STABILITY IN A HEBBIAN NETWORK OF KURAMOTO OSCILLATORS WITH SECOND-ORDER COUPLINGS FOR BINARY PATTERN RETRIEVE

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Abstract. We study the stability of an oscillatory associative memory network consisting of N coupled Kuramoto oscillators with applications in binary pattern retrieve. In this model, the coupling function consists of a Hebbian term and a second-order Fourier term with nonnegative strength ε. In [Physica D 197 (2004) 134-148] Nishikawa et al. studied the stability using the approach of linearization; the criteria for stability/instability is given by the spectrum of linearization which is a matrix of order N. In recent literature [SIAM J. Appl. Dyn. Syst. 14 (2015) 188-201], Hölzel et al. considered the model with $\varepsilon = 0$ and introduced the orthogonality of binary patterns so that the eigenvalues of linearization can be calculated. In this paper, we will present conditions for stability/instability based on the gradient formulation. First, we use the potential estimate to derive a criteria for stability/instability by the spectrum of a matrix of order $N - 1$. This potential estimate also gives convergence rate under some conditions. Second, we focus on the special case with mutually orthogonal memorized patterns. We find a sufficient and necessary condition for a binary pattern to be stable for any $\varepsilon > 0$. For any other binary pattern we prove that there exists a critical value of ε below which it is unstable. A lower bound for this critical strength is provided. A significant advantage of the results in this case is that the conditions for stability/instability is easy to verify and the lower bound of ε is easy to compute. Thirdly, when the memorized patterns are not mutually orthogonal, we suggest a framework to transform it into the case of orthogonal memorized patterns. Simulations are presented to illustrate our results.

1. Introduction

General background.- The famous Hopfield model of associative memory [16] provides basic ideas for the origin of neural computing and has attracted a lot of interest. The physical significance of Hopfield's work lies in his proposal of the energy function and his idea that memories are dynamically stable attractors, naturally bringing concepts and tools from statistical and nonlinear physics into neuroscience and information sciences as well as engineering. In this model, neurons in the network are assumed to be discrete values (e.g. $+1$ and -1) and a set of patterns is stored such that when a new pattern is presented, the network responds by producing a stored pattern that most closely resembles the new pattern. This is the basic mechanism for the binary pattern recognition using an associative memory network. Such models typically consist of coupled oscillators interacting with each other according to a Hebbian rule, and the patterns are stored as phase-locked states. The

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network with a coupling matrix determined by Hebbian rule was studied in some literature, see, for example, [2, 3, 13, 14, 15, 29]. One advantage of the type of this model is that it can be naturally implemented using oscillatory devices including phase-locked loop circuits [17], laser oscillators [18], and MEMS resonators [19]. There are also other mechanisms for pattern recognition problem, for example, the face recognition can be formulated as sparse representation or sparse signal reconstruction using optimization algorithms [27, 28].

The equation of motion for a network of coupled oscillators can be reduced to a phase model under fairly moderate conditions. Assuming that interactions are weak and that the oscillators have stable limit cycles with nearly identical periods, Kuramoto [20] has shown that the equations of motion for a network of N oscillators can be reduced to equations for the phase variables $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N)^T$:

$$\dot{\varphi}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} \Gamma_{ij}(\varphi_j - \varphi_i), \qquad i = 1, 2, \dots, N,$$

where φ_i is the phase angle of an individual oscillator, ω_i represents the intrinsic natural frequency of the i-th oscillator drawn from some given distribution function $g = g(\omega)$, N is the size of the network and $\Gamma_{ij}(\varphi)$ is a 2π-periodic function determining the coupling between oscillators i and j. Many of the previous studies of weakly coupled oscillators with nearly identical frequencies have focused on the sinusoidal coupling functions, i.e., $\Gamma_{ij}(\varphi) = \sin\varphi$, which gives the classic Kuramoto model [6, 8, 10, 20].

