover, for $x \in B_{\rho_n}(x_{\epsilon_n})$ and n large enough, we have $k_{\epsilon_n(x)} u_1(x) < u_2(x)$, and further,

$$\begin{aligned}
& -f(x, k_{\epsilon_n} u_1(x)) + f(x, u_2(x)) \\
= \quad & -\partial_u f(x,0)[k_{\epsilon_n} u_1(x) - u_2(x)] + k_{\epsilon_n} u_1(x) \left[\partial_u f(x,0) - \frac{f(x, k_{\epsilon_n} u_1(x))}{k_{\epsilon_n} u_1(x)} \right] \\
& -u_2(x) \left[\partial_u f(x,0) - \frac{f(x, u_2(x))}{u_2(x)} \right] \\
\leq \quad & -\partial_u f(x,0)[k_{\epsilon_n} u_1(x) - u_2(x)] + [k_{\epsilon_n} u_1(x) - u_2(x)] \left[\partial_u f(x,0) - \frac{f(x, k_{\epsilon_n} u_1(x))}{k_{\epsilon_n} u_1(x)} \right] \\
\leq \quad & -\partial_u f(x,0)[k_{\epsilon_n} u_1(x) - u_2(x)] + [k_{\epsilon_n} u_1(x) - u_2(x)] \eta k_{\epsilon_n} u_1(x) \\
= \quad & [-\partial_u f(x,0) + \eta k_{\epsilon_n} u_1(x)][k_{\epsilon_n} u_1(x) - u_2(x)],
\end{aligned} \tag{5.8}$$

which is nonpositive for $x \in B_{\rho_n}(x_{\epsilon_n})$ and sufficiently large n since $\partial_u f(x,0)$ decays faster than $u_1(x)$ as $x \to +\infty$. Thus, Lemma 5.1(ii) is proved.

Proof of Lemma 5.2(ii). Recall that $\sigma^2 + c\sigma + \beta < 0$. Thus, $\sigma < 0$ and

$$\sigma \in \left(\frac{-c - \sqrt{c^2 - 4\beta}}{2}, \frac{-c + \sqrt{c^2 - 4\beta}}{2} \right).$$

Assume, by contradiction, that $u_i, i = 1, 2$ are two solutions and they decay faster than $e^{\sigma x}$ as $x \to +\infty$. Define $v_i := u_i e^{-\sigma x}$. Then $v_i(\pm\infty) = 0$ and

$$v''_i + (c + 2\sigma) v'_i + (\sigma^2 + c\sigma) v_i + f(x, v_i e^{\sigma x}) e^{-\sigma x} = 0.$$

Next for v_i equation we proceed as in the proof of Lemma 5.1. Indeed, for $\epsilon > 0$ similarly we define the number k_ϵ, function w_ϵ, vanishing point x_ϵ of w_ϵ and the limiting function w. Then $w_\epsilon \geq 0, w_\epsilon(\pm\infty) = \epsilon$ and $w_\epsilon(x_\epsilon) = 0$. Further, if x_ϵ is bounded, then the limiting function w satisfies

$$w'' + (c + 2\sigma)w' + (\sigma^2 + c\sigma)w \leq -\partial_u f(x, \xi(x))e^{-\sigma x}w,$$

where $\xi(x)$ is in between $f(x, u_i(x))e^{-\sigma x}, i = 1, 2$. Then the strong maximum principle leads to a contradiction. If there is a sequence $x_{\epsilon_n} \to \infty$, then we compute to have

$$\begin{aligned}
& w''_{\epsilon_n} + (c + 2\sigma)w'_{\epsilon_n} \\
= \quad & -(\sigma^2 + c\sigma)(k_{\epsilon_n}v_1 - v_2) - k_{\epsilon_n}f(x, u_1)e^{-\sigma x} + f(x, u_2)e^{-\sigma x} \\
\leq \quad & -(\sigma^2 + c\sigma)(k_{\epsilon_n}v_1 - v_2) - f(x, k_{\epsilon_n}u_1)e^{-\sigma x} + f(x, u_2)e^{-\sigma x} \\
= \quad & -(\sigma^2 + c\sigma)(k_{\epsilon_n}v_1 - v_2) - \frac{f(x, k_{\epsilon_n}u_1)}{k_{\epsilon_n}u_1}k_{\epsilon_n}v_1 + \frac{f(x, u_2)}{u_2}v_2 \\
\leq \quad & -(k_{\epsilon_n}v_1 - v_2)[\sigma^2 + c\sigma + \frac{f(x, u_2)}{u_2}], \quad \text{when } k_{\epsilon_n}u_1(x) \leq u_2(x).
\end{aligned} \tag{5.9}$$

