ON A SINGULARLY PERTURBED SEMI-LINEAR PROBLEM WITH ROBIN BOUNDARY CONDITIONS

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ABSTRACT. This paper is concerned with a semi-linear elliptic problem with Robin boundary condition:

$$\begin{cases} \varepsilon\Delta w - \lambda w^{1+\chi} = 0, & \text{in } \Omega \\ \nabla w \cdot \vec{n} + \gamma w = 0, & \text{on } \partial\Omega \end{cases} \tag{$*$}$$

where $\Omega \in \mathbb{R}^N (N \geq 1)$ is bounded domain with smooth boundary. The problem $(*)$ is derived from the well-known singular Keller-Segel system. When $\gamma > 0$, we show there is only trivial solution $w = 0$. When $\gamma < 0$ and $\Omega = B_R(0)$ is a ball, we show that problem $(*)$ has a non-constant solution which converges to zero uniformly as ε tends to zero. The main idea of this paper is to transform the Robin problem $(*)$ to a nonlocal Dirichelt problem by a Cole-Hopf type transformation and then use the shooting method to obtain the existence of the transformed nonlocal Dirichlet problem. With the results for $(*)$, we get the existence of non-constant stationary solutions to the singular Keller-Segel system.

MSC 2010: 35A01, 35J15, 35J66, 35Q92, 92C17

KEYWORDS: Keller-Segel model, Robin boundary condition, nonlocal ODE, Cole-Hopf transformation, Blowup radius

1. INTRODUCTION

To describe the propagation of traveling bands of chemotactic bacteria observed in the celebrated experiment of Adler [1], Keller and Segel proposed the following singular chemotaxis system in the seminal work [14]

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla \ln w), & \text{in } \Omega \\ w_t = \varepsilon\Delta w - uw^m, & \text{in } \Omega \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the bacterial density and $w(x,t)$ the oxygen/nutrient concentration at position $x \in \mathbb{R}^N$ and at time $t > 0$, respectively. $\varepsilon \geq 0$ is the chemical diffusion coefficient, $\chi > 0$ denotes the chemotactic coefficient and $m \geq 0$ the oxygen consumption rate. The system (1.1) has been well-known as the singular Keller-Segel model/system nowadays as a cornerstone for the modeling of chemotactic movement of bacteria attracted by nutrient/oxygen.

The prominent feature of the Keller-Segel system (1.1) is the use of a logarithmic sensitivity function $\ln w$, which was experimentally verified later in [12]. This logarithm results in a mathematically unfavorable singularity which, however, has been proved to be necessary to generate traveling wave solutions (cf. [22]) that were the first type analytical results developed for the Keller-Segel system (1.1). When $0 \leq m < 1$, Keller and Segel [14] have shown that the model (1.1) with $\varepsilon = 0$ can generate traveling bands qualitatively in agreement with the experiment findings of [1], and later the existence results of traveling wave solutions were extended to any $\varepsilon \geq 0$ and $0 \leq m \leq 1$ (cf. [13, 22, 24, 28]), where the wave profile of (u, w) is of (pulse, front) for $0 \leq m < 1$ and of (front, front) for $m = 1$. When $m > 1$, it was proved that the system (1.1) did not admit any type of traveling wave solutions (e.g., see [28, 30]). Though the Keller-Segel model (1.1) with $m = 1$ can not reproduce the pulsating wave profile to interpret the experiment of [1], it was later employed to describe the boundary movement

of bacterial chemotaxis [25] and migration of endothelial cells toward the signaling molecule vascular endothelial growth factor (VEGF) during the initiation of angiogenesis (cf. [15]).

Aside from the existence of traveling wave solutions, the logarithmic singularity become a source of difficulty in studying the Keller-Segel system (1.1), such as stability of traveling waves, global well-posedness and so on. When $m = 1$, a Cole-Hopf type transformation was cleverly used to remove the singularity, which consequently led to a lot of interesting analytical works, for instance the stability of traveling waves (cf. [2, 3, 5, 11, 18, 20, 21]), global well-posedness and/or asymptotic behavior of solutions (see [4, 16, 17, 19, 23, 29, 32] in one dimensional bounded or unbounded space and [6, 7, 19, 26, 27, 31] in multidimensional spaces) and boundary layer solutions [8–10]. Even for the case $m = 1$, the model in multi-dimensional space still remains poorly understand and in particular no results on the large-data solutions have even been obtained. The paper will continue to consider the Keller-Segel system with $m = 1$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with the following boundary conditions

$$\begin{cases} (\nabla u - \chi u \nabla \ln w) \cdot \vec{n} = 0, & \text{on } \partial\Omega \\ \alpha \nabla w \cdot \vec{n} + \gamma w = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\vec{n}$ denotes the unit outward normal vector to $\partial\Omega$, $\gamma \in \mathbb{R}/\{0\}$ and $\alpha > 0$. The zero-flux boundary condition for u means that no cells can crosses the boundary of the habitat, and w is prescribed by Robin boundary condition which become Neumann boundary condition if $\gamma = 0$ and Dirichlet boundary condition if $\alpha = 0$. When $\gamma = 0$, namely the Neumann boundary condition prescribed for w, there are some well-posedness results available (cf. [27, 29, 32, 33]). However as $\gamma \neq 0$, as we know, no results have been developed for the problem (1.1)-(1.2). In general, Robin boundary condition is harder than the Neumann or Dirichlet boundary condition due to the loss of Maximum principle and the integrability. In this paper, we shall consider the stationary problem of (1.1)-(1.2) which reads