The model and pattern retrieve.- In this paper, we consider a network that can be used in binary pattern retrieve. More precisely, for a given initial pattern, we want to recognize a binary pattern ξ ($\xi_i = \pm 1, i = 1, \dots, N$) out of memorized patterns $\{\xi^1, \xi^2, \dots, \xi^M\}$. Typically, the retrieved pattern should be closest to the initial one among the memorized patterns. For this aim, we use the coupling function $\Gamma_{ij}(\varphi)$ with the Hebbian rule and second-order Fourier term, namely,

$$\Gamma_{ij}(\varphi) = C_{ij}\sin\varphi + \varepsilon \sin 2\varphi,$$

where ε is a nonnegative constant and C_{ij} is set to $C_{ij} = \sum_{k=1}^{M} \xi_i^k \xi_j^k$ which encodes the memorized patterns and is an application of the Hebbian rule. We will consider coupled oscillators with identical frequencies

$$\dot{\varphi}_i = \frac{1}{N}\sum_{j=1}^{N} C_{ij}\sin(\varphi_j - \varphi_i) + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin 2(\varphi_j - \varphi_i), \qquad i = 1, 2, \dots, N. \tag{1.1}$$

Here, $\varepsilon > 0$ is the strength of second-order Fourier term, which can be regarded as an adjustable parameter that influences the stability of equilibriums and leads to rich dynamical properties.

Let $\eta = (\eta_1, \eta_2, \dots, \eta_N)^T$ be an N-dimensional vector of 1's and -1's representing a binary pattern. There is a unique (up to constant translation) phase-locked solution corresponding to the pattern η, which is characterized by

$$\left|\varphi_i^* - \varphi_j^*\right| = \begin{cases} 0, & \eta_i = \eta_j; \\ \pi, & \eta_i \neq \eta_j. \end{cases} \tag{1.2}$$

We denote this phase-locked state (up to constant translation) by $\varphi^*(\eta)$. Thanks to the global phase shift invariance, for convenience in the following context we will say a binary pattern η is stable (unstable) if the corresponding phase-locked solution $\varphi^*(\eta)$ is stable (unstable). We notice that for binary patterns, η and $-\eta$ can be regarded as the same pattern. In fact, if the memorized pattern ξ^k is replaced by $-\xi^k$, the coupling term C_{ij} does not change.

The network can be used to identify a binary pattern $\eta\,(\eta_i = \pm 1, i = 1, 2, \ldots, N)$ as one of a given set of M memorized binary patterns $\xi^k\left(\xi_i^k = \pm 1, i = 1, 2, \ldots, N, k = 1, 2, \ldots, M\right)$. The pattern η is regarded as the defective pattern. In [14, 24], the authors introduced the *overlap* as follows:

$$m(\eta) = \left|\frac{1}{N}\sum_{i=1}^{N} \eta_i e^{\sqrt{-1}\varphi_i}\right|.$$

The overlap $m(\xi^k)$ measures the closeness of the solution to the memorized pattern ξ^k and it is a convenient way to check whether a given pattern is currently represented by the state of the network. Due to the global phase shift invariance in system (1.1), $m(\eta)$ is invariant under global rotations. In [14], two-step pattern recognition is as follows: First, the coupling matrix is chosen as $C_{ij} = \eta_i\eta_j$ for a defective pattern η. As a result, the phases φ_i will evolve towards a distribution reflecting this pattern, i.e., the overlap $m(\eta)$ will approach 1. Second, after this initialization of the network, the coupling coefficients are set to $C_{ij} = \sum_{k=1}^{M} \xi_i^k \xi_j^k$, which is an application of the Hebbian rule.If recognition is successful, the network evolves towards a memorized pattern which is closest to its initial state. For example, if the initial state is a slightly defective copy of ξ^1, the desired final state of the network would be ξ^1 which is encoded by the overlap $m(\xi^1) = 1$.

