ON A SINGULAR INTEGRAL OF CHRIST-JOURNÉ TYPE WITH HOMOGENEOUS KERNEL

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Abstract. In this paper, we prove that the following singular integral defined by

$$T_{\Omega,a}f(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\cdot m_{x,y}a\cdot f(y)dy$$

is bounded on $L^p(\mathbb{R}^d)$ for $1<p<\infty$ and is of weak type (1,1), where $\Omega\in L\log^+ L(\mathbb{S}^{d-1})$ and $m_{x,y}a =: \int_0^1 a(sx+(1-s)y)ds$ with $a\in L^\infty(\mathbb{R}^d)$ satisfying some restricted conditions.

1. Introduction

In 1965, A. P. Calderón [2] introduced the commutator $[A,S]$ on $\mathbb{R}$ which is defined by

$$[A,S]f(x) = A(x)Sf(x) - S(Af)(x),$$

where $A\in Lip(\mathbb{R})$ and the operator $S := \frac{d}{dx}\circ H$, H denotes the Hilbert transform defined by

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,dy.$$

Note that the commutator $[A,S]$ can be rewritten as $[A,\sqrt{-\Delta}]$, where $\Delta = \frac{d^2}{dx^2}$ is the Laplacian operator on $\mathbb{R}$. Therefore, the study of the commutator $[A,S]$ plays an important role in the theory of linear partial differential equations, Cauchy integral along Lipschitz curve in $\mathbb{C}$ and the Kato square root problem on $\mathbb{R}$ (see [3], [4], [14], [21], [22], [23], [6], [7] for the details).

By a formal computation, we see that

$$[A,S]f(x) = (-1)\,\text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(x)-A(y)}{x-y}\frac{f(y)}{x-y}\,dy.$$

The operator $[A,S]$ is the so called *Calderón commutator* . In [2], A. P. Calderón proved that if $A\in Lip(\mathbb{R})$, then the Calderón commutator $[A,S]$ is bounded on $L^p(\mathbb{R})$ for all $1<p<\infty$.

In 1987, Christ and Journé [9] introduced a variant singular integral of the Calderón commutator in higher dimensions as follows

$$T_af(x) = \text{p.v.}\int_{\mathbb{R}^d} K(x-y)\cdot m_{x,y}a\cdot f(y)dy, \tag{1.1}$$

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where K is the with the *standard Calderón-Zygmund convolution kernel* , which means that K satisfies the following conditions:

(k1) $|K(x)| \le C|x|^{-d}$;
(k2) $\int_{R<|x|<2R} K(x)dx = 0,$ for all $R > 0$;
(k3) $|K(x-h) - K(x)| \le C|h|^{\nu}|x|^{-d-\nu}$ if $|x| > 2|h|$, where $0 < \nu \le 1$.

Here and in the sequel, for $a \in L^\infty(\mathbb{R}^d)$,

$$m_{x,y}a = \int_0^1 a(sx + (1-s)y)ds.$$

When the dimension $d = 1$, we have

$$m_{x,y}a = \frac{\int_0^x a(z)dz - \int_0^y a(z)dz}{x-y} =: \frac{A(x) - A(y)}{x-y}.$$

Obviously, $A'(x) = a(x) \in L^\infty(\mathbb{R})$. So, if taking $K(x) = -\frac{1}{\pi x}$, we see that

$$T_a f(x) = (-1)\,\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{A(x)-A(y)}{x-y}\frac{f(y)}{x-y}dy.$$

Hence, when $d = 1$, the operator T_a is just the *Calderón commutator* $[A, S]$. In [9], Christ and Journé showed that T_a is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

In 1995, by taking $K(x) = \Omega(x)|x|^{-d}\,(x \ne 0)$, S. Hofmann [20] discussed the following singular integral of Christ-Journé type with homogeneous kernel:

$$T_{\Omega,a}f(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\cdot m_{x,y}a \cdot f(y)dy, \tag{1.2}$$

where

$$\Omega(rx') = \Omega(x'), \text{ for any } r > 0 \text{ and } x' \in \mathbb{S}^{d-1} \tag{1.3}$$

and Ω satisfies

$$\int_{\mathbb{S}^{d-1}} \Omega(x')d\sigma(x') = 0. \tag{1.4}$$

In [20], S. Hofmann gave the weighted L^p boundedness of $T_{\Omega,a}$ if $\Omega \in L^\infty(\mathbb{S}^{d-1})$ satisfies (1.3), (1.4) and $a \in L^\infty(\mathbb{R}^d)$. Recently, the weak type estimates for the singular integral T_a defined by (1.1) are also discussed. In 2012, Grafakos and Honzík [18] proved that T_a is of weak type (1,1) in dimension $d = 2$. Further, Seeger [25] showed that T_a is of weak type (1,1) for all dimension $d \ge 2$. In 2015, the author [11] gave a weighted weak (1,1) boundedness of T_a for dimension $d = 2$ with power weight $\omega(x) = |x|^\alpha$ for $-2 < \alpha < 0$ and later extended to more general $A_1(\mathbb{R}^d)$ weight for dimension $d \ge 2$ in [12].

It is well known that if $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ and satisfies (1.3) and (1.4), the singular integral operator with rough kernel defined by

$$T_\Omega(f)(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}f(y)dy \tag{1.5}$$

is bounded from $L^p(\mathbb{R}^d)$ to itself for $1 < p < \infty$ (see [5]) and is of weak type (1,1) (see [24]). Now a natural question is that whether similar results hold for $T_{\Omega,a}$ defined in (1.2) if $\Omega \in$

$L\log^+ L(\mathbb{S}^{d-1})$. In this paper, we give some partial answer to this question. Our main result is as follows.

Theorem 1.1. *Suppose $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ and satisfies (1.3) and (1.4). Let $a \in L^1(\mathbb{R}^d)$ and satisfy $\hat{a} \in L^1(\mathbb{R}^d)$.*

(a) *For $1 < p < \infty$, we have*

$$\|T_{\Omega,a}f\|_p \le C\|\hat{a}\|_1\|\Omega\|_{L\log^+ L}\|f\|_p;$$

(b) *For $p = 1$, we have*

$$m(\{x \in \mathbb{R}^d : |T_{\Omega,a}f(x)| > \lambda\}) \le \frac{C}{\lambda}\|\hat{a}\|_1\|f\|_1.$$

The constant C above is depended on the dimension d and Ω.