$$\begin{cases} 0 = \Delta u - \chi \nabla \cdot (u \nabla \ln w), & \text{in } \Omega \\ 0 = \varepsilon \Delta w - uw, & \text{in } \Omega \\ (\nabla u - \chi u \nabla \ln w) \cdot \vec{n} = 0, & \text{on } \partial\Omega \\ \nabla w \cdot \vec{n} + \gamma w = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where we have assumed $\alpha = 1$ without loss of generality. With the zero-flux boundary condition on u, we can solve from the first equation of (1.3) that

$$u = \lambda w^{\chi} \tag{1.4}$$

where $\lambda > 0$ is a constant of integration. Then substituting (1.4) into the second equation of (1.3), we reduce the stationary problem to a scalar semi-linear elliptic equation with Robin boundary condition:

$$\begin{cases} \varepsilon \Delta w - \lambda w^{1+\chi} = 0, & \text{in } \Omega \\ \nabla w \cdot \vec{n} + \gamma w = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The reduction from a system to a scalar equation in the above procedure is a key step to attack the stationary problem. Clearly $w = 0$ is naturally a solution of (1.5). What we are concerned with is whether the semi-linear Robin problem (1.5) admits non-constant solutions. This is a nontrivial problem in general due to homogeneous Robin boundary conditions for which the available methods are very limited. First, we show that the problem (1.5) has only trivial solution $w = 0$ if $\gamma > 0$ by the maximum principle directly. The conclusion for the case $\gamma < 0$ becomes elusive due to the loss of maximum principle. In this case, we shall consider the radially symmetric solution in a ball $\Omega = B_R(0)$ with radius $R > 0$. The key finding of this paper is that with radially symmetric case, a Cole-Hopf type transformation can be used to relegate the homogeneous Robin boundary condition to a non-homogeneous Dirichlet boundary condition and also reduce the second-order equation into a first-order one

for which shooting method becomes available for analysis. We shall demonstrate this idea in details later and state our main results of this paper as follows.

Theorem 1.1. *Let $\varepsilon > 0$ and $0 \neq \gamma \in \mathbb{R}$. Then the following results hold.*

(i) *If $\gamma > 0$, the system (1.3) has only a trivial solution $w = 0$.*

(ii) *If $\gamma > 0$ and $\Omega = B_R(0) = \{x \in \mathbb{R}^N |\ r = |x| < R\}$ with $R > 0$, then (1.3) admits an analytic radial solution $(u(r), w(r)$ with $u(r) = \lambda w(r)$, which is unique up to the constant $\lambda > 0$ through (1.4). Moreover, the solution $w(r)$ converges uniformly to 0 as $\varepsilon \to 0$ with the following convergence rate*

$$\|u(r)\|_{C[0,R]} \leq C\varepsilon, \qquad \|w(r)\|_{C[0,R]} \leq C\varepsilon^{\frac{1}{\chi}} \tag{1.6}$$

where $C > 0$ is a constant independent of ε.

Sketch of main ideas. Assume the solution of (1.5) is analytic and radially symmetric:

$$w(x) = w(|x|) = w(r),\ r \in (0, R). \tag{1.7}$$

Substituting the ansatz (1.7) into (1.5), we get the following boundary value problem

$$\begin{cases} \varepsilon w_{rr} + \varepsilon \frac{N-1}{r} w_r = \lambda w^{1+\chi}, & r \in (0, R) \\ w(0) = w_0,\ w_r(0) = 0, & \\ w_r + \gamma w = 0, & r = R \end{cases} \tag{1.8}$$

where we have imposed the boundary condition $w(0) = w_0$ which will be determined afterwards, and the condition $w_r(0) = 0$ follows from the analyticity of $w(r)$ at $r = 0$. In order to treat the Robin boundary condition, we introduce the following Cole-Hopf type transformation

$$v = \frac{w_r}{w} \tag{1.9}$$

and transform (1.8) into a boundary value problem of a first-order ODE:

$$\begin{cases} \varepsilon v_r + \varepsilon \frac{N-1}{r} v + \varepsilon v^2 = w^\chi, & r \in (0, R) \\ v(0) = 0, & \\ v = -\gamma, & r = R. \end{cases} \tag{1.10}$$

From (1.9), one can solve w in terms of v as

$$w(r) = w_0 e^{\int_0^r v(s)ds}$$

which turns (1.10) into a boundary value problem for a nonlocal ODE

$$\begin{cases} \varepsilon v_r + \varepsilon \frac{N-1}{r} v + \varepsilon v^2 = u_0 e^{\chi \int_0^r v(s)ds}, & r \in (0, R) \\ v(0) = 0, & \\ v = -\gamma, & r = R. \end{cases} \tag{1.11}$$

where

$$u_0 = \lambda w_0^\chi. \tag{1.12}$$

Hence by the Cole-Hopf transformation (1.9), we not only relegate the Robin problem to a Dirichlet problem but also reduce the order of the equation. However this is not gotten for free. The price that we paid is the generation of a nonlocal term with an exponential nonlinearity, which brings new obstacles to analysis. However we find that the classical shooting method (phase-plane analysis) may be applicable to the first-order ODE problem (1.11) though additional efforts are needed to handle the nonlocal term and exponential nonlinearity.