Other than the above two-step process, the model (1.1) can be also used to recognize a binary pattern from a non-binary "pattern" (which is typically a gray scale image, as a defective copy of a binary pattern), in the following way: we transform the non-binary "pattern" into an initial phase vector typically in $[0, \pi]^N$ which reflect the non-binary pattern, and then the model (1.1) will evolve towards a binary pattern which is close to the initial state. This is reasonable especially when we are recognizing some standard patterns such as Arabic numbers and/or letters. This idea will be illustrated in Subsection 5.3 with a simulation. In this way, we do not need the initialization step and the process looks simpler. However, it is necessary to do an initialization step in the two-step process, since the defective pattern here is a binary pattern which is an equilibrium and will stay for ever.

As far as the authors know, there are few analytical results on the associative memory network of Kuramoto oscillators. Recently, Hölzel et al. [14] considered a Hebbian network of Kuramoto oscillators described by system (1.1) with $\varepsilon = 0$. The Hebbian term reflects the set of memorized patterns such that these patterns stand out among other binary patterns. When memorized patterns are mutually orthogonal, they showed that these patterns have some stability in some sense by subtly finding out the eigenspecturm of linearization, see [14]. Precisely, the memorized patterns $\{\xi^1, \xi^2, \ldots, \xi^M\}$ are non-isolated equilibriums and they are part of a single, connected set of degenerate stationary states which comprises all straight lines connecting any pair of memorized patterns. Despite of this, as indicated in [25], the memorized patterns of such oscillatory networks are typically *unstable.*

A way to avoid this undesirable property and enhance the stability of memorized patterns is to add the second-order Fourier term in (1.1). However, this term will enhance the stability of all binary patterns, not only the memorized ones. Actually, if ε is sufficiently large, then all binary patterns become stable (see Remark 3.1). We believe that in binary pattern retrieve, one expects to recognize a memorized pattern (or related ones) and most of others should be unstable. Therefore, to make the memorized patterns stand out among others, we need to seek a balance between the Hebbian term and second-order Fourier term. Fortunately, this can be realized by controlling the strength ε of second-order Fourier term, typically it should be positive but not too large. We acknowledge that the system (1.1) was invented in earlier literature and some interesting work were performed mainly by numerical simulations, for example, [9, 24, 25]. Rigorous study can be found in [24], where Nishikawa et al. performed a linearization stability analysis and criteria for stability of any given binary pattern is given by assuming the spectrum of the linearization are negative.

Contributions.- In this paper, we will perform rigorous analysis for (1.1) and the main results are three-fold. First, we use the energy method to obtain a sufficient condition leading to the stability of binary patterns. This is based on the theory of Łojasiewicz inequality for analytic potential, by which we also give a condition under which the convergence is exponentially fast. Second, we pay special attention to the special case that the memorized patterns are mutually orthogonal. We show that the memorized orthogonal patterns are *ε-independently stable* (stable for any $\varepsilon > 0$, see Definition 4.1). A necessary and sufficient condition for the ε-independent stability of binary patterns is provided. Surprisingly we find that there may exist other ε-independently stable patterns except the memorized ones. For a pattern η which is not ε-independently stable, we prove that there is a critical strength ε^*_η such that η is unstable for $\varepsilon < \varepsilon^*_\eta$. A lower bound for the critical strength is given as well. A notable feature in the study for orthogonal memorized patterns is that the conditions for stability/instability is easy to verify compared to that in [24]. We also consider the stability of equilibrium which is the middle state of memorized patterns, which further explains the advantage to include the second-order Fourier term with ε. Finally, we give a new idea so that nonorthogonal memorized patterns can be transformed to orthogonal memorized patterns and the cost is the size of network becomes larger. This new idea is illustrated in Subsection 5.4 with a simulation.

Organization of paper.- In Section 2, we give some preliminaries for the gradient system approach and matrix theory. In Section 3, we present sufficient conditions for stability of binary patterns and study the convergence rate. In Section 4, we consider the case that the memorized patterns are mutually orthogonal and discuss how the nonorthogonal binary patterns can be transformed to orthogonal binary patterns. In Section 5 we provide numerical examples, and Section 6 is devoted to be a brief summary.