Remark 1.2. It is clear that the conditions $a \in L^1(\mathbb{R}^d)$ and $\hat{a} \in L^1(\mathbb{R}^d)$ imply $a \in L^\infty(\mathbb{R}^d)$. It seems difficult to get the L^p and weak (1,1) boundedness of $T_{\Omega,a}$ with $a \in L^\infty(\mathbb{R}^d)$ only by the method presented in this paper. So, it is still an open question whether the commutator $T_{\Omega,a}$ is L^p bounded for $1 < p < \infty$ and is of weak type (1,1) for $a \in L^\infty(\mathbb{R}^d)$ and $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ with (1.3) and (1.4).

The proof of part (a) is quite simple. We use the Fourier inversion formula of a and the problem can be reduced to the L^p boundedness of T_Ω. The main content of this paper is dedicated to the proof of part (b) in Theorem 1.1. The proof is based on a variant Calderón-Zygmund decomposition. More precisely, we make a Calderón-Zygmund type decomposition of a L^1 function with some parameters and the constants that appears in the estimate are independent of these parameters. For the rest of the proof, we use some nice ideas from Seeger's works ([24], [25]). Recall that when the dimension $d = 1$, $m_{x,y}a$ can be rewritten as $\frac{A(x)-A(y)}{x-y}$ which has some smoothness about variable x, y. For dimension $d \ge 2$, $m_{x,y}a$ has no smoothness about x, y since $a \in L^\infty(\mathbb{R}^d)$. Note that the kernel K satisfying (k1)-(k3) has some smoothness and the commutator T_a defined in (1.1) has only one rough factor $m_{x,y}a$. However, for the commutator $T_{\Omega,a}$, it is much harder to establish the weak (1,1) boundedness since it involves two rough factors: Ω and $m_{x,y}a$.

Besides the higher dimensional variant form of the Calderón commutator defined in (1.2), there are some other kinds of the Calderón commutators in higher dimensions. For example, in [2], A. P. Calderón considered the following commutator:

$$\mathfrak{T}_{\Omega,A}f(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} \cdot \frac{A(x)-A(y)}{|x-y|} \cdot f(y)dy, \tag{1.6}$$

where $A \in Lip(\mathbb{R}^d)$ and Ω satisfies (1.3) and

$$\int_{\mathbb{S}^{d-1}} \Omega(x')x'^\alpha d\sigma(x') = 0, \quad \text{for all} \quad \alpha \in \mathbb{Z}_+^d \quad \text{with} \quad |\alpha| = 1.$$

Calderón showed that $\mathfrak{T}_{\Omega,A}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ if $\nabla A \in L^\infty(\mathbb{R}^d)$ and $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$. Recently the authors of this paper established a weak type (1,1) criteria for singular integral with rough kernel in [13] and used this criteria to show $\mathfrak{T}_{\Omega,A}$ is weak type (1,1) bounded if $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$. However this criteria is not efficient to the operator $T_{\Omega,a}$ discussed in this paper if $a \in L^\infty$. For more topics about singular integral with rough kernel, we refer to see [5],[1],[8],[10],[19],[26],[15],[27],[16].

This paper is organized as follows. In Section 2, we complete the proof of part (a) in Theorem 1.1 and part (b) based on some lemmas, their proofs are given in Section 3 and 4, respectively. Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. For a Lebesgue measurable set $E \subset \mathbb{R}^d$, we denote its measure by $|E|$ or $m(E)$. $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform of f defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx.$$

$\mathbb{Z}_+^d$ denotes the space of nonnegative multi-indices and $\mathbb{Z}_+$ denote the set of all nonnegative integers. Moreover, $\|\Omega\|_q := \big(\int_{\mathbb{S}^{d-1}} |\Omega(x')|^q d\sigma(x')\big)^{\frac{1}{q}}$ and $\|\Omega\|_{L\log^+ L} := \int_{\mathbb{S}^{d-1}} |\Omega(x')| \log(2 + |\Omega(x')|)d\sigma(x')$.

2. Proof of Theorem 1.1

2.1. **Proof of part (a) in Theorem 1.1.** Using the inversion Fourier formula, we write

$$m_{x,y}a = \frac{1}{(2\pi)^d}\int_0^1 \int_{\mathbb{R}^d} \widehat{a}(\eta) e^{is\langle \eta,x\rangle} e^{i(1-s)\langle y,\eta\rangle} d\eta ds.$$

Therefore by Fubini's theorem, we have

$$\begin{aligned} T_{\Omega,a}(f)(x) &= \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\Big(\frac{1}{(2\pi)^d}\iint_{[0,1]\times\mathbb{R}^d} \widehat{a}(\eta) e^{is\langle x,\eta\rangle} e^{i(1-s)\langle y,\eta\rangle} ds d\eta\Big) f(y)dy \\ &= \iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) T_\Omega(W^{\eta,s}f)(x)\, d\eta ds \end{aligned} \tag{2.1}$$

where $a^{x,s}(\eta) = \frac{1}{(2\pi)^d}\widehat{a}(\eta)e^{is\langle x,\eta\rangle}$, $W^{\eta,s}(y) = e^{i(1-s)\langle y,\eta\rangle}$ and T_Ω is defined by (1.5). Now applying Minkowski's inequality, the above inequality and T_Ω is bounded on $L^p(\mathbb{R}^d)$, we have

$$\|T_{\Omega,a}(f)\|_p \le \iint\limits_{[0,1]\times\mathbb{R}^d} |\hat{a}| \|T_\Omega(W^{\eta,s}f)\|_p \, d\eta ds \le C\|\hat{a}\|_1 \|\Omega\|_{L\log^+ L}\|f\|_p.$$

□

2.2. **Proof of part (b) in Theorem 1.1.** We will finish the proof of part (b) based on some lemmas, their proofs are given in Section 3 and Section 4, respectively. We only focus on dimension $d \ge 2$. By using scaling arguments, we may assume $\|\Omega\|_{L\log^+ L(\mathbb{S}^{d-1})} = \|\widehat{a}\|_{L^1(\mathbb{R}^d)} = 1$. Write $T_{\Omega,a}$ in the form (2.1). In the following, we try to make a Calderón-Zygmund decomposition of $W^{\eta,s}f$ with the underlying cubes independent of η, s.