We claim that the right-hand side of (5.9) is nonpositive for $|x|$ sufficiently large and $k_{\epsilon_n}u_1(x) \leq u_2(x)$. It then suffices to show that $\sigma^2 + c\sigma + \partial_u f(+\infty, 0) < 0$ and $\sigma^2 + c\sigma + \frac{f(+\infty,\alpha)}{\alpha} < 0$. The former is as assumed, and the later is also true because $f(+\infty, \alpha) = 0$. Therefore, when $|x|$ large enough and $k_{\epsilon_n}u_1(x) \leq u_2(x)$, one has

$$w''_{\epsilon_n} + (c + 2\sigma)w'_{\epsilon_n} \leq 0, \quad w_{\epsilon_n} \geq 0, \quad w_{\epsilon_n}(x_{\epsilon_n}) = 0.$$

Then the strong maximum principle leads to a contradiction as in the proof Lemma 5.1. The proof is complete.

We end up this section with the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. (i) Since $\lambda_1 < 0$, $S \neq \emptyset$. Since $\beta < 0$, by Lemma 5.1(i) we see that S is a singleton that has limit zero at $+\infty$. (ii) Since $\beta > 0$ and $c < 2\sqrt{\beta}$, there is no solution having limit zero at $+\infty$ owing to Lemma 4.2. Thus, all solutions have positive limits at $+\infty$. By Lemma 5.2(i), we see that S is a singleton whose limit at $+\infty$ is the unique positive zero of $f(+\infty, \cdot)$. (iii) To show S is a continuum, by Theorem 1.1, it suffices to construct two different solutions. We first construct a solution having positive limit at $+\infty$. Clearly, $\bar{u} \equiv \sup_{x\in\mathbb{R}} \partial_u f(x, 0)$ is a super solution. As for the sub-solution, let $\bar{x}$ and δ be two positive numbers that will be specified later. Define

$$\underline{u}(x) := \delta \max\{0, 1 - e^{-c(x-\bar{x})}\}.$$

Then $\underline{u}(x) = 1 - e^{-c(x-\bar{x})}$ for $x > \bar{x}$ and

$$\underline{u}'' + c\underline{u}' + f(x, \underline{u}) = f(x, \underline{u}) \geq 0, \quad x > \bar{x} \tag{5.10}$$

provided that $\bar{x}$ is large enough and δ is small enough since $f(x,0) \equiv 0$ and $\partial_u f(-\infty,0) > 0$. Hence, there exists a solution u with $\underline{u} \le u^* \le \bar{u}$. Clearly, $u^*(+\infty)$ is positive. Next we construct a solution with zero limit at $+\infty$. Clearly, there are a family of compactly supported sub-solutions due to $\lambda_1 < 0$. As for the super solution, we note that for $\epsilon > 0$ there exists $\bar{x} > 0$ such that $f(x,u) \le \partial_u f(x,0)u \le (\beta+\epsilon)u$ for $x \ge \bar{x}$ and $u \ge 0$. This yields that

$$\bar{u}(x) := \min\left\{\sup_{x\in\mathbb{R}} \partial_u f(x,0), ke^{\frac{-c-\sqrt{c^2-4(\beta+\epsilon)}}{2}x}\right\}$$

is a super solution as long as k is sufficiently large. Therefore, we have a solution u with $u(+\infty) = 0$ and the decay rate $\gamma_u \le \frac{-c-\sqrt{c^2-4(\beta+\epsilon)}}{2}$. Further, by Lemma 4.2 we obtain $\gamma_u = \frac{-c-\sqrt{c^2-4\beta}}{2}$. Therefore, S is a contimuum. Further, by Lemma 5.2, the constructed two solutions are exactly the maximal and minimal solutions, respectively. And all other solutions decay to zero at $+\infty$ with the smaller exponential rate $\frac{-c+2\sqrt{c^2-4\beta}}{2}$ among the two candidates established in Lemma 4.2. The proof is complete.

Proof of Theorem 1.4. By Theorems 1.1 and 1.2, we immediately see that S is a continuum without the minimal solution. For the decay rate, we argue by the way of contradiction that if there is one solution with decay rate $\frac{-c-2\sqrt{c^2-4\beta}}{2}$, then by the ordering lemma, the fact of non-existence of minimal solution and the *a priori* estimates in Lemma 4.2, we infer that there are infinitely many solutions with decay rate $\frac{-c-2\sqrt{c^2-4\beta}}{2}$. This contradicts Lemma 5.2(ii). Therefore, all solutions expect for the maximal solution decays with the exponential rate $\frac{-c+2\sqrt{c^2-4\beta}}{2}$. The maximal solution has positive limit at $+\infty$. The proof is complete.