2. Preliminaries

In this section, we first review the coupled Kuramoto oscillators with associative memory patterns and give the gradient system with analytic potential; then we study some crucial propositions and lemmas, which will be used in the paper.

We consider the following dynamical equations for the binary pattern recognition:

$$\dot{\varphi}_i = \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\sin(\varphi_j-\varphi_i) + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin 2(\varphi_j-\varphi_i), \qquad i = 1,2,\dots,N. \tag{2.1}$$

Let $\varphi = (\varphi_1,\dots,\varphi_N)$ and

$$f(\varphi) = -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\cos(\varphi_j-\varphi_i) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j-\varphi_i), \tag{2.2}$$

then f is the potential of (2.1) and (2.1) can be written as

$$\dot{\varphi} = -\nabla f(\varphi). \tag{2.3}$$

Therefore, the nice theory of gradient system is available for this model. Next we introduce some related works which will be helpful in this paper. The gradient inequality was first developed by Łojasiewicz [23].

Lemma 2.1. [5, 23] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a real analytic function.*

(1) For any $x_* \in \mathbb{R}^N$, there exist a neighborhood $\mathcal{N}(x_*)$ of x_* and some constants $c = c(x_*) > 0$ and $r = r(x_*) \in (0, \frac{1}{2}]$ such that

$$|f(x) - f(x_*)|^{1-r} \le c\|\nabla f(x)\|, \ \forall\ x \in \mathcal{N}(x_*). \tag{2.4}$$

(2) Let $x(\cdot)$ be a solution of (2.3). If $\{x(t)\}_{t\in\mathbb{R}^+}$ is bounded, then there exists an equilibrium x_∞ such that $x(t) \to x_\infty$. Furthermore, if $r = r(x_\infty) = \frac{1}{2}$, then $\|x(t) - x_\infty\| \le Ce^{-\lambda(t-T)}$ for some $C, T, \lambda > 0$. If $r = r(x_\infty) < \frac{1}{2}$, then $\|x(t) - x_\infty\| \le Ct^{-\frac{r}{1-2r}}$ for some $C > 0$.

The inequality (2.4) is referred as the celebrated Łojasiewicz's inequality and the constant $r \in \left(0, \frac{1}{2}\right]$ is called the Łojasiewicz exponent of f at x_*. This inequality reveals a fundamental relation between the potential and its gradient near the equilibrium, and provides a powerful tool to derive the convergence of a trajectory towards a single equilibrium. This approach was applied to the synchronization analysis of Kuramoto model in some recent literature such as [7, 11, 12, 22]. Based on Łojasiewicz inequality, Absil et al. [4] gave a sufficient and necessary condition for the stability of equilibriums of a gradient system.

Lemma 2.2. [4] *Let f be real analytic in a neighbourhood of $\varphi^* \in \mathbb{R}^n$. Then, φ^* is a stable equilibrium of (2.3) if and only if φ^* is a local minimum of f. Furthermore, it is asymptotically stable if and only if it is a strict local minimum.*

The following lemma immediately implies that any solution of system (2.3) converges to a certain equilibrium point.

Lemma 2.3. [21] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be real analytic and satisfy $f(x + 2\pi K) = f(x)$ for any $K \in \mathbb{Z}^N$. Then for any solution $x(\cdot)$ of (2.3), there exists $x^* \in \Gamma := \{x^* | \ \nabla f(x^*) = 0\}$ such that $x(t) \to x^*$.*

The following lemma for eigenvalues of a matrix will be also used.

Lemma 2.4. [26] *Let $A, B \in M_n$ be Hermitian and let the respective eigenvalues $\{\lambda_i(A + B)\}_{i=1}^n$,$\{\lambda_i(A)\}_{i=1}^n$ and $\{\lambda_i(B)\}_{i=1}^n$ be arranged in increasing order. Then*

$$\lambda_{i-j+1}(A) + \lambda_j(B) \le \lambda_i(A + B), \quad j = 1, 2, \dots, i,$$

for each $i = 1, 2, \dots, n$.