Lemma 2.1. *Fix η, s. Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Set $\Omega_\lambda = \{x \in \mathbb{R}^d : M(f)(x) > \lambda\}$ where M is the Hardy-Littlewood maximal operator. Then we have the following conclusions:*

(i) $\Omega_\lambda = \bigcup Q$, *$Q$'s are disjoint dyadic cubes. Set $\mathcal{Q}$ be the collection of all these cubes.*

(ii) $m(\Omega_\lambda) \le C\lambda^{-1}\|f\|_1$.

(iii) $fW^{\eta,s} = g^{\eta,s} + b^{\eta,s}$.

(iv) $b^{\eta,s} = \sum_{Q\in\mathcal{Q}} b_Q^{\eta,s}$, $\text{supp} b_Q^{\eta,s} \subset Q$, $\int b_Q^{\eta,s} = 0$, $\|b_Q^{\eta,s}\|_1 \le C\lambda|Q|$, $\|b^{\eta,s}\|_1 \le C\|f\|_1$.

(v) $\|g^{\eta,s}\|_2^2 \le C\lambda\|f\|_1$.

Here all the constants C in (i)-(v) are independent of η, s.

Proof. We first make a Whitney decomposition of the set Ω_λ. Then there exists a family of dyadic closed cubes $\{Q_j\}_j$ (see [17]) such that

(a) $\bigcup Q_j = \Omega_\lambda$ and Q_j's have disjoint interior.

(b) $\sqrt{d} \cdot l(Q_j) \le dist(Q_j, \Omega_\lambda^c) \le 4\sqrt{d} \cdot l(Q_j)$, where $l(Q_j)$ denotes the side's length of Q_j.

By the weak type (1,1) bound of M, we have

$$m(\Omega_\lambda) \le \frac{C}{\lambda}\|f\|_1. \tag{2.2}$$

We write $fW^{\eta,s} = g^{\eta,s} + b^{\eta,s}$, where

$$g^{\eta,s} = fW^{\eta,s}\chi_{\Omega_\lambda^c} + \sum_Q \frac{1}{|Q|}\int_Q f(x)W^{\eta,s}(x)dx\chi_Q,$$

$$b^{\eta,s} = \sum_Q \Big\{fW^{\eta,s} - \frac{1}{|Q|}\int_Q f(x)W^{\eta,s}(x)dx\Big\}\chi_Q =: \sum_Q b_Q^{\eta,s}.$$

So, $b_Q^{\eta,s}$ is supported in Q and $\int b_Q^{\eta,s} = 0$. Let tQ denote the cube with t times the side length of Q and the same center. We first claim that

$$\frac{1}{|Q|}\int_Q |f(x)|dx \le C\lambda, \tag{2.3}$$

where C is only dependent of the dimension d. In fact, by the Whitney decomposition's property (b) we have $9\sqrt{d}Q \cap \Omega_\lambda^c \ne \emptyset$. Thus by the definition of Ω_λ^c, there exists $x_0 \in 9\sqrt{d}Q$ such that $Mf(x_0) \le \lambda$. Using the property of the maximal function, we have $\frac{1}{|9\sqrt{d}Q|}\int_{9\sqrt{d}Q}|f(x)|dx \le C'\lambda$, where C' is only dependent of the dimension d. Hence we have the estimate

$$\frac{1}{|Q|}\int_Q |f(x)|dx \le \frac{(9\sqrt{d})^d}{|9\sqrt{d}Q|}\int_{9\sqrt{d}Q}|f(x)|dx \le C\lambda.$$

For $b_Q^{\eta,s}$ and $b^{\eta,s}$, by (2.2) and (2.3) we have

$$\|b_Q^{\eta,s}\|_1 \le 2\int_Q |f(x)|dx \le C\lambda|Q|,$$

$$\|b^{\eta,s}\|_1 \le C\|f\|_1 + \lambda m(\Omega_\lambda) \le C\|f\|_1.$$

Note that $|f(x)| \le \lambda$ almost everywhere in $(\Omega_\lambda)^c$, by (2.2) and (2.3), we have

$$\|g^{\eta,s}\|_2^2 \le C\lambda\|f\|_1 + C\lambda^2 m(\Omega_\lambda) \le C\lambda\|f\|_1.$$

∎

By the property (iii) in Lemma 2.1 and (2.1), we have

$$m(\{x:|T_{\Omega,a}(f)(x)|>\lambda\})\le m\Big(\Big\{x:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(g^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big) \tag{2.4}$$
$$+\,m\Big(\Big\{x:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big).$$

Notice that T_Ω is bounded from $L^p(\mathbb{R}^d)$ to itself with bound $\|\Omega\|_{L\log^+L}$. Hence, combining this with Chebyshev's inequality, Minkowski's inequality, and the property (v) in Lemma 2.1,

$$m\Big(\Big\{x:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(g^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)$$
$$\le\frac{4}{\lambda^2}\Big(\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\cdot\|T_\Omega(g^{\eta,s})\|_2 d\eta ds\Big)^2\le\frac{C}{\lambda}\|f\|_1.$$

For $Q\in\mathcal{Q}$, denote by $l(Q)$ the side length of cube Q. Set $E^*=\bigcup_{Q\in\mathcal{Q}}2^{200}Q$. Then we have

$$m\Big(\Big\{x:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)$$
$$\le m(E^*)+m\Big(\Big\{x\in(E^*)^c:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big).$$

By the property (ii) in Lemma 2.1, the set E^* satisfies

$$m(E^*)\le Cm(\Omega_\lambda)\le\frac{C}{\lambda}\|f\|_1.$$

Thus, to complete the proof of part (b) in Theorem 1.1, it remains to show

$$m\Big(\Big\{x\in(E^*)^c:\Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)\le\frac{C}{\lambda}\|f\|_1, \tag{2.5}$$

where C is only dependent of the dimension d.