6 Indeterminacy when $\beta = 0$ or $\beta = \frac{c^2}{4}$

In this section, we assume that $f(s,u) = u(a(s)-u)$ and

$$a(-\infty) = \alpha > 0, \quad a(+\infty) = \beta \in \mathbb{R}. \tag{6.1}$$

With this special nonlinearity, we explain the indeterminacy stated in Proposition 1.1 by introducing the rate of convergence of $a(s)$ to β as an additional condition.

Lemma 6.1. *Assume that $\beta = \frac{c^2}{4} \in (0,\alpha)$. Then the following statements hold.*

(i) If $a(x) > \beta, \forall x \in \mathbb{R}$, then S is a singleton.

(ii) If there exists $\bar{x}$ such that $a(x) < \beta, \forall x \ge \bar{x}$, then S is a continuum.

Proof. For both situations, S is not empty due to $\lambda_1 < 0$. (i). We first show that a positive solution must have limit β at $+\infty$. For this purpose, we construct a subsolution for the parabolic equation in moving coordinate. Fix any $x^* \in \mathbb{R}$, then by assumption we have

$$a(x) \geq \inf_{x \leq x^*} a(x) > \beta, \forall x \leq x^*.$$

Define $a^* := \min_{x \leq x^*} a(x)$. Choose $R \gg 1$ such that $\beta + \left(\frac{\pi}{R}\right)^2 - a^* < 0$. Then all nontrivial solutions of

$$\begin{cases} \zeta_t = \zeta_{xx} + c\zeta_x + \zeta(a^* - \zeta), & x \in (-R + x^*, x^*), \\ \zeta(-R + x^*) = \zeta(x^*) = 0 \end{cases} \tag{6.2}$$

converges to a^* uniformly for x in any compact sets of $(-R + x^*, x^*)$. Define

$$\underline{w}(t, x) = \begin{cases} \zeta(t, x), & x \in (-R + x^*, x^*) \\ 0, & x \notin (-R + x^*, x^*). \end{cases} \tag{6.3}$$

Then $\underline{w}(t, x)$ is a subsolution of $w_t = w_{xx} + cw_x + w(a(x) - w)$. Let $u \in S$ and choose $\underline{w}(0, x) \leq u(x)$, then we have $u(x) \geq \beta$ for x in any compacts of $(-R+x^*, x^*)$. Recall that x^* is arbitrary. Thus, $u(x) \geq \beta, \forall x \in \mathbb{R}$, whence, $u(+\infty) = \beta$. By Lemma 5.2(i), such solution is unique.

(ii). From the construction in the proof of Theorem 1.3(iii) we already have a solution having limit β at $+\infty$. By Theorem 1.1, it suffices to construct some $u \in S$ with $u(+\infty) = 0$. Indeed, since $\lambda_1 < 0$, there exists a family of small compactly supported sub-solutions. Since $a(x) \leq \beta$ for all large x, $\min\{\sup_{x \in \mathbb{R}} a(x), ke^{-\sqrt{\beta}x}\}$ is a super solution as long as k is large enough. This pair of super and sub-solutions give rise to $u \in S$ with $u(+\infty) = 0$. □

With the additional assumption that $a'(x) \leq 0$ for $x \in \mathbb{R}$, we know from [**?**] that

$$\lambda_1 = -\alpha + \frac{c^2}{4}, \quad \lambda_1' = \begin{cases} -\alpha, & c < 0, \\ \lambda_1, & c \in [0, 2\sqrt{\alpha - \beta}], \\ -\beta, & c > 2\sqrt{\alpha - \beta}. \end{cases} \tag{6.4}$$

Next, we consider the critical situation $\beta = 0$ with two sub-cases:

(i) $\beta = 0, c \geq 2\sqrt{\beta}$.

(ii) $\beta = 0, c < 2\sqrt{\beta}$.

Lemma 6.2 (Case (i)). *$S = \emptyset$ if $a(x) = o(e^{-\gamma x})$ for some $\gamma > c$ and $S \neq \emptyset$ if $a(x) = O(x^{-\gamma})$ for some $\gamma \in (0, 1)$.*

From the expression of λ_1', this case corresponds to $\lambda_1' = 0$. Thus, this lemma implies that $\lambda_1' = 0$ is not sufficient to determine the existence or non-existence of solutions for $u'' + cu' + a(x)u = u^2$ in $\mathbb{R}$.