3. The stability of equilibrium: General case

In this section, we first present a framework for stability/instability of equilibrium corresponding to a binary pattern. We also study the Łojasiewicz exponent of the potential at these equilibriums which implies exponential convergence.

The tool for deriving stability/instability is to use Lemma 2.2. So we first consider the potential difference.

Lemma 3.1. *Let η be a binary pattern, and let $\varphi^*(\eta)$ be the phase-locked solution of* (2.1) *satisfying* (1.2)*, then there exists a neighborhood $\mathcal{N}(\varphi^*)$ such that for any $\varphi \in \mathcal{N}(\varphi^*)$,*

$$f(\varphi) - f(\varphi^*) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 2\varepsilon\right)\gamma_{ji}^2 - \frac{1}{3N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 8\varepsilon\right)\gamma_{ji}^4 + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + \frac{\varepsilon}{2}\right)o\left(\gamma_{ji}^5\right)$$

where $\gamma_{ji} = \frac{\varphi_j - \varphi_i - \varphi_j^* + \varphi_i^*}{2}$.

Proof. For any $\varphi \in \mathcal{N}(\varphi^*)$, we have

$$\varphi_j - \varphi_i = \varphi_j^* - \varphi_i^* + 2\gamma_{ji}.$$

It follows from (1.2) that

$$\sin(\varphi_j^* - \varphi_i^* + \gamma_{ji}) = \eta_i\eta_j \sin\gamma_{ji}, \quad \sin(2(\varphi_j^* - \varphi_i^*) + 2\gamma_{ji}) = \sin 2\gamma_{ji}.$$

Then

$$\begin{aligned}
&f(\varphi) - f(\varphi^*) \\
&= \left[-\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\cos(\varphi_j - \varphi_i) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i)\right] \\
&\quad - \left[-\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\cos(\varphi_j^* - \varphi_i^*) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j^* - \varphi_i^*)\right] \\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\left[\cos(\varphi_j^* - \varphi_i^*) - \cos(\varphi_j - \varphi_i)\right] \\
&\quad + \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\cos 2(\varphi_j^* - \varphi_i^*) - \cos 2(\varphi_j - \varphi_i)\right] \\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\left(-2\sin\frac{\varphi_j^* - \varphi_i^* + \varphi_j - \varphi_i}{2}\sin\frac{\varphi_j^* - \varphi_i^* - \varphi_j + \varphi_i}{2}\right) \\
&\quad + \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(-2\sin(\varphi_j^* - \varphi_i^* + \varphi_j - \varphi_i)\sin(\varphi_j^* - \varphi_i^* - \varphi_j + \varphi_i)\right) \\
&= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\sin\frac{2(\varphi_j^* - \varphi_i^*) + 2\gamma_{ji}}{2}\sin\frac{-2\gamma_{ji}}{2} \\
&\quad - \frac{\varepsilon}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(2\left(\varphi_j^* - \varphi_i^*\right) + 2\gamma_{ji}\right)\sin\left(-2\gamma_{ji}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\sin(\varphi_j^* - \varphi_i^* + \gamma_{ji})\sin\gamma_{ji} + \frac{\varepsilon}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sin^2 2\gamma_{ji}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\eta_i\eta_j \sin^2\gamma_{ji} + \frac{\varepsilon}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sin^2 2\gamma_{ji} \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\eta_i\eta_j \left(\gamma_{ji} - \frac{\gamma_{ji}^3}{3!} + o\left(\gamma_{ji}^4\right)\right)^2 + \frac{\varepsilon}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(2\gamma_{ji} - \frac{(2\gamma_{ji})^3}{3!} + o\left(\gamma_{ji}^4\right)\right)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\eta_i\eta_j \left(\gamma_{ji}^2 - \frac{\gamma_{ji}^4}{3} + o\left(\gamma_{ji}^5\right)\right) + \frac{\varepsilon}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(4\gamma_{ji}^2 - \frac{16\gamma_{ji}^4}{3} + o\left(\gamma_{ji}^5\right)\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 2\varepsilon\right)\gamma_{ji}^2 - \frac{1}{3N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 8\varepsilon\right)\gamma_{ji}^4 \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + \frac{\varepsilon}{2}\right) o\left(\gamma_{ji}^5\right).
\end{aligned}
$$

□

Corollary 3.1. *Let $\varepsilon > \frac{M-2}{2}$, then $\{\varphi^*(\xi^k)\}_{k=1}^M$ are asymptotically stable phase-locked states of* (2.1).