Denote $\mathfrak{Q}_k=\{Q\in\mathcal{Q}:\,l(Q)=2^k\}$ and let $B_k^{\eta,s}=\sum\limits_{Q\in\mathfrak{Q}_k}b_Q^{\eta,s}$. Then $b^{\eta,s}$ can be rewritten as $b^{\eta,s}=\sum\limits_{j\in\mathbb{Z}}B_j^{\eta,s}$. Taking a smooth radial function ϕ on $\mathbb{R}^d$ such that supp $\phi\subset\{x:\frac{1}{4}\le|x|\le1\}$ and $\sum_j\phi_j(x)=1$ for all $x\in\mathbb{R}^d\backslash\{0\}$, where $\phi_j(x)=\phi(2^{-j}x)$. Now we define the operator T_j as

$$T_j(f)(x)=\int_{\mathbb{R}^d}\frac{\Omega(x-y)}{|x-y|^d}\phi_j(x-y)f(y)dy. \tag{2.6}$$

Then we have $T_\Omega=\sum\limits_j T_j$. We write

$$T_\Omega(b^{\eta,s})(x)=\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}T_j(B_{j-n}^{\eta,s})(x).$$

Note that $T_j(B^{\eta,s}_{j-n})(x) = 0$ for $x \in (E^*)^c$ and $n < 100$. Therefore

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) T_\Omega(b^{\eta,s})(x)\, d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big)$$
$$= m\Big(\Big\{x \in (E^*)^c : \Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j \sum_{n\ge 100} T_j(B^{\eta,s}_{j-n})(x)\, d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big).$$

Hence, to finish the proof of of part (b), it suffices to verify the following estimate:

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j \sum_{n\ge 100} T_j(B^{\eta,s}_{j-n})(x)\, d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le \frac{C}{\lambda}\|f\|_1. \tag{2.7}$$

2.3. **Some key estimates.**

In the following we will show that (2.7) holds if Ω is restricted in some subset of $\mathbb{S}^{d-1}$. More precisely, for a fixed $n \ge 100$, denote $D^\iota = \{\theta \in \mathbb{S}^{d-1} : |\Omega(\theta)| \ge 2^{\iota n}\|\Omega\|_1\}$, where $0 < \iota < \frac{\gamma}{2}$ will be chosen later. The operator $T^n_{j,\iota}$ is defined by

$$T^n_{j,\iota}(f)(x) = \text{p.v.}\int_{\mathbb{R}^d} \Omega\chi_{D^\iota}\Big(\frac{x-y}{|x-y|}\Big)\frac{\phi_j(x-y)}{|x-y|^d}\cdot f(y)dy.$$

We have the following result, which will be proved in next section.

Lemma 2.2. *Under the conditions of Theorem 1.1 with $0 < \iota < \gamma/2$, we have*

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint\limits_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j \sum_{n\ge 100} T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)\, d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le C\frac{\|f\|_1}{\lambda}.$$

Thus, by Lemma 2.2, to finish the proof of Theorem 1.1, it suffices to verify (2.7) for the kernel function Ω, which satisfies $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ in each $T_j(B^{\eta,s}_{j-n})$.

In the following, we need to give a partition of unity on the unit surface $\mathbb{S}^{d-1}$. Fix $n \ge 100$. Let $\Theta_n = \{e^n_v\}_v$ be a collection of unit vectors on $\mathbb{S}^{d-1}$ which satisfies the following two conditions:

(a) $|e^n_v - e^n_{v'}| \ge 2^{-n\gamma-4}$, if $v \ne v'$;

(b) If $\theta \in \mathbb{S}^{d-1}$, there exists a e^n_v such that $|e^n_v - \theta| \le 2^{-n\gamma-4}$.

The constant $0 < \gamma < 1$ in (a) and (b) will be chosen later. To do this, we may simply take a maximal collection $\{e^n_v\}_v$ for which (a) holds. Notice that there are $C2^{n\gamma(d-1)}$ elements in the collection $\{e^n_v\}_v$. For every $\theta \in \mathbb{S}^{d-1}$, there only exists finite e^n_v such that $|e^n_v - \theta| \le 2^{-n\gamma-4}$. Now we can construct an associated partition of unity on the unit surface $\mathbb{S}^{d-1}$. Let ζ be a smooth, nonnegative, radial function with $\zeta(u) = 1$ for $|u| \le \frac{1}{2}$ and $\zeta = 0$ for $|u| > 1$. Set

$$\tilde{\Gamma}^n_v(\xi) = \zeta\Big(2^{n\gamma}\Big(\frac{\xi}{|\xi|} - e^n_v\Big)\Big)$$

and define

$$\Gamma^n_v(\xi) = \tilde{\Gamma}^n_v(\xi)\Big(\sum_v \tilde{\Gamma}^n_v(\xi)\Big)^{-1}.$$

Then it is easy to see that Γ_v^n is homogeneous of degree 0 with

$$\sum_v \Gamma_v^n(\xi) = 1, \text{ for all } \xi \neq 0 \text{ and all } n.$$

Now we define operator $T_j^{n,v}$ by

$$T_j^{n,v}(h)(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\phi_j(x-y)\Gamma_v^n(x-y)\cdot h(y)dy. \tag{2.8}$$

For convenience, define the kernel of $T_j^{n,v}$ as $K_j^{n,v}(x) = \frac{\Omega(x)}{|x|^d}\phi_j(x)\Gamma_v^n(x)$. Therefore, for fixed $n \geq 100$ we have

$$T_j = \sum_v T_j^{n,v}.$$

In the sequel, we need to separate the phase of the kernel into different direction. Hence we define a multiple operator by

$$\widehat{G_{n,v}h}(\xi) = \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)\hat{h}(\xi),$$

where h is a Schwartz function and Φ is a smooth, nonnegative, radial function such that $0 \leq \Phi(x) \leq 1$ and $\Phi(x) = 1$ on $|x| \leq 2$, $\Phi(x) = 0$ on $|x| > 4$. Now we can split $T_j^{n,v}$ into two parts:

$$T_j^{n,v} = G_{n,v}T_j^{n,v} + (I - G_{n,v})T_j^{n,v}.$$

The following lemma gives the L^2 estimate involving $G_{n,v}T_j^{n,v}$, which will be proved in next section.

Lemma 2.3. *For $n \geq 100$, $\|\Omega\|_\infty \leq 2^{\iota n}\|\Omega\|_1$ with $0 < \iota < \gamma/2$, there exists a constant C such that*

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_v \sum_j G_{n,v}T_j^{n,v}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_2^2 \leq C2^{-n\gamma+2n\iota}\lambda\|f\|_1,$$

where constant C is independent of n, λ and f.

The terms involving $(I - G_{n,v})T_j^{n,v}$ are more complicated. In Section 4, we shall prove following lemma.

Lemma 2.4. *For $\|\Omega\|_\infty \leq 2^{\iota n}\|\Omega\|_1$ in $T_j^{n,v}$, then*

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_{n\geq 100} \sum_v \sum_j (I - G_{n,v})T_j^{n,v}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_1 \leq C\|f\|_1$$

where C is independent of λ and f .