To employ the shooting method, we treat (1.11) as an initial value problem starting from $r = 0$. Since the solution $v(r)$ of (1.11) is analytic at a neighbourhood of $r = 0$, we insert its

Taylor expansion $v(r)=\sum_{k=0}^{\infty}a_k r^k$ into (1.11) and deduce that the coefficients $a_{2k}=0$ and that a_{2k+1} with $k\in\mathbb{N}$ are determined by $\frac{u_0}{\varepsilon N}$. In particular, $a_0=0$, $a_1=\frac{u_0}{\varepsilon N}$ and thus

$$v(r)=\frac{u_0}{\varepsilon N}r+O(r^2),\qquad v_r(r)=\frac{u_0}{\varepsilon N}+O(r), \tag{1.13}$$

for r sufficiently close to 0. Hence the initial value problem relevant to (1.11) reads:

$$\begin{cases} \varepsilon v_r+\varepsilon\frac{N-1}{r}v=-\varepsilon v^2+u_0e^{\chi\int_0^r v(s)\,ds}, \quad r\in(0,\infty),\\ v(0)=0,\quad v_r(0)=\frac{u_0}{\varepsilon N}. \end{cases} \tag{1.14}$$

Now for given u_0, we shall show that the solution of (1.14) will blow up at a finite $r=R_*$. Then we trace back to find the condition for u_0 such that the solution of (1.11) exists for given $R>0$. With the existence for $v(r)$, we get the solution of (1.8) and hence the radial solution of (1.5).

The rest of this paper is organized as follows. In section 2, we shall focus on the auxiliary problem (1.14) and prove the existence and uniqueness of (1.14). In section 3, we shall use the Cole-Hopf transformation (1.9) to prove Theorem 1.1.

2. Blowup solutions of (1.14)

In this section, we shall exploit the nonlocal problem (1.14) and prove the following results.

Theorem 2.1. *Suppose $u_0>0$ and $\varepsilon>0$. Let $\tilde v(r)$ be the solution of (1.14) with $\varepsilon=1$ and $u_0=1$. Then $\tilde v(r)$ blows up at a finite number $\tilde R>0$ and there exists $R_*=\tilde R\sqrt{\frac{\varepsilon}{u_0}}>0$ such that (1.14) admits an unique solution $v(r)$ in $(0,R_*)$ which is analytic at $r=0$ and can be explicitly expressed as*

$$v(r)=\sqrt{\frac{u_0}{\varepsilon}}\tilde v\left(\sqrt{\frac{u_0}{\varepsilon}}\,r\right),\ r\in(0,R_*). \tag{2.1}$$

Moreover $v_r(r)>0$ on $[0,R_*)$ and blows up at $R_*<\infty$.

Remark 2.1. *The upper and lower bounds of $\tilde R$ are given in Proposition 2.1.*

Before proceeding, we introduce the main difficulties encountered and ideas employed to overcome them in the proof of Theorem 2.1. Indeed, with $u_0>0$ the term $u_0e^{\chi\int_0^r v(s)\,ds}$ on the right-hand side of (1.14) will enhance the blow-up process as v increases, while $-\varepsilon v^2$ is a damping term preventing the blow-up. Hence which of them will dominate the dynamics as v is large is crucial to determine whether the blow-up radius R_* is finite or not. For our problem, we first heuristically employ a formal analysis (see Remark 2.2) to (1.14) to investigate the asymptotic behavior of the term $u_0e^{\chi\int_0^r v(s)\,ds}$ as $v\to\infty$ to see whether it grows fast enough to dominates or cancel out the damping effect of $-\varepsilon v^2$. Then by a delicate (formal) analysis, we find that $u_0e^{\chi\int_0^r v(s)\,ds}=\frac{\chi}{2}v^2+\varepsilon v^2+o(v^2)$ (for large v), which substituted into (1.14) indeed cancels out $-\varepsilon v^2$ and gives a $\frac{\chi}{2}v^2$-growth rate to v, i.e. $v_r\sim\frac{\chi}{2}v^2$. Then the blow-up radius $R_*<\infty$ follows immediately from this $\frac{\chi}{2}v^2$-growth of v. This procedure will be elaborated in Remark 2.2. Motivated by this formal analysis, we first justify our speculation for a special case $\varepsilon=1$ and $u_0=1$ in Proposition 2.1. Then with a scaling-invariant property of (1.14), we prove the similar results for the general case $\varepsilon>0$ and $u_0>0$. We start by presenting some preliminary results on (1.14).

2.1. Some preliminary results.

Lemma 2.1. *Suppose $u_0>0$ and $\varepsilon>0$. Then (1.14) admits a unique solution $v(r)$, which can be extended to a maximal interval $[0,R_*)$ such that either $R_*=\infty$ or $|v(r)|\to\infty$ as $r\to R_*$.*

The proof of Lemma 2.1 follows from the classical ordinary differential equation theory. We thus omit its proof.

Lemma 2.2. *Assume $u_0 > 0$ and $\varepsilon > 0$. If $v(r)$ is the unique solution of (1.14) in the maximal interval $[0, R_*)$, then $v_r(r) > 0$ for $r \in [0, R_*)$ and $v(r) > 0$ for $r \in (0, R_*)$.*

Proof. We first prove that $v_r(r) > 0$ for all $r \in [0, R_*)$. Indeed, it follows from (1.14) that $v_r(r) > 0$ for r small enough by continuity of v_r. We claim that

$$v_r(r) > 0 \qquad \text{for } r \in [0, R_*). \tag{2.2}$$

If this is false, we denote by r_1 the smallest value of $r > 0$ such that $v_r(r) = 0$. Then one derives

$$v_r(r) > 0, \quad \text{for } r \in (0, r_1); \qquad v_r(r_1) = 0, \quad v_{rr}(r_1) \le 0. \tag{2.3}$$

However, differentiating (1.14) with respect to r leads to

$$\begin{aligned}\varepsilon v_{rr}(r_1) =& -\varepsilon\frac{N-1}{r_1}v_r(r_1) + \varepsilon\frac{N-1}{r_1^2}v(r_1) - 2\varepsilon v(r_1)v_r(r_1) + \chi u_0 v(r_1)e^{\chi\int_0^{r_1} v(s)\,ds}\\ =&\varepsilon\frac{N-1}{r_1^2}v(r_1) + \chi u_0 v(r_1)e^{\chi\int_0^{r_1} v(s)\,ds} > 0.\end{aligned}$$

The above result contradicts with the last inequality in (2.3). Hence (2.2) holds true. $v(r) > 0$ for $r \in (0, R_*)$ follows directly from (2.2) and the initial condition $v(0) = 0$ in (1.14). The proof is finished.