Proof. For any $i, j \in \{1, 2, \ldots, N\}, \quad k \in \{1, 2, \ldots, M\}$, we have

$$
\begin{aligned}
&C_{ij}\xi_i^k\xi_j^k + 2\varepsilon \\
&= \left(\xi_i^1\xi_j^1 + \cdots + \xi_i^{k-1}\xi_j^{k-1} + \xi_i^k\xi_j^k + \xi_i^{k+1}\xi_j^{k+1} + \cdots + \xi_i^M\xi_j^M\right)\xi_i^k\xi_j^k + 2\varepsilon \\
&= \xi_i^1\xi_j^1\xi_i^k\xi_j^k + \cdots + \xi_i^{k-1}\xi_j^{k-1}\xi_i^k\xi_j^k + 1 + \xi_i^{k+1}\xi_j^{k+1}\xi_i^k\xi_j^k + \cdots + \xi_i^M\xi_j^M\xi_i^k\xi_j^k + 2\varepsilon \\
&\geq -(M-1) + 1 + 2\varepsilon.
\end{aligned}
$$

Then the desired result follows from Lemmas 2.2 and 3.1. □

Next, we derive a sufficient condition for stability/instability of binary patterns. We introduce the following matrices with suitable dimensions:

$$
D := \begin{pmatrix}
\sum_{j\neq 2} C_{2j}\eta_2\eta_j & -C_{23}\eta_2\eta_3 & -C_{24}\eta_2\eta_4 & \ldots & -C_{2N}\eta_2\eta_N \\
-C_{23}\eta_2\eta_3 & \sum_{j\neq 3} C_{3j}\eta_3\eta_j & -C_{34}\eta_3\eta_4 & \ldots & -C_{3N}\eta_3\eta_N \\
\ldots & \ldots & \ldots & \ddots & \ldots \\
-C_{2N}\eta_2\eta_N & -C_{3N}\eta_3\eta_N & -C_{4N}\eta_4\eta_N & \ldots & \sum_{j\neq N} C_{Nj}\eta_N\eta_j
\end{pmatrix},
$$

$$
\mathbb{1} := (1, 1, \ldots, 1)^{\mathrm{T}}, \quad E := \mathbb{1}\mathbb{1}^{\mathrm{T}}, \quad I = \operatorname{diag}(1, 1, \ldots, 1).
$$

Theorem 3.1. *$\varphi^*(\eta)$ is asymptotically stable equilibrium if $\lambda_{\min}(D - 2\varepsilon E + 2\varepsilon N I) > 0$. Furthermore, $\varphi^*(\eta)$ is unstable if $\lambda_{\min}(D - 2\varepsilon E + 2\varepsilon N I) < 0$.*