Proof. Since $\lambda_1 \geq 0$, the minimal positive solution does not exist according to Theorem 1.2. To show that $S = \emptyset$, it then suffices to prove that S contains at most one solution, which has been proved in Lemma 5.1(ii) under the condition $a(x) = o(e^{-\gamma x})$ as $x \to +\infty$ for some $\gamma > c$.

If $a(x) = O(x^{-\gamma})$ for some $\gamma \in (0, 1)$, then one can construct a pair of super- and sub-solutions to show $S \neq \emptyset$. Indeed, fix

$$\epsilon \in (0, c) \text{ and } p > 1, \tag{6.5}$$

and define

$$\underline{u}(x) = \begin{cases} x^{-p}(1 - Me^{-\epsilon x}), & x > x_M := \epsilon^{-1} \ln M, \\ 0, & x \leq x_M, \end{cases} \tag{6.6}$$

where $M \gg 1$ will be specified later. Then we compute to obtain for $x > x_M$,

$$\begin{aligned} \underline{u}'(x) &= -px^{-p-1}(1 - Me^{-\epsilon x}) + x^{-p}M\epsilon e^{-\epsilon x} \\ \underline{u}''(x) &= p(p+1)x^{-p-2}(1 - Me^{-\epsilon x}) - 2px^{-p-1}M\epsilon e^{-\epsilon x} - x^{-p}M\epsilon^2 e^{-\epsilon x} \end{aligned}$$

and

$$\begin{aligned} & \underline{u}''(x) + c\underline{u}'(x) + a(x)\underline{u}(x) - \underline{u}^2(x) \\ = \; & x^{-p-1}(-pc + p(p+1)x^{-1} + xa(x) - x^{-p+1}) + Mx^{-p}e^{-\epsilon x}p(c+2)x^{-1} \\ & + Mx^{-p}e^{-\epsilon x}[\epsilon(c - \epsilon - a(x)) - p(p+1)x^{-2} + x^{-p}(2 - Me^{-\epsilon x})] \\ \geq \; & 0 \end{aligned}$$

provided that M is sufficiently large. This, together with the super solution $\bar{u} \equiv \sup_{x\in\mathbb{R}} a(x)$, gives rise to a solution. The second part is proved. □

Lemma 6.3 (Case (ii))**.** *S is a continuum if $a(x) = O(x^{-\gamma})$ for some $\gamma \in (0, 1)$ and S is a singleton if $a(x) = o(e^{-\gamma x})$ for some $\gamma > c$.*

Proof. Since $\lambda_1 < 0$, we know from Theorem 1.2 that $S \neq \emptyset$ and the minimal solution exists. In the case where $a(x) = o(e^{-\gamma x})$ for some $\gamma > c$, by Lemma 5.1(ii) we obtain S is a singleton. In the case where $a(x) = O(x^{-\gamma})$ for some $\gamma \in (0, 1)$, it suffices to construct two different solutions. Using the same super and sub-solutions as in proof of Lemma 6.2, we obtain a solution which decays slower than a polynomial rate at $+\infty$. Meanwhile, since $\lambda_1 < 0$, there are a family of small compactly supported sub solutions, and since $a(x) = o(e^{-\gamma x})$ for some $\gamma > c$, we see that $\min\{\sup_{x\in\mathbb{R}} a(x), ke^{-cx}\}$ is a super solution when k is sufficiently large. Thus, there is another solution which decays exponentially at $+\infty$. □

7 Initial value problem

We consider the initial value problem in the moving coordinate $\xi = x - ct$.

$$\begin{cases} v_t = v_{\xi\xi} + cv_\xi + f(\xi, u), & t > 0, \xi \in \mathbb{R}, \\ v(0, \cdot) = v_0 \in BC(\mathbb{R}, \mathbb{R}^+). \end{cases} \tag{7.1}$$

Clearly, if S is a singleton, then $\lambda_1 < 0$ and hence $v(t,\xi)$ is convergent to the unique forced wave locally uniformly in $\xi \in \mathbb{R}$. If S is empty, then $\lambda_1 \geq 0$. Choosing large constants as initial values, then the solutions are non-increasingly converges to the unique steady state zero of (7.1). By the comparison principle, we see all solutions converges to zero uniformly in $\mathbb{R}$.