□

Another property of (1.14) is its invariance under some appropriate scalings. Precisely, suppose that $v(r)$ is a solution of (1.14) with data $u_0 > 0$ and $\varepsilon > 0$. Then it is easy to verify for any $\beta > 0$ that $g(r) := \sqrt{\beta}v(\sqrt{\beta}r)$ is still a solution of (1.14) by replacing u_0 with βu_0. This property is crucial to prove Theorem 2.1. Indeed, we shall first study the solution $\tilde{v}(r)$ of (1.14) with fixed data $\varepsilon = 1$ and $u_0 = 1$ (see Proposition 2.1), of which the results on blow-up property and blow-up radius will be converted to the solutions $v(r)$ with general data $u_0 > 0$ and $\varepsilon > 0$, which equals to $\sqrt{\frac{u_0}{\varepsilon}}\tilde{v}(\sqrt{\frac{u_0}{\varepsilon}}r)$ thanks to the above scaling-invariant property. Details are given in the proof of Theorem 2.1.

2.2. **Blowup solutions of (1.14) with $\varepsilon = 1$ and $u_0 = 1$.** To prove Theorem 2.1, we first study the solution (denoted by $\tilde{v}(r)$) of (1.14) corresponding to data $\varepsilon = 1$ and $u_0 = 1$, which reads:

$$\begin{cases} \tilde{v}_r + \frac{N-1}{r}\tilde{v} = -\tilde{v}^2 + e^{\chi\int_0^r \tilde{v}(s)\,ds}, \quad r \in \mathbb{R}_+,\\ \tilde{v}(0) = 0, \quad \tilde{v}_r(0) = \frac{}{N}. \end{cases} \tag{2.4}$$

Before establishing the result on (2.4), we write out some variants of (2.4) for later use. Differentiating (2.4) with respect to r, one derives

$$\chi e^{\chi\int_0^r \tilde{v}(s)\,ds}\tilde{v} = \tilde{v}_{rr} + \frac{N-1}{r}\tilde{v}_r - \frac{N-1}{r^2}\tilde{v} + 2\tilde{v}\tilde{v}_r,$$

which, along with (2.4) leads to

$$\tilde{v}_{rr} + \frac{N-1}{r}\tilde{v}_r - \frac{N-1}{r^2}\tilde{v} = (\chi - 2)\tilde{v}\tilde{v}_r + \chi\tilde{v}^3 + \frac{\chi(N-1)}{r}\tilde{v}^2. \tag{2.5}$$

Denoting $\tilde{u}(r) = e^{\chi\int_0^r \tilde{v}(s)\,ds}$, then we get another variant of (2.4) as follows

$$\begin{cases} \tilde{v}_r + \frac{N-1}{r}\tilde{v} + \tilde{v}^2 = \tilde{u}, \quad r \in \mathbb{R}_+,\\ \tilde{u}_r = \chi\tilde{u}\tilde{v},\\ \tilde{u}(0) = 1, \quad \tilde{v}(0) = 0, \quad \tilde{v}_r(0) = \frac{}{N}. \end{cases} \tag{2.6}$$

For (2.4) we have the following result.

Proposition 2.1. *The unique solution $\tilde{v}(r)$ of (2.4) blows up at a finite $\tilde{R} < \infty$, that is $\lim_{r\to\tilde{R}} \tilde{v}(r) = \infty$. The solution $\tilde{v}(r)$ is strictly positive and $v_r(r) > 0$ for any $r \in (0, \tilde{R})$. Moreover, the blow-up radius $\tilde{R}$ satisfies*

$$\tilde{R} \geq \sqrt{\frac{2}{\chi}}. \tag{2.7}$$

If we further assume that $\chi \geq 2$, then $\tilde{R}$ also satisfies:

$$\tilde{R} \leq \pi\sqrt{\frac{2N}{\chi}}. \tag{2.8}$$

Remark 2.2. Before giving the proof for Proposition 2.1, we briefly discuss the main ideas motivated in the proof of Proposition 2.1. Let $[0, \tilde{R})$ with $\tilde{R} \leq \infty$ be the maximal interval of existence for $\tilde{v}(r)$. Then from Lemma 2.1, we have $\tilde{R} = \infty$ or $\lim_{r\to\tilde{R}} \tilde{v}(r) = \infty$, Hence to prove Proposition 2.1, we just need to rule out the case $\tilde{R} = \infty$ and prove the blowup radius $\tilde{R}$ is finite, which is the main difficulty encountered. Indeed, the two terms $-\tilde{v}^2$ and $e^{\chi\int_0^r \tilde{v}(s)\,ds}$ on the right-hand side of (2.4) have opposite effects on the blow-up process of $\tilde{v}$. Thus the asymptotic behavior of the term $e^{\chi\int_0^r \tilde{v}(s)\,ds}$ as $\tilde{v} \to \infty$ would be very helpful to determine whether the blow-up radius $\tilde{R}$ is finite or not. If $e^{\chi\int_0^r \tilde{v}(s)\,ds}$ dominates over $-\tilde{v}^2$ then $\tilde{R} < \infty$, otherwise $\tilde{R} = \infty$. Hence in the following we shall study the asymptotic behavior of $e^{\chi\int_0^r \tilde{v}(s)\,ds}$ by applying a formal analysis to the equations in (2.6) to gain some insights into the proof of Proposition 2.1 and shall formally derive