Proof. Since $\gamma_{ji} = \gamma_{1i} - \gamma_{1j}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 2\varepsilon\right)\gamma_{ji}^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j + 2\varepsilon\right)\left(\gamma_{1i} - \gamma_{1j}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\eta_i\eta_j\gamma_{1i}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\eta_i\eta_j\gamma_{1j}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\left(-2C_{ij}\eta_i\eta_j-4\varepsilon\right)\gamma_{1i}\gamma_{1j}\\
&\quad+2\varepsilon\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{1i}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{1j}^2\right)\\
&=2\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\eta_i\eta_j\gamma_{1i}^2+2\sum_{i=1}^{N}\left(-C_{ii}\eta_i\eta_i-2\varepsilon\right)\gamma_{1i}^2+2\sum_{i=1}^{N}\sum_{\substack{j=1\\i<j}}^{N}\left(-2C_{ij}\eta_i\eta_j-4\varepsilon\right)\gamma_{1i}\gamma_{1j}\\
&\quad+4\varepsilon N\sum_{i=1}^{N}\gamma_{1i}^2\\
&=2\sum_{i=1}^{N}\left[\sum_{j=1}^{N}C_{ij}\eta_i\eta_j-C_{ii}\eta_i\eta_i-2\varepsilon+2\varepsilon N\right]\gamma_{1i}^2+2\sum_{i=1}^{N}\sum_{\substack{j=1\\i<j}}^{N}\left(-2C_{ij}\eta_i\eta_j-4\varepsilon\right)\gamma_{1i}\gamma_{1j}\\
&=2\sum_{i=1}^{N}\left[\sum_{\substack{j=1\\j\neq i}}^{N}C_{ij}\eta_i\eta_j+2\varepsilon\left(N-1\right)\right]\gamma_{1i}^2+2\sum_{i=1}^{N}\sum_{\substack{j=1\\i<j}}^{N}\left(-2C_{ij}\eta_i\eta_j-4\varepsilon\right)\gamma_{1i}\gamma_{1j}\\
&=2\Gamma^{\mathrm{T}}\left(D-2\varepsilon E+2\varepsilon NI\right)\Gamma,
\end{aligned}
$$

where $\Gamma := \begin{pmatrix}\gamma_{12} & \gamma_{13} & \dots & \gamma_{1N}\end{pmatrix}^{\mathrm{T}}$. Since $\lambda_{\min}(D-2\varepsilon E+2\varepsilon NI) > 0$, we see that $D-2\varepsilon E+2\varepsilon NI$ is positively definite. On the other hand, Lemma 3.1 tells that

$$
f(\varphi)-f(\varphi^*)=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+2\varepsilon\right)\gamma_{ji}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{O}\left(\gamma_{ji}^4\right).
$$

By Lemma 2.2, $\varphi^*(\eta)$ is asymptotically stable equilibrium point. By similar argument, we see that $\varphi^*(\eta)$ is unstable if $\lambda_{\min}(D-2\varepsilon E+2\varepsilon NI) < 0$. □

Corollary 3.2. *If $\lambda_{\min}(D)+2\varepsilon > 0$, $\varphi^*(\eta)$ is asymptotically stable equalibrium of* (2.1).

Proof. The eigenvalues of matrix $-2\varepsilon E$ are

$$
-2\varepsilon\left(N-1\right),\underbrace{0,0,\dots,0}_{N-2}.
$$

By Lemma 2.4 we have

$$
\lambda_{\min}(D)-2\varepsilon\left(N-1\right)\leq\lambda_{\min}(D-2\varepsilon E).
$$

This implies

$$
\lambda_{\min}(D)+2\varepsilon\leq\lambda_{\min}(D-2\varepsilon E+2\varepsilon NI).
$$

We use Theorem 3.1 to obtain the desired result. □

Remark 3.1. *By Corollary 3.1 or 3.2, any binary pattern η is asymptotically stable for* (2.1) *if ε is sufficiently large.*

We note that in [24], Nishikawa et al. gave a sufficient condition for stability by considering the Jacobian matrix of order N. In our paper, the sufficient condition is given through a matrix of order $N-1$. Moreover, with the potential approach we can further consider the Łojasiewicz exponent which gives the convergence rate. The main result is as follows.

Theorem 3.2. *Let η be a binary pattern, and let $\varphi^*(\eta)$ be the phase-locked solution of* (2.1) *satisfying* (1.2). *If $\min\limits_{1\le i,j\le N}\{C_{ij}\eta_i\eta_j+2\varepsilon\}>0$, then there exists a positive constant C such that*

$$|f(\varphi^*)-f(\varphi)|^{\frac{1}{2}}\le C\|\nabla f(\varphi)\|_\infty.$$

Therefore, the convergence towards such an equilibrium is exponentially fast.