2.4. **Proof of part (b) in Theorem 1.1.**

We now complete the proof of (2.7) with $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ in each T_j. By Chebyshev's inequality, we have

$$\begin{aligned}
&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j\sum_{n\ge100}T_j^n(B_{j-n}^{\eta,s})(x)d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)\\
&\le\frac{16}{\lambda^2}\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_v\sum_j G_{n,v}T_j^{n,v}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_2^2\\
&+\frac{4}{\lambda}\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_v\sum_j (I-G_{n,v})T_j^{n,v}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_1\\
&=: I+II.
\end{aligned}$$

Using Lemma 2.4, we can get the desired estimate of II. Notice that we choose $0<\iota<\frac{\gamma}{2}$. For I, by Minkowski's inequality and Lemma 2.3, we have

$$\begin{aligned}
I&\le C\lambda^{-2}\Big(\sum_{n\ge100}\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_v\sum_j G_{n,v}T_j^{n,v}(B_{j-n}^{\eta,s})d\eta ds\Big\|_2\Big)^2\\
&\le C\lambda^{-2}\Big(\sum_{n\ge100}(2^{-n\gamma+2n\iota}\lambda\|f\|_1)^{\frac12}\Big)^2\le C\lambda^{-1}\|f\|_1.
\end{aligned}$$

Combining with Lemma 2.2, we hence complete the proof of part (b) in Theorem 1.1 once Lemmas 2.2-2.4 hold. □

3. Proofs of Lemmas 2.2 and 2.3

3.1. **Proof of Lemma 2.2.**

Denote the kernel of operator $T_{j,\iota}^n$ by

$$K_{j,\iota}^n(y):=\Omega\chi_{D^\iota}(\frac{y}{|y|})\frac{\phi_j(y)}{|y|^d}.$$

It is easy to see that

$$\Big|\int_{\mathbb{R}^d}K_{j,\iota}^n(y)dy\Big|\le C\int_{D^\iota}\int_{2^{j-2}}^{2^j}|\Omega(\theta)|r^{-1}drd\sigma(\theta)\le C\int_{D^\iota}|\Omega(\theta)|d\sigma(\theta).$$

Therefore by Chebyshev's inequality, Minkowski's inequality and the property (iv) in Lemma 2.1, we get

$$\begin{aligned}
&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_{n\geq100}\sum_{j\in\mathbb{Z}}T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)\\
&\leq\frac{C}{\lambda}\Big\|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_{n\geq100}\sum_{j\in\mathbb{Z}}T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)d\eta ds\Big\|_1\\
&\leq\frac{C}{\lambda}\sum_{n\geq100}\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\sum_j\|B^{\eta,s}_{j-n}\|_1d\eta ds\int_{D^\iota}|\Omega(\theta)|d\sigma(\theta)\\
&\leq\frac{C}{\lambda}\|\hat{a}\|_1\|f\|_1\int_{\mathbb{S}^{d-1}}\mathrm{card}\big\{n\in\mathbb{N}:\,n\geq100,2^{\iota n}\leq|\Omega(\theta)|/\|\Omega\|_1\big\}|\Omega(\theta)|d\sigma(\theta)\\
&\leq\frac{C}{\lambda}\|\hat{a}\|_1\|f\|_1.
\end{aligned}$$

□

3.2. **Proof of Lemma 2.3.**

We will use some idea from [24] in the proof of Lemma 2.3. As usually, we adopt the TT^* method in the L^2 estimate. Moreover, we also use some orthogonality argument based on the following observation of the support of $\mathcal{F}(G_{n,v}T^{n,v}_j)$: For a fixed $n\geq100$, one has

$$\sup_{\xi\neq0}\sum_v|\Phi^2(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle)|\leq C2^{n\gamma(d-2)}. \tag{3.1}$$

In fact, by the homogeneity of Φ, it suffices to take the supremum over the surface $\mathbb{S}^{d-1}$. For $|\xi|=1$ and $\xi\in\operatorname{supp}\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle)$, denote by $\xi^\perp$ the hyperplane perpendicular to ξ. Thus

$$\mathrm{dist}(e^n_v,\xi^\perp)\leq C2^{-n\gamma}. \tag{3.2}$$

Since the mutual distance of e^n_v's is bounded by $2^{-n\gamma-4}$, there are at most $C2^{n\gamma(d-2)}$ vectors satisfy (3.2). We hence get (3.1).

By applying Minkowski's inequality, Plancherel's theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big\|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_v\sum_jG_{n,v}T^{n,v}_j(B^{\eta,s}_{j-n})(x)d\eta ds\Big\|_2^2\\
&\leq\Big(\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\Big\|\sum_v\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle)\mathcal{F}\Big(\sum_jT^{n,v}_j(B^{\eta,s}_{j-n})\Big)(\xi)\Big\|_2d\eta ds\Big)^2\\
&\leq C2^{n\gamma(d-2)}\Big(\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\Big\|\sum_v\Big|\mathcal{F}\Big(\sum_jT^{n,v}_j(B^{\eta,s}_{j-n})\Big)\Big|^2\Big\|_1^{\frac12}d\eta ds\Big)^2\\
&\leq C2^{n\gamma(d-2)}\Big(\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\Big(\sum_v\Big\|\sum_jT^{n,v}_j(B^{\eta,s}_{j-n})\Big\|_2^2\Big)^{\frac12}d\eta ds\Big)^2.
\end{aligned} \tag{3.3}$$

Next we will show that for a fixed e^n_v,η,s,

$$\Big\|\sum_jT^{n,v}_j(B^{\eta,s}_{j-n})\Big\|_2^2\leq C2^{-2n\gamma(d-1)+2n\iota}\lambda\|f\|_1. \tag{3.4}$$

Then by $\mathrm{card}(\Theta_n) \le C2^{n\gamma(d-1)}$, and apply (3.3) and (3.4) we get