Proof of Theorem 1.5. The first two statements are obvious according to above discussions. In the following, we prove the third statement. Indeed, in the case where $\gamma > \frac{c-\sqrt{c^2-4\beta}}{2}$, define

$$\psi_+(x) = M\min\{1, ke^{\frac{-c+\sqrt{c^2-4(\beta+\epsilon)}}{2}x}\},$$

where the positive numbers M, k and ϵ will be specified later. Choose ϵ small enough such that

$$\gamma > \frac{c-\sqrt{c^2-4(\beta+\epsilon)}}{2}.$$

Then $\psi_+ \geq \psi_\gamma$ provided that M and k are large enough. We compute in the half interval $x > \frac{2}{-c+\sqrt{\beta+\epsilon}}\ln\frac{1}{k}$:

$$\begin{aligned}
\psi_+'' + c\psi_+' + f(x,\psi_+) \quad &\leq \left(\frac{-c+\sqrt{c^2-4(\beta+\epsilon)}}{2}\right)^2 \\
&\quad + \frac{-c+\sqrt{c^2-4(\beta+\epsilon)}}{2}c + \partial_u f(x,0) \\
&= \partial_u f(x,0) - (\beta+\epsilon),
\end{aligned}$$

which is negative for large x. Therefore, $v(t,x;\psi_+)$ is non-increasing in $t \geq 0$ provided that k is large enough. Let $v_* := \lim_{t\to\infty} v(t,x;\psi_\gamma)$. Since $\lambda_1 < 0$, $v_* \not\equiv 0$ and $v_* \in S$. It decays faster than the exponential function $e^{\frac{-c+\sqrt{c^2-4(\beta+\epsilon)}}{2}x}$ due to $v^* \leq \psi_+$. By Theorem 1.3(iii), v_* is the minimal element in S.

In the case where $\gamma < \frac{c-\sqrt{c^2-4\beta}}{2}$, define

$$\psi_-(x) := \max\{0, \delta e^{\frac{-c+\sqrt{c^2-4(\beta-\epsilon)}}{2}x}(1-Me^{-\eta x})\},$$

where positive numbers δ, ϵ and M will be specified later. Choose $\eta > 0$ such that

$$\left(\frac{-c+\sqrt{c^2-4\beta}}{2}-\eta\right)^2 + \left(\frac{-c+\sqrt{c^2-4\beta}}{2}-\eta\right)c + \beta := C_\eta < 0.$$

Choose ϵ small enough such that

$$\left(\frac{-c+\sqrt{c^2-4(\beta-\epsilon)}}{2}-\eta\right)^2 + \left(\frac{-c+\sqrt{c^2-4(\beta-\epsilon)}}{2}-\eta\right)c + \beta - \epsilon < \frac{1}{3}C_\eta.$$

By using (2.3), we may choose δ small enough and M large enough such that

$$f(x,\underline{u}(x)) \geq (1-\frac{1}{\beta}\epsilon)\partial_u f(-\infty,0)\underline{u}(x) = (1-\frac{1}{\beta}\epsilon)\beta\underline{u}(x), \quad x > \frac{1}{\eta}\ln M.$$

A further computation for $x > \frac{1}{\eta}\ln M$ yields

$$\begin{aligned}
& \frac{1}{\delta}e^{\frac{c-\sqrt{c^2-4(\beta-\epsilon)}}{2}x}[\psi_-'' + c\psi_-' + f(x,\psi)] \\
\geq \quad & -(\beta-\epsilon) - Me^{-\eta x}[\frac{1}{3}C_\eta - (\beta-\epsilon)] + (1-\frac{1}{\beta}\epsilon)\beta(1-Me^{-\eta x}) \\
= \quad & -\frac{1}{3}C_\eta Me^{-\eta x} \geq 0.
\end{aligned}$$

Choose δ smaller if necessary such that $\psi_\gamma > \psi_-$. Then $v(t,x;\psi) \geq v(t,x;\psi_-)$, which converges to some $v^* \in S$ locally uniformly since any large constant is a super solution. Clearly, $v^* \geq \psi_-$, and hence, v^* is the maximal element in S thanks to Theorem 1.3(iii). Note that $v(t,x;M) \downarrow v^*$ for any big constant M as $t \to +\infty$, and therefore, so is $v(t,x;\psi_\gamma)$ by the comparison argument.

8 Forced heteroclinic orbits

As shown in Theorem 1.3(iii), in appropriate region of parameters, there are minimal and maximal forced waves, denoted by $u_{\min}$ and $u_{\max}$, respectively. In this section, we construct two heteroclinic orbits connecting zero to these two special forced waves. It then suffices to consider the heteroclinc solutions in the moving frame, that is, of the following equation

$$v_t = v_{xx} + cv_x + v(a(x)-v), \quad t\in\mathbb{R}, x\in\mathbb{R} \tag{8.1}$$

subject to the condition

$$v(-\infty,x) = 0, \quad v(+\infty,x) = u_{\min}(x)$$

or

$$v(-\infty,x) = 0, \quad v(+\infty,x) = u_{\max}(x).$$

where all the limits are uniform in $x\in\mathbb{R}$.