$$\tilde{v}_r(r) = \frac{\chi}{2}\tilde{v}^2(r) + o(\tilde{v}^2), \quad r \in (R_1, \tilde{R}) \tag{2.9}$$

for some large R_1. Once (2.9) is justified, the conclusion $\tilde{R} < \infty$ immediately follows thanks to this $\frac{\chi}{2}\tilde{v}^2$-growth rate of $\tilde{v}$ (see the proof of Proposition 2.1). Actually, instead of (2.9), we shall strictly prove in the proof of Proposition 2.1 the following sharper result

$$\tilde{v}_r(r) > \frac{\chi}{2}\tilde{v}^2(r), \quad r \in [0, \tilde{R}). \tag{2.10}$$

We next briefly introduce the formal analysis to derive the key estimate (2.9). Indeed, by Lemma 2.2 we know the solutions $\tilde{v}(r)$ and $\tilde{u}(r)$ of (2.6) are strictly increasing in $r > 0$. Hence we can define the inverse function of $\tilde{v}(r)$ as $r = f(\tilde{v})$. We further denote $g(\tilde{v}) := \tilde{u}(r) = \tilde{u}(f(\tilde{v}))$. Then from (2.6) one deduces that $(f, g)(\tilde{v})$ satisfies:

$$\begin{cases} f + (N-1)f_{\tilde{v}}\tilde{v} = -\tilde{v}^2 f f_{\tilde{v}} + g f f_{\tilde{v}}, \\ g_{\tilde{v}} = \chi f_{\tilde{v}} g\tilde{v}, \end{cases} \tag{2.11}$$

where $f_{\tilde{v}} := \frac{df}{d\tilde{v}}$, $g_{\tilde{v}} := \frac{dg}{d\tilde{v}}$. We assume that the blow-up radius $\tilde{R}$ of $\tilde{v}(r)$ is finite. Then $\lim_{r\to\tilde{R}} \tilde{v}(r) = \infty$ and $\lim_{\tilde{v}\to\infty} f(\tilde{v}) = \tilde{R} < \infty$. Hence $f(\tilde{v})$ has the following asymptotic expansion when $\tilde{v}$ is large:

$$f(\tilde{v}) = \tilde{R} + \frac{b_1}{\tilde{v}} + o\Big(\frac{1}{\tilde{v}}\Big), \qquad f_{\tilde{v}}(\tilde{v}) = \frac{(-b_1)}{\tilde{v}^2} + o\Big(\frac{1}{\tilde{v}^2}\Big), \tag{2.12}$$

where b_1 is a constant to be determined. Substituting (2.12) into the first equation of (2.11) gives

$$0 = \underbrace{g\tilde{R}\cdot\frac{(-b_1)}{\tilde{v}^2}}_{S_1} + \cdots + \underbrace{\tilde{R}b_1}_{S_2} + \cdots + \underbrace{(N-1)\cdot\frac{b_1}{\tilde{v}}}_{S_3} + \cdots - \underbrace{\tilde{R}}_{S_4} + \cdots, \tag{2.13}$$

where S_1, S_2, S_3 and S_4 are respectively the lowest order terms with respect to $\frac{1}{\tilde{v}}$ among the expansions corresponding to each part of the first equation in (2.11) and we have omitted the higher order terms converging to 0 faster than the terms S_1, S_2, S_3 and S_4 as $\tilde{v} \to \infty$. We

proceed to derive the value of constant b_1 by finding a valid balance among terms S_1, S_2, S_3 and S_4 to make (2.13) hold as $\frac{1}{\tilde{v}} \to 0$ with fixed $0 < \tilde{R} < \infty$. Noting that S_3 is of order $O(\frac{1}{\tilde{v}})$ and that S_2, S_4 are of order $O(1)$ with respect to $\frac{1}{\tilde{v}}$, we only need to find a balance among terms S_1, S_2 and S_4 since S_3 is a higher order term comparing with S_2 and S_4. Hence as $\frac{1}{\tilde{v}} \to 0$ there are the following two possible balancing to make (2.13) hold :

(i) $S_2 \sim S_4$ and S_1 is higher-order term. Then we get $b_1 = 1$, which substituted into (2.12) indicates that $f_{\tilde{v}} < 0$ when $\tilde{v}$ is large. On the other hand, from Lemma 2.2 and $f_{\tilde{v}} = \frac{1}{\tilde{v}_r}$ we deduce that $f_{\tilde{v}} > 0$ for all $\tilde{v} \in [0, \infty)$. Combining the above arguments, we arrive at a contradiction and thus this balancing is impossible.

(ii) $S_1 \sim S_2 \sim S_4$. In this case, one deduces $g \cdot \frac{(-b_1)}{\tilde{v}^2} + b_1 - 1 = 0$.