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_v \sum_j G_{n,v} T_j^{n,v}(B_{j-n}^{\eta,s}) d\eta ds \Big\|_2^2 \le C2^{-n\gamma+2n\iota}\lambda\|f\|_1,$$

which is just desired bound of Lemma 2.3. Thus, to finish the proof of Lemma 2.3, it is enough to prove (3.4). By applying $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$, then

$$\begin{aligned} |T_j^{n,v}(B_{j-n}^{\eta,s})(x)| &\le C2^{-jd}2^{\iota n}\|\Omega\|_1 \int_{\mathbb{R}^d} \phi_j(x-y)\Gamma_v^n(x-y)|B_{j-n}^{\eta,s}(y)|dy \\ &\le C2^{\iota n} H_j^{n,v} * |B_{j-n}^{\eta,s}|(x), \end{aligned}$$

where $H_j^{n,v}(x) := 2^{-jd}\chi_{E_j^{n,v}}(x)$ and $\chi_{E_j^{n,v}}(x)$ is a characteristic function of the set

$$E_j^{n,v} := \{x \in \mathbb{R}^d : |\langle x, e_v^n\rangle| \le 2^j, |x - \langle x, e_v^n\rangle e_v^n| \le 2^{j-n\gamma}\}.$$

For a fixed e_v^n, we write

$$\begin{aligned} \Big\| \sum_j T_j^{n,v}(B_{j-n}^{\eta,s}) \Big\|_2^2 &\le C2^{2\iota n} \sum_j \int_{\mathbb{R}^d} H_j^{n,v} * H_j^{n,v} * |B_{j-n}^{\eta,s}|(x) \cdot |B_{j-n}^{\eta,s}(x)| dx \\ &\quad + C2^{2\iota n} \sum_j \sum_{i=-\infty}^{j-1} \int_{\mathbb{R}^d} H_j^{n,v} * H_i^{n,v} * |B_{i-n}^{\eta,s}|(x) \cdot |B_{j-n}^{\eta,s}(x)| dx. \end{aligned} \tag{3.5}$$

Observe that $\|H_i^{n,v}\|_1 \le C2^{-id}m(E_i^{n,v}) \le C2^{-n\gamma(d-1)}$, therefore for any $i \le j$,

$$H_j^{n,v} * H_i^{n,v}(x) \le 2^{-n\gamma(d-1)}2^{-jd}\chi_{\widetilde{E}_j^{n,v}},$$

where $\widetilde{E}_j^{n,v} = E_j^{n,v} + E_j^{n,v}$. Hence for a fixed j, n, e_v^n and x, we have

$$\begin{aligned} &H_j^{n,v} * H_j^{n,v} * |B_{j-n}^{\eta,s}|(x) + \sum_{i=-\infty}^{j-1} H_j^{n,v} * H_i^{n,v} * |B_{i-n}^{\eta,s}|(x) \\ &\le C2^{-n\gamma(d-1)}2^{-jd} \sum_{i\le j} \int_{x+\widetilde{E}_j^{n,v}} |B_{i-n}^{\eta,s}(y)| dy \\ &\le C2^{-n\gamma(d-1)}2^{-jd} \sum_{i\le j} \sum_{\substack{Q\in\mathfrak{Q}_{i-n} \\ Q\cap\{x+\widetilde{E}_j^{n,v}\}\ne\emptyset}} \int_{\mathbb{R}^d} |b_Q^{\eta,s}(y)| dy \\ &\le C2^{-n\gamma(d-1)}2^{-jd} \sum_{i\le j} \sum_{\substack{Q\in\mathfrak{Q}_{i-n} \\ Q\cap\{x+\widetilde{E}_j^{n,v}\}\ne\emptyset}} \lambda|Q| \\ &\le C2^{-n\gamma(d-1)}2^{-jd}2^{jd-n\gamma(d-1)}\lambda \\ &\le C\lambda 2^{-2n\gamma(d-1)}, \end{aligned} \tag{3.6}$$

where in third inequality above, we use $\int |b_Q^{\eta,s}(y)|dy \le C\lambda|Q|$ (see the property (iv) in Lemma 2.1) and in the fourth inequality we use the fact that the cubes in $\mathcal{Q}$ are disjoint (see the property

(i) in Lemma 2.1). By (3.5), (3.6) and $\sum_j \|B^{\eta,s}_{j-n}\|_1 \le C\|f\|_1$, we obtain

$$\Big\|\sum_j T^{n,v}_j(B^{\eta,s}_{j-n})\Big\|_2^2 \le C\lambda 2^{-2n\gamma(d-1)+2n\iota}\sum_j \|B^{\eta,s}_{j-n}\|_1 \le C\lambda 2^{-2n\gamma(d-1)+2n\iota}\|f\|_1,$$

which is just (3.4) and we complete the proof of Lemma 2.3. □

4. Proof of Lemma 2.4

To prove Lemma 2.4, we have to face with some oscillatory integrals which come from the term $(I-G_{n,v})T^{n,v}_j$.

Before stating the proof of Lemma 2.4, let us give some notations. We introduce a frequency decomposition. Let ψ be a radial C^∞ function such that $\psi(\xi)=1$ for $|\xi|\le 1$, $\psi(\xi)=0$ for $|\xi|\ge 2$ and $0\le\psi(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Define $\beta(\xi)=\psi(\xi)-\psi(2\xi), \beta_k(\xi)=\beta(2^k\xi)$, then β_k is supported in $\{\xi : 2^{-k-1}\le|\xi|\le 2^{-k+1}\}$. Define the convolution operators Λ_k with Fourier multipliers β_k. That is, $\widehat{\Lambda_k f}(\xi)=\beta_k(\xi)\hat{f}(\xi)$. Then by the construction of β_k, we have

$$I=\sum_{k\in\mathbb{Z}}\Lambda_k$$

where I is the identity. Write $(I-G_{n,v})T^{n,v}_j=\sum_k (I-G_{n,v})\Lambda_k T^{n,v}_j$. By using Minkowski's inequality,

$$\begin{aligned}(4.1)\qquad &\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_v\sum_j (I-G_{n,v})T^{n,v}_j(B^{\eta,s}_{j-n})(x)d\eta ds\Big\|_1\\ &\le \sum_{n\ge100}\sum_v\sum_j\sum_k\sum_{l(Q)=2^{j-n}}\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\cdot\|(I-G_{n,v})\Lambda_k T^{n,v}_j(b^{\eta,s}_Q)\|_1 d\eta ds.\end{aligned}$$

Lemma 4.1. *There exists $N>0$, such that for any $N_1\in\mathbb{Z}_+$*

$$\|(I-G_{n,v})\Lambda_k T^{n,v}_j(b^{\eta,s}_Q)\|_1 \le C2^{-n\gamma(d-1)+n\iota+(-j+k)N_1+n\gamma(N_1+2N)}\|b^{\eta,s}_Q\|_1, \tag{4.2}$$

where C is a constant only dependent of N_1.