Heteroclinic orbit from 0 to $u_{\max}$. We plan to construct a sequence of appropriate solutions defined for $-n \leq t < +\infty$ and pass to the limit $n\to\infty$. Indeed, let $v^n(t,x)$ be the solution of

$$\begin{cases} v_t = v_{xx} + cv_x + v(a(x)-v), & t > -n, x\in\mathbb{R} \\ v(-n,x) \equiv \delta_n, \end{cases} \tag{8.2}$$

where δ_n is a positive number that will be specified inductively in the following.

Define $\eta^+ := \sup_{x\in\mathbb{R}} a(x)$ and $\eta^- = \inf_{x\in\mathbb{R}} a(x)$. Assume that $\eta_- > 0$. Fix $\delta_0 \in (0, \frac{1}{2}\min\{\eta^-, u_{\min}(0)\})$. Let $w^\pm(t)$ be the unique entire solution of the ordinary differential equation

$$\begin{cases} w' = w(\eta^\pm - w), t \in \mathbb{R} \\ w(0) = \delta_0. \end{cases} \tag{8.3}$$

In fact, $w^\pm(t)$ is explicitly given as follows

$$w^\pm(t) = \frac{\eta^\pm}{1 + \frac{\eta^\pm - \delta_0}{\delta_0} e^{-\eta^\pm t}} = \frac{1}{\frac{1}{\eta^\pm} + \left(\frac{1}{\delta_0} - \frac{1}{\eta^\pm}\right) e^{-\eta^\pm t}}. \tag{8.4}$$

Observe that

$$\frac{d}{d\eta}\left(\frac{1}{\eta} + (\frac{1}{\delta_0} - \frac{1}{\eta})e^{-\eta t}\right) = \frac{1}{\eta^2}(e^{-\eta t} - 1) + \left(\frac{1}{\eta} - \frac{1}{\delta_0}\right) te^{-\eta t}, \tag{8.5}$$

which has the same sign as t for $\eta > \delta_0$. Thus, $w^+(t) > w^-(t)$ for $t > 0$ and $w^+(t) < w^-(t)$ for $t < 0$. Moreover, any translation of $w^+(w^-)$ in time is a spatially homogeneous super (sub-) solution of (8.2).

Now we are ready to look for $\delta_1 \in [w^+(-1), w^-(-1)]$. For this purpose, define a family of numbers parameterised by $\sigma \in [0, 1]$:

$$\delta_1^\sigma := w^+(-1) + [w^-(-1) - w^+(-1)]\sigma, \quad \sigma \in [0, 1]. \tag{8.6}$$

Let $v(t, x; \delta_1^\sigma), t \geq -1$ be the solution of (8.2) with the initial condition $v(-1, x) = \delta_1^\sigma$. Define the map $F : [0, 1] \to \mathbb{R}$ by

$$F(\sigma) = v(0, 0; \delta_1^\sigma). \tag{8.7}$$

Since $w^\pm$ are (strict) super and sub-solutions, respectively, one has

$$v(t, x; w^+(-1)) < w^+(t), \quad v(t, x; w^-(-1)) > w^-(t), \quad \forall t > -1 \tag{8.8}$$

and in particular,

$$F(0) = v(0, 0; w^+(-1)) < w^+(0) = \delta_0 = w^-(0) < v(0, 0; w^+(-1)) = F(1) \tag{8.9}$$

Now for each $t > -1$, $v(t, x; \delta_1^\sigma)$ is strictly increasing in σ and continuous in σ locally uniformly in $x \in \mathbb{R}$. In particular, $F(\sigma)$ is strictly increasing and continuous in σ. Thus, there exists a unique $\sigma_1 \in (0, 1)$ such that $F(\sigma_1) = \delta_0$. Then $\delta_1^{\sigma_1}$ is the number δ_1 that we look for.

Using the same argument we may construct δ_n inductively. With the initial value δ_n, the solution v_n defined for $t \geq -n$ has the following properties: (i) it is increasing in $t \geq -n$; (ii) $v_n(0, 0) = \delta_0$; (iii) v_n is uniformly bounded in n, t, x; (iv) $v_n(t, x) \geq w^-(t)$ for $t \geq -n$ and $x \in \mathbb{R}$. By the parabolic estimates, one then obtains a limit v which is increasing in $t \in \mathbb{R}$ and solves (8.2). Also $v(\pm\infty, x)$ are stationary solutions. Since $v(-\infty,0) \le v(0,0,x) = \delta_0$, we have $v(-\infty,x) = 0$ locally uniformly thanks to the choice of δ_0. Since $v(+\infty,x) \ge w^-(+\infty) = \eta^-$, we have $v(+\infty,x) = u_{\max}(x)$ locally uniformly.