Hence only the balancing in (ii) is possible, which leads to

$$g(\tilde{v}) = \tilde{v}^2 \left(1 - \frac{1}{b_1}\right) + o(\tilde{v}^2). \tag{2.14}$$

Then inserting (2.14) and (2.12) into the second equation of (2.11) we deduce that

$$g_{\tilde{v}}(\tilde{v}) = \chi(1 - b_1)\tilde{v} + o(\tilde{v}), \tag{2.15}$$

which, along with (2.14) and the L'Höpital's rule leads to

$$1 - \frac{1}{b_1} = \lim_{\tilde{v}\to\infty} \frac{g(\tilde{v})}{\tilde{v}^2} = \lim_{\tilde{v}\to\infty} \frac{g_{\tilde{v}}(\tilde{v})}{2\tilde{v}} = \frac{\chi(1 - b_1)}{2}.$$

Then we solve from the above equality and get $b_1 = -\frac{2}{\chi}$ or $b_1 = 1$. Since $b_1 = 1$ contradicts the fact $f_{\tilde{v}} > 0$, we conclude that $b_1 = -\frac{2}{\chi}$. Hence

$$\tilde{u} = g(\tilde{v}) = \tilde{v}^2 \left(1 + \frac{\chi}{2}\right),$$

which substituted into the first equation of (2.6) entails that

$$\tilde{v}_r(r) = \frac{\chi}{2}\tilde{v}^2 + o(\tilde{v}^2), \qquad \text{for } \tilde{v} \text{ large enough.}$$

Hence we derive (2.9) from the above equality.

With the formal analysis of Remark 2.2 in hand, we next rigorously justify (2.10) and thus prove Proposition 2.1.

Proof of Proposition 2.1. The proof is divided into three steps.

Step 1 (blowup). Let $[0, \tilde{R})$ with $\tilde{R} \le \infty$ be the maximal interval of existence for $\tilde{v}(r)$. From Lemma 2.2 we know that $\tilde{v}(r)$ is monotonically increasing in r. Hence if we let

$$l = \lim_{r\to\tilde{R}} \tilde{v}(r),$$

it follows from the fact $\tilde{v}(r) > 0$ (see Lemma 2.2) that $l \ge 0$. Now we prove that

$$l = \infty. \tag{2.16}$$

Indeed if (2.16) is false and then $< \infty$, it follows from Lemma 2.1 that $\tilde{R} = \infty$. Thus

$$l = \lim_{r\to\infty} \tilde{v}(r) < \infty. \tag{2.17}$$

By (2.17), we claim that one can choose a number sequence $\{r_k\}_{k\in\mathbb{N}}$ such that

$$r_{k+1} \ge r_k + 1, \qquad \tilde{v}_r(r_k) < \frac{1}{k}. \tag{2.18}$$

Indeed if (2.18) is false, then there exists some $k_0 \in \mathbb{N}$ such that

$$\tilde{v}_r(r) \ge \frac{1}{k_0} \quad \text{for all } r \ge r_{k_0}.$$

This indicates that

$$\lim_{r\to\infty}\tilde{v}(r)\geq\lim_{r\to\infty}\left[\tilde{v}(r_{k_0})+\frac{1}{k_0}\cdot(r-r_{k_0})\right]=\infty,$$

which contradicts (2.17). Hence (2.18) holds true under (2.17). Then from (2.18) we deduce that

$$\lim_{k\to\infty}r_k=\infty,\qquad\lim_{k\to\infty}\tilde{v}_r(r_k)=0,$$

which, along with (2.4) and (2.17), raises the following contradiction:

$$0=\lim_{k\to\infty}\left(\tilde{v}_r+\frac{N-1}{r}\tilde{v}\right)(r_k)=\lim_{k\to\infty}(-\tilde{v}^2+e^{\chi\int_0^{r_k}\tilde{v}(s)\,ds})(r_k)=\infty.$$

Hence our assumption (2.17) is false and we have proved (2.16) which gives

$$\lim_{r\to\tilde{R}}\tilde{v}(r)=\infty. \tag{2.19}$$

Step 2 (finiteness of blowup radius $\tilde{R}$). Let $\tilde{R}$ be the blow-up radius of $\tilde{v}(r)$ as in (2.19). Define the function

$$F(r)=\tilde{v}_r(r)-\frac{\chi}{2}\tilde{v}^2(r),\quad r\in[0,\tilde{R}).$$

Then it follows from the data $\tilde{v}_r(0)=\frac{}{N}$, $\tilde{v}(0)=0$ and the continuity of $F(r)$ that $F(r)>0$ for r close to 0 enough. We claim that

$$F(r)>0,\qquad\text{for } r\in[0,\tilde{R}), \tag{2.20}$$

which will be proved by the argument of contradiction. Indeed, assume that (2.20) is false and denote r_1 the smallest value of r satisfying $F(r)=0$. Then we have

$$F(r)>0,\quad\text{for } r\in(0,r_1);\qquad F(r_1)=[\tilde{v}_r-\frac{\chi}{2}\tilde{v}^2](r_1)=0,\quad F_r(r_1)\leq 0. \tag{2.21}$$