Proof. Denote $h_{k,n,v}(\xi)=(1-\Phi(2^{n\gamma}\langle e^n_v,\xi/|\xi|\rangle))\beta_k(\xi)$. Then

$$\|(I-G_{n,v})\Lambda_k T^{n,v}_j(b^{\eta,s}_Q)\|_1 \le \|\mathcal{F}^{-1}(h_{k,n,v}\widehat{K^{n,v}_j})\|_1\|b^{\eta,s}_Q\|_1.$$

Write

$$\mathcal{F}^{-1}(h_{k,n,v}\widehat{K^{n,v}_j})(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}h_{k,n,v}(\xi)\int_{\mathbb{R}^d} e^{-i\xi\cdot\omega}K^{n,v}_j(\omega)d\omega d\xi.$$

In order to separate the rough kernel, we change to polar coordinates $\omega=r\theta$, then the integral above can be written as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma^n_v(\theta)\Big\{\int_{\mathbb{R}^d}\int_0^\infty e^{i(\langle x-r\theta,\xi\rangle)}h_{k,n,v}(\xi)\cdot\frac{\phi_j(r)}{r}drd\xi\Big\}d\sigma(\theta). \tag{4.3}$$

Since $\theta \in \text{supp } \Gamma_v^n$, then $|\theta - e_v^n| \le 2^{-n\gamma}$. By the support of Φ, we see $|\langle e_v^n, \xi/|\xi|\rangle| \ge 2^{1-nr}$. Thus,

$$|\langle \theta, \xi/|\xi|\rangle| \ge |\langle e_v^n, \xi/|\xi|\rangle| - |\langle e_v^n - \theta, \xi/|\xi|\rangle| \ge 2^{-n\gamma}. \tag{4.4}$$

Notice that ϕ_j is supported in $[2^{j-2}, 2^j]$, we can integrate by parts N_1 times with r. Hence the integral (4.3) can be rewritten as

$$\frac{1}{(2\pi)^d} \int_{\mathbb{S}^{d-1}} \Omega(\theta)\Gamma_v^n(\theta) \bigg\{ \int_{\{2^{-k-1} \le |\xi| \le 2^{-k+1}\}} \int_{2^{j-2}}^{2^j} e^{i(\langle x - r\theta, \xi\rangle)} h_{k,n,v}(\xi) \times (i\langle \theta, \xi\rangle)^{-N_1} \cdot \partial_r^{N_1}[\phi_j(r) r^{-1}] dr d\xi \bigg\} d\sigma(\theta),$$

since $h_{k,n,v}$ is supported in $\{2^{-k-1} \le |\xi| \le 2^{-k+1}\}$. Integrating by parts in ξ, the integral in curly brackets above can be rewritten as

$$\int_{\{2^{-k-1} \le |\xi| \le 2^{-k+1}\}} \int_{2^{j-2}}^{2^j} e^{i\langle x - r\theta, \xi\rangle} \frac{(I - 2^{-2k}\Delta_\xi)^N \big[(i\langle \theta, \xi\rangle)^{-N_1} h_{k,n,v}(\xi)\big]}{(1 + 2^{-2k}|x - r\theta|^2)^N} \cdot \partial_r^{N_1}[\phi_j(r) r^{-1}] dr d\xi. \tag{4.5}$$

We first give an exploit estimate of the term in (4.5). Note that $2^{j-2} \le r \le 2^j$, we get

$$\big|\partial_r^{N_1}[\phi_j(r) r^{-1}]\big| \le C 2^{-j(1+N_1)}. \tag{4.6}$$

In the following, we claim that

$$\big|(I - 2^{-2k}\Delta_\xi)^N [\langle \theta, \xi\rangle^{-N_1} h_{k,n,v}(\xi)]\big| \le C 2^{(n\gamma+k)N_1 + 2n\gamma N}. \tag{4.7}$$

In fact, by (4.4), it is easy to see that

$$|(-i\langle \theta, \xi\rangle)^{-N_1} \cdot h_{k,n,v}(\xi)| \le C|\langle \theta, \xi\rangle|^{-N_1} \le C 2^{(n\gamma+k)N_1}.$$

Using product rule, we get

$$|\partial_{\xi_i} h_{k,n,v}(\xi)| = | - \partial_{\xi_i}[\Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle)] \cdot \beta_k(\xi) + \partial_{\xi_i}\beta_k(\xi) \cdot (1 - \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle))| \le C 2^{n\gamma+k}.$$

Therefore by induction, we have $|\partial_\xi^\alpha h_{k,n,v}(\xi)| \le C 2^{(n\gamma+k)|\alpha|}$ for any multi-indices $\alpha \in \mathbb{Z}_+^d$. By using product rule again and (4.4), we have

$$\begin{aligned}\big|\partial_{\xi_k}^2 (\langle \theta, \xi\rangle)^{-N_1} h_{k,n,v}(\xi))\big| = \big|&\langle \theta, \xi\rangle^{-N_1-2} \cdot N_1(N_1+1)\theta_k^2 \cdot h_{k,n,v} \\ &+ 2\langle \theta, \xi\rangle^{-N_1-1} \cdot (-N_1) \cdot \theta_k \partial_{\xi_k} h_{k,n,v}(\xi) + \langle \theta, \xi\rangle^{-N_1} \partial_{\xi_k}^2 h_{k,n,v}(\xi)\big| \\ \le{}& C 2^{(n\gamma+k)(N_1+2)}.\end{aligned}$$

Hence we conclude that

$$2^{-2k}\big|\Delta_\xi[(\langle \theta, \xi\rangle)^{-N_1} h_{k,n,v}(\xi)]\big| \le C 2^{(n\gamma+k)N_1 + 2n\gamma}.$$

Proceeding by induction, we get (4.7). Now we choose $N = [d/2] + 1$. Since we need to get the L^1 estimate of (4.3), by the support of $h_{k,n,v}$,

$$\int_{\{2^{-k-1} \le |\xi| \le 2^{-k+1}\}} \int \Big(1 + 2^{-2k}|x - r\theta|^2\Big)^{-N} dx d\xi \le C.$$