Next, we show that the limit at $+\infty$ holds uniformly in $x \in \mathbb{R}$. We first prove that $\lim_{t\to+\infty} v(t,x) = u_{\max}(x)$ uniformly for x in the left half line. Assume for the sake of contradiction that there exists $t_n \to +\infty$, $x_n \to -\infty$ and $\epsilon > 0$ such that $|v(t_n,x_n) - u_{\max}(x_n)| \ge \epsilon$ for all n. Since $u_{\max}(-\infty) = \alpha$, at least one of the following inequalities holds:

$$\liminf_{n\to+\infty} v(t_n,x_n) < \alpha, \quad \limsup_{n\to+\infty} v(t_n,x_n) > \alpha. \tag{8.10}$$

Next we show that neither of these two inequalities cannot hold. Indeed, we find that, as $n \to \infty$ and up to subsequences, $v(t,x+x_n)$ converges to a function $\tilde{v}(t,x)$ uniformly in $[0,\rho] \times \bar{B}_\rho$ for any $\rho > 0$. Moreover, $\tilde{v}$ satisfies $\tilde{v}_t = \tilde{v}_{xx} + c\tilde{v}_x + \tilde{v}(\alpha - \tilde{v})$. Since

$$\tilde{v}(-1,x) = \lim_{n\to\infty} v(-1,x+x_n) \ge \lim_{n\to\infty}\lim_{m\to\infty} v_m(-1,x+x_n) \ge w^-(-1), \tag{8.11}$$

by the comparison principle one has $\tilde{v}(t,x) \ge \zeta(t)$ for $t \ge -1$ and $x \in \mathbb{R}$, where $\zeta(t)$ is the solution of $\zeta' = \zeta(\alpha - \zeta)$ with initial condition $\zeta(-1) = w^-(-1) > 0$. Hence,

$$\liminf_{n\to+\infty} v(t_n,x_n) \ge \liminf_{t\to\infty} \inf_{x\in\mathbb{R}} \tilde{v}(t,x) \ge \lim_{t\to\infty} \zeta(t) = \alpha, \tag{8.12}$$

which implies that the first inequality of (8.10) cannot hold. On the other hand, $\tilde{v} \le \zeta(t)$ with $\zeta(0) = \sup_{x\in\mathbb{R}} a(x)$ due to the fact that $v_n(t,x) \le \sup_{x\in\mathbb{R}} a(x)$. This then implies that

$$\limsup_{n\to+\infty} v(t_n,x_n) \le \limsup_{t\to\infty} \sup_{x\in\mathbb{R}} \tilde{v}(t,x) \le \alpha, \tag{8.13}$$

and hence, the second inequality of (8.10) cannot hold either. For the uniform convergence for x in the right half line, the same arguments applies.

Finally, we investigate the uniform convergence as $t \to -\infty$. Assume, in addition, that $a'(x) \le 0$. Then the forced speed c of interest satisfies $2\sqrt{\beta} \le c < 2\sqrt{\alpha}$. Each $v_n(t,x)$ is non-increasing in x, so is $v(t,x)$. With these additional informations, we assume for the sake of contradiction that there exist $\epsilon > 0$, $t_n \to -\infty$ and x_n such that $v(t_n,x_n) \ge \epsilon$. Since $v(t,x)$ is non-increasing in x and $v(t,0) \le \delta_0$ for $t \le 0$, one may choose $x_n \le 0$ so that x_n is non-increasing in n and $v(t_n,x_n) \in [\epsilon,\delta_0]$. If $x_n \to -\infty$, then $\tilde{w}_n(x) := v(t_n,x+x_n)$ converges, up to subsequences, locally uniformly to w^* which satisfies $w^*_{xx} + cw^*_x + w^*(\alpha - w^*) = 0$ with $w^*(0) \in [\epsilon,\delta_0]$. This leads to a contradiction in view of the range of c. If $x_n \to x^*$, then w^* satisfies $w^*_{xx} + cw^*_x + w^*(a(x+x^*) - w^*) = 0$, which implies that $w^*(x) \ge u_{\min}(x+x^*)$. In particular, $w^*(0) \ge u_{\min}(x^*) \ge u_{\min}(0) > \delta_0$, a contradiction. The proof is complete.