Thus it follows from (2.5) and (2.21) that

$$\begin{aligned}F_r(r_1)=&\tilde{v}_{rr}(r_1)-\chi\tilde{v}\tilde{v}_r(r_1)\\=&-\frac{N-1}{r_1}\tilde{v}_r+\frac{N-1}{r_1^2}\tilde{v}+(\chi-2)\tilde{v}\tilde{v}_r+\chi\tilde{v}^3+\frac{\chi(N-1)}{r_1}\tilde{v}^2\\=&-\frac{\chi(N-1)}{r_1}\frac{\tilde{v}^2}{2}+\frac{N-1}{r_1^2}\tilde{v}+(\chi-2)\tilde{v}\frac{\chi\tilde{v}^2}{2}+\chi\tilde{v}^3+\frac{\chi(N-1)}{r_1}\tilde{v}^2\\=&\frac{\chi(N-1)}{2r_1}\tilde{v}^2+\frac{N-1}{r_1^2}\tilde{v}+\frac{\chi^2}{2}\tilde{v}^3\\>&0,\end{aligned} \tag{2.22}$$

which contradicts the last inequality of (2.21). Hence (2.20) holds true. That is

$$\tilde{v}_r(r)>\frac{\chi}{2}\tilde{v}^2(r),\quad r\in[0,\tilde{R}). \tag{2.23}$$

Let $R_2\in(0,\tilde{R})$. Then solving (2.23) immediately yields that

$$\tilde{v}(r)>\frac{1}{\frac{1}{\tilde{v}(R_2)}-\frac{\chi(r-R_2)}{2}},\qquad r>R_2.$$

Noting that the function on the right-hand side of the above inequality blows up at $\frac{2}{\chi\tilde{v}(R_2)}+R_2$, we conclude that

$$\tilde{R}\leq\frac{2}{\chi\tilde{v}(R_2)}+R_2<\infty.$$

Step 3 (bounds of blowup radius). We first prove (2.7). Define $h(r) = \int_0^r \tilde{v}(s)\,ds$. Then it follows from (2.4) that

$$h_{rr} < e^{\chi h},$$

which multiplied with h_r and then upon integration over $(0, r)$ gives rise to

$$\frac{h_r^2(r)}{2} - -\frac{}{\chi} e^{\chi h(r)} < -\frac{}{\chi} < 0, \tag{2.24}$$

where $h_r(0) = \tilde{v}(0) = 0$ has been used. Noting that $h_r(r) = \tilde{v}(r) > 0$, from (2.24) we deduce that $h_r(r) < \sqrt{\frac{2}{\chi}}\, e^{\frac{\chi}{2}h}$ and thus

$$h(r) < \frac{2}{\chi} \ln\Big(1 - \sqrt{\frac{\chi}{2}}\, r\Big)^{-1}.$$

From this one concludes that $\tilde{R} \geq \sqrt{\frac{2}{\chi}}$ and derives (2.7). We proceed to prove (2.8). Since $\chi \geq 2$, it follows from (2.5) that

$$\Big(\tilde{v}_r + \frac{N-1}{r}\tilde{v}\Big)_r = \tilde{v}_{rr} + \frac{N-1}{r}\tilde{v}_r - \frac{N-1}{r^2}\tilde{v} > 0, \tag{2.25}$$

which upon integration over $(0, r)$, along with (1.13) leads to

$$\frac{(r^{N-1}\tilde{v})_r}{r^{N-1}} = \tilde{v}_r + \frac{N-1}{r}\tilde{v} > .$$

Then integrating the above inequality over $(0, r)$, we have $\tilde{v}(r) > \frac{}{N} r$. This along with (2.25) yields

$$\frac{(r^{N-1}\tilde{v}_r)_r}{r^{N-1}} = \tilde{v}_{rr} + \frac{N-1}{r}\tilde{v}_r > \frac{N-1}{r^2}\tilde{v} > \frac{(N-1)}{Nr}.$$

Integrating this inequality over $(0, r)$ one has

$$\tilde{v}_r(r) > \frac{}{N}. \tag{2.26}$$

Combining (2.26) and (2.23), we have that

$$\tilde{v}_r(r) > \frac{\chi}{4}\tilde{v}^2(r) + \frac{}{2N},$$

which gives rise to

$$\tilde{v}(r) > \sqrt{\frac{2}{N\chi}} \tan\Big(\sqrt{\frac{\chi}{8N}}\, r\Big), \qquad r > 0. \tag{2.27}$$

where the function on the right-hand side of (2.27) is strictly increasing in r and blows up at $\pi\sqrt{\frac{2N}{\chi}}$. Noting that $\tilde{v}(r)$ is also monotonically increasing in r (see Lemma 2.2), thus from (2.27) we conclude that $\tilde{v}(r)$ blows up at some finite $\tilde{R}$ and the blow-up radius $\tilde{R}$ satisfies

$$\tilde{R} \leq \pi\sqrt{\frac{2N}{\chi}}.$$

The proof of Proposition 2.1 is completed.

2.3. **Proof of Theorem 2.1.** We are now in a position to prove Theorem 2.1 by the results of Proposition 2.1. Let $\tilde{v}(r)$ be the solution of (1.14) with $\varepsilon = u_0 = 1$. Then from Proposition 2.1, we know that $\tilde{v}(r)$ blows up at $\tilde{R} < \infty$. Define $g(r) = \sqrt{\frac{u_0}{\varepsilon}}\,\tilde{v}(\sqrt{\frac{u_0}{\varepsilon}}r)$. Then it follows from (2.4) that

$$\begin{cases} \varepsilon g_r(r) + \varepsilon\dfrac{N-1}{r}g(r) = -\varepsilon g^2(r) + u_0 e^{\chi\int_0^r g(s)\,ds}, \quad r \in \Big(0,\, \tilde{R}\sqrt{\dfrac{\varepsilon}{u_0}}\Big), \\ g(0) = 0, \quad g_r(0) = \dfrac{u_0}{\varepsilon N}. \end{cases} \tag{2.28}$$

It is easy to check that systems (1.14) and (2.28) are the same, and hence by the uniqueness of solutions it follows that $v(r) = g(r) = \sqrt{\frac{u_0}{\varepsilon}}\,\tilde{v}(\sqrt{\frac{u_0}{\varepsilon}}r)$ and the blow-up radius of $v(r)$ is $R_* = \tilde{R}\sqrt{\frac{\varepsilon}{u_0}}$. The proof is completed.