Integrating with r, we get a bound 2^j. Note that we suppose that $\|\Omega\|_\infty \le 2^{n\iota}\|\Omega\|_1$. Next integrating with θ, we get a bound $2^{-n\gamma(d-1)+n\iota}\|\Omega\|_1$. Combining (4.6), (4.7) and above estimates, (4.2) is bounded by

$$2^{-j(1+N_1)+(n\gamma+k)N_1+2n\gamma N+j-n\gamma(d-1)+n\iota}\|\Omega\|_1 \le C2^{-n\gamma(d-1)+n\iota}2^{(-j+k)N_1+n\gamma(N_1+2N)}.$$

Hence we complete the proof of Lemma 4.1 with $N=[d/2]+1$. □

Lemma 4.2. *There exists $N>0$, such that*

$$\|(I-G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})\|_1 \le C2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1.$$

Proof. The proof of this lemma is similar to that of Lemma 4.1. However we will not integrate by part with r, but use some cancellation of $b_Q^{\eta,s}$. Denote $h_{k,n,v}(\xi)=(1-\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle))\beta_k(\xi)$. Then

(4.8)
$$(I-G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})(x)=\int_{\mathbb{R}^d}\Big(\mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y)-\mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y_Q)\Big)b_Q^{\eta,s}(y)dy$$

where y_Q is the center of Q. Here we use the cancellation of $b_Q^{\eta,s}$ (see the property (iv) in Lemma 2.1). By making a change to polar coordinate and integrating by parts with ξ, $\mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y)$ could be rewritten as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta)\bigg\{\int_{\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}}\int_{2^{j-2}}^{2^j}e^{i\langle x-y-r\theta,\xi\rangle}\times\frac{(I-2^{-2k}\Delta_\xi)^N\big[h_{k,n,v}(\xi)\big]}{(1+2^{-2k}|x-y-r\theta|^2)^N}\cdot\phi_j(r)r^{-1}drd\xi\bigg\}d\sigma(\theta).$$

Here we choose $N=[d/2]+1$. Thus (4.8) can be rewritten as two parts: $I(x)+II(x)$, where

$$I(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta)\bigg\{\int_\xi\int_r e^{i\langle x-r\theta,\xi\rangle}\Big(e^{-i\langle y,\xi\rangle}-e^{-i\langle y_Q,\xi\rangle}\Big)\times\frac{(I-2^{-2k}\Delta_\xi)^N\big[h_{k,n,v}(\xi)\big]}{(1+2^{-2k}|x-y-r\theta|^2)^N}\phi_j(r)r^{-1}drd\xi\bigg\}d\sigma(\theta)\cdot b_Q^{\eta,s}(y)dy$$

and

$$II(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta)\bigg\{\int_\xi\int_r e^{i\langle x-y_Q-r\theta,\xi\rangle}(I-2^{-2k}\Delta_\xi)^N\big[h_{k,n,v}(\xi)\big]\phi_j(r)r^{-1}\times\Big((1+2^{-2k}|x-y-r\theta|^2)^{-N}-(1+2^{-2k}|x-y_Q-r\theta|^2)^{-N}\Big)drd\xi\bigg\}d\sigma(\theta)\cdot b_Q^{\eta,s}(y)dy.$$

Note that $y\in Q$ and y_Q is the center of Q, then $|y-y_Q|\le C2^{j-n}$. By applying (4.7) with $N_1=0$, we get

$$\big|(I-2^{-2k}\Delta_\xi)^N\big(h_{k,n,v}(\xi)\big)\big|\le C2^{2n\gamma N} \tag{4.9}$$

Notice that $|e^{-i\langle y,\xi\rangle}-e^{-i\langle y_Q,\xi\rangle}|\le C2^{j-n-k}$. Now integrating with the variables in the order as we do in proving Lemma 4.1, we may obtain that the L^1 norm of $I(x)$ is bounded by $2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1$.

For $II(x)$, using the following observation

$$\begin{aligned}\Big|\Psi(y)-\Psi(y_Q)\Big| &= \Big|\int_0^1 \langle y-y_Q, \nabla\Psi(ty+(1-t)y_0)\rangle dt\Big| \\ &\le C|y-y_Q|\int_0^1 \frac{N2^{-2k}|x-(ty+(1-t)y_Q)-r\theta|}{(1+2^{-2k}|x-(ty+(1-t)y_Q)-r\theta|^2)^{N+1}}dt\end{aligned}$$

where $\Psi(y)=(1+2^{-2k}|x-y-r\theta|^2)^{-N}$, we may also get the L^1 norm of $II(x)$ is bounded by $2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1$. Thus we finish the proof of Lemma 4.2 □

4.1. **Proof of Lemma 2.4.**

Let us come back to the proof of Lemma 2.4. Denote by $[x]$ the integral part of x. Let ε_0 satisfy $0<\varepsilon_0<1$ and will be chosen later. By (4.1),

$$\begin{aligned}&\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_v\sum_j (I-G_{n,v})T_j^{n,v}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_1 \\ &\le \sum_{n\ge100}\sum_v\sum_j\sum_{k<j-[n\varepsilon_0]}\sum_{l(Q)=2^{j-n}}\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\cdot\|(I-G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})\|_1 d\eta ds \\ &+\sum_{n\ge100}\sum_v\sum_j\sum_{k\ge j-[n\varepsilon_0]}\sum_{l(Q)=2^{j-n}}\iint_{[0,1]\times\mathbb{R}^d}|\hat{a}(\eta)|\cdot\|(I-G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})\|_1 d\eta ds\end{aligned}$$

Now using Lemma 4.1 with $N=[d/2]+1$ for the first term, Lemma 4.2 with $N=[d/2]+1$ for the second term, the fact $[n\varepsilon_0]\le n\varepsilon_0<[n\varepsilon_0]+1$, the property (iv) in Lemma 2.1 and $\mathrm{card}(\Theta_n)\le C2^{n\gamma(d-1)}$, the above sum is bounded by

$$\sum_{n\ge100}(2^{n\tau_1}+2^{n\tau_2})\|\hat{a}\|_1\|f\|_1,$$

where

$$\tau_1=-\varepsilon_0 N_1+\iota+\gamma\big(N_1+2\big([d/2]+1\big)\big),\qquad \tau_2=2\gamma\big([d/2]+1\big)+\varepsilon_0+\iota-1.$$

Choose $0<\iota\ll\gamma\ll\varepsilon_0\ll1$ and N_1 large enough such that

$$\max\{\tau_1,\tau_2\}<0.$$

Therefore the sum for $n\ge100$ is convergent and we finish the proof of Lemma 2.4, thus we prove part (b) in Theorem 1.1. □

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