Heteroclinic orbit from 0 to $u_{\min}$. We plan to apply the Dancer-Hess connecting orbit lemma (see, for instance [**?**]) for strictly monotone dynamical systems, and then show that the limit of such a heteroclinc orbit at infinity is uniform in $x \in \mathbb{R}$. Assume that $a'(x) \leq 0$. Then $u_{\min}$ is non-increasing.

Let E be the ordered metric space of all continuous and non-increasing functions endowed with the compact open topology and the natural ordering. We use U to denote the set $\{\phi \in E : 0 \leq \phi \leq u_{\min}\}$. For any $\phi \in U$, let $Q_t[\phi]$ be the solution of $v_t = v_{xx} + cv_x + v(a(x) - v)$ with the initial condition $v(0, x) = \phi(x)$. By the strong maximum principle, we see that $\{Q_t\}_{t\geq 0} : U \to U$ defines a strictly monotone semiflow in the following sense: (i) $Q_t[\phi]$ is jointly continuous in (t, ϕ) with respect to the compact open topology of the product space $\mathbb{R} \times U$; (ii) $Q_t[\phi](x) > Q_t[\psi](x)$ for all $t > 0, x \in \mathbb{R}$ if $\phi \geq, \not\equiv \psi$. It then follows from the Dancer-Hess connecting orbit lemma that the autonomous semiflow $\{Q_t\}_{t\geq 0} : U \to U$ admits an entire monotone orbit $\Gamma(t)$ connecting two steady states 0 and $u_{\min}$ in the sense that $\Gamma(t) = Q_s[\Gamma(t - s)], t \in \mathbb{R}, s \geq 0$ and

$$\text{either} \quad \Gamma(-\infty) = 0 \text{ and } \Gamma(+\infty) = u_{\min} \quad \text{or} \quad \Gamma(-\infty) = u_{\min} \text{ and } \Gamma(+\infty) = 0, \tag{8.14}$$

where the limit is with respect to the compact open topology. By the strong maximum principle, we know $\Gamma(t)$ is positive everywhere for any $t \in \mathbb{R}$. Furthermore, we know there exists a compactly supported initial function $\underline{\phi}$ such that $Q_t[\underline{\phi}]$ is increasing in time. So we infer that $\Gamma(t)$ is increasing in t with $\Gamma(-\infty) = 0$ and $\Gamma(+\infty) = u_{\min}$.

Next we show that these two limits are uniform in $x \in \mathbb{R}$. Indeed, since $\Gamma(t) \leq u_{\min}$ and $u_{\min}(+\infty) = 0$, one may conclude that theses two limits are uniform for x in the right half line. For the left half line as $t \to +\infty$, we assume for the sake of contradiction that there exists $\epsilon > 0$, $t_n \to +\infty$ and $x_n \to -\infty$ such that $\Gamma(t_n)(x_n) < \alpha - \epsilon$. Let $y \in \mathbb{R}$ be a point at which $u_{\min}$ attains the value $\alpha - \frac{\epsilon}{2}$. Then

$$\alpha - \epsilon \geq \liminf_{n\to\infty} \Gamma(t_n)(x_n) \geq \lim_{n\to\infty} \Gamma(t_n)(y) = u_{\min}(y) = \alpha - \frac{\epsilon}{2}, \tag{8.15}$$

a contradiction. For the left half line as $t \to -\infty$, we assume again for the sake of contradiction that there exists $\epsilon > 0$, $t_n \to -\infty$ and $x_n \to -\infty$ such that $\Gamma(t_n)(x_n) > \epsilon$. Then by the monotonicity and the fact $\Gamma(t) \leq u_{\min}$ for all t, we obtain

$$\Gamma(t)(-\infty) > \epsilon, \quad \Gamma(t)(+\infty) \leq u_{\min}(+\infty) = 0 \tag{8.16}$$

for all very negative time. Hence, one could find $t_n \to -\infty$ and x_n such that x_n is decreasing in n and $\Gamma(t_n)(x_n) = \epsilon$. Then $\Gamma(t_n)(x + x_n)$ converges locally uniformly, up to subsequences, to a solution w^* of $w'' + cw' + w(\lim_{n\to\infty} a(x + x_n) - w) = 0$ with $w^*(0) = \epsilon$. If x_n decreases to $-\infty$, then such w^* does not exist due to the range of c. If x_n decreases to a bounded number x^*, then $w^*(x) \geq u_{\min}(x + x^*)$, and in particular,

$$\epsilon = w^*(0) \geq u_{\min}(x^*) \geq u_{\min}(x_n) > \Gamma(t_n)(x_n) = \epsilon, \tag{8.17}$$

a contradiction. The proof is thereby complete.