□

3. Proof of Theorem 1.1

We first prove the following results.

Proposition 3.1. *Assume $R > 0$ and $\varepsilon > 0$. Let $\tilde{v}(r)$ be the solution of (1.14) with $\varepsilon = u_0 = 1$. Then for any $\gamma < 0$, there exists a u_0 uniquely determined by γ and ε through the identity*

$$\sqrt{\frac{u_0}{\varepsilon}}\tilde{v}\left(\sqrt{\frac{u_0}{\varepsilon}}\,R\right) = -\gamma \tag{3.1}$$

such that the problem (1.11) admits a unique solution $v(r)$ given in (2.1).

Proof. From Theorem 2.1 we know that the solutions $v(r)$ of (1.14) is strictly positive when $r > 0$ and in particular $v(R) > 0$. Hence for $\gamma > 0$, (1.11) does not admit a solution. We next consider the case of $\gamma < 0$. With the fixed $R > 0$ in (1.11), we define

$$f(z) = z\tilde{v}(zR) \qquad \text{for } z > 0,$$

where $\tilde{v}(r)$ is defined in Theorem 2.1. Then from Proposition 2.1 we deduce that $f(z)$ is monotonically increasing in z, that is

$$f_z(z) = \tilde{v}(zR) + R\tilde{v}_r(zR) > 0. \tag{3.2}$$

By Proposition 2.1 and the continuity of $\tilde{v}(r)$, we further get

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}[z\tilde{v}(zR)] = 0, \qquad \lim_{z\to\infty} f(z) = \lim_{z\to\infty}[z\tilde{v}(zR)] = \infty,$$

which, along with (3.2) and the continuity of $\tilde{v}(r)$ implies that there exists a unique z_γ depending on $\gamma < 0$, such that

$$f(z_\gamma) = z_\gamma\tilde{v}(z_\gamma R) = -\gamma. \tag{3.3}$$

For fixed $\gamma < 0$ and $\varepsilon > 0$, we take u_0 such that

$$\sqrt{\frac{u_0}{\varepsilon}} = z_\gamma, \tag{3.4}$$

Then (3.1) follows from (3.3) and (3.4). By Theorem 2.1, (3.3) and (3.4) we further deduce that the solution $v(r) = \sqrt{\frac{u_0}{\varepsilon}}\,\tilde{v}\left(\sqrt{\frac{u_0}{\varepsilon}}r\right)$ solved from (1.14) with u_0 defined in (3.4) is the unique solution of (1.11). The proof is completed.

□

Proof of Theorem 1.1. If $\gamma > 0$, by the maximum principle, it can be easily verified that the problem (1.5) only admits the trivial solution $u = w = 0$. Next we consider the case

$\gamma < 0$. By Proposition 3.1 and transformation (1.9), it follows that the boundary problem (1.3) admits a unique radial solution $(u, w)(r)$ explicitly expressed as

$$\begin{aligned} w(r) &= w_0 e^{\int_0^r \sqrt{\frac{u_0}{\varepsilon}}\tilde{v}\left(\sqrt{\frac{u_0}{\varepsilon}}s\right)ds} = \left(\frac{u_0}{\lambda}\right)^{\frac{1}{\chi}} e^{\int_0^{\sqrt{\frac{u_0}{\varepsilon}}r}\tilde{v}(\tau)d\tau}, \\ u(r) &= \lambda w^{\chi}(r) = u_0 e^{\chi\int_0^{\sqrt{\frac{u_0}{\varepsilon}}r}\tilde{v}(\tau)d\tau}, \end{aligned} \tag{3.5}$$

where (1.12) and the change of variable $\tau = \sqrt{\frac{u_0}{\varepsilon}}s$ have been used and u_0 is the value of $u(r)$ at $r = 0$. We proceed to prove (1.6). In fact, it follows from (3.4) and (3.5) that

$$\begin{aligned} w(r) &= \varepsilon^{\frac{1}{\chi}}\lambda^{-\frac{1}{\chi}} z_\gamma^{\frac{2}{\chi}} e^{\int_0^{z_\gamma r}\tilde{v}(\tau)d\tau}, \\ u(r) &= \varepsilon z_\gamma^2 e^{\chi\int_0^{z_\gamma r}\tilde{v}(\tau)d\tau}. \end{aligned} \tag{3.6}$$

Note that the function $\tilde{v}(r)$ (defined in Theorem 2.1) in (3.6) is continuous in r and independent of ε. One can find a constant $C > 0$ independent of ε, such that

$$\|w(r)\|_{C[0,R]} \le C\varepsilon^{\frac{1}{\chi}}, \qquad \|u(r)\|_{C[0,R]} \le C\varepsilon,$$

where the constant C depends on χ, λ, γ and R. Hence

$$\lim_{\varepsilon\to 0}\|w(r)\|_{C[0,R]} = \lim_{\varepsilon\to 0}\|u(r)\|_{C[0,R]} = 0.$$

This completes the proof of Theorem 1.1.

Acknowledgements

Q. Hou is supported by China Postdoctoral Science Foundation (No. 2019M651269), the National Natural Science Foundation of China No.11901139). Z. Wang is supported by the Hong Kong RGC GRF grant No. PolyU 153031/17P (Project ID P0005368) and an internal grant P0001905.

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