Proof. It is easy to see

$$\cos(\varphi_j^*-\varphi_i^*+\gamma_{ji})=\eta_i\eta_j\cos\gamma_{ji}.$$

Set

$$x_j:=\varphi_j^*-\varphi_j,\quad x_M:=\max_{1\le j\le N}\{x_j\},\quad x_m:=\min_{1\le j\le N}\{x_j\},\quad x_{ji}:=x_j-x_i,\quad \alpha:=\max_{1\le i,j\le N}|\gamma_{ji}|,$$

we can easily get

$$2\alpha=x_M-x_m,\quad \sum_{j=1}^N x_{Mj}\ge 2\alpha,\quad 2\gamma_{ji}=x_i-x_j=x_{ij}.$$

Then we have

$$\begin{aligned}
&\|\nabla f(\varphi)\|_\infty\\
&=\max_{1\le i\le N}\Big|-\frac{1}{N}\sum_{j=1}^N C_{ij}\sin(\varphi_j-\varphi_i)-\frac{\varepsilon}{N}\sum_{j=1}^N\sin 2(\varphi_j-\varphi_i)\Big|\\
&=\max_{1\le i\le N}\Big|\frac{1}{N}\sum_{j=1}^N C_{ij}\left(\sin(\varphi_j-\varphi_i)-\sin(\varphi_j^*-\varphi_i^*)\right)+\frac{\varepsilon}{N}\sum_{j=1}^N\left(\sin 2(\varphi_j-\varphi_i)-\sin 2(\varphi_j^*-\varphi_i^*)\right)\Big|\\
&=\max_{1\le i\le N}\Big|\frac{2}{N}\sum_{j=1}^N C_{ij}\cos\left(\varphi_j^*-\varphi_i^*+\gamma_{ji}\right)\sin\gamma_{ji}+\frac{2\varepsilon}{N}\sum_{j=1}^N\cos 2\gamma_{ji}\sin 2\gamma_{ji}\Big|\\
&=\max_{1\le i\le N}\Big|\frac{2}{N}\sum_{j=1}^N C_{ij}\eta_i\eta_j\cos\gamma_{ji}\sin\gamma_{ji}+\frac{\varepsilon}{N}\sum_{j=1}^N\sin 4\gamma_{ji}\Big|\\
&=\max_{1\le i\le N}\Big|\frac{1}{N}\sum_{j=1}^N C_{ij}\eta_i\eta_j\sin 2\gamma_{ji}+\frac{\varepsilon}{N}\sum_{j=1}^N\sin 4\gamma_{ji}\Big|\\
&=\max_{1\le i\le N}\Big|\frac{1}{N}\sum_{j=1}^N C_{ij}\eta_i\eta_j\left(2\gamma_{ji}-\frac{(2\gamma_{ji})^3}{3!}+\frac{(2\gamma_{ji})^5}{5!}+o\left(\gamma_{ji}^6\right)\right)\\
&\qquad+\frac{\varepsilon}{N}\sum_{j=1}^N\left(4\gamma_{ji}-\frac{(4\gamma_{ji})^3}{3!}+\frac{(4\gamma_{ji})^5}{5!}+o\left(\gamma_{ji}^6\right)\right)\Big|
\end{aligned}$$

$$= \max_{1\le i\le N}\Big|\frac{2}{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+2\varepsilon\right)\gamma_{ji}-\frac{4}{3N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+8\varepsilon\right)\gamma_{ji}^3$$
$$+\frac{4}{15N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+32\varepsilon\right)\gamma_{ji}^5+\frac{1}{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+\varepsilon\right)o\left(\gamma_{ji}^6\right)\Big|.$$

Therefore,

$$\begin{aligned}\max_{1\le i\le N}\Big|\frac{2}{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+2\varepsilon\right)\gamma_{ji}\Big| &= \max_{1\le i\le N}\Big|\frac{1}{N}\sum_{j=1}^{N}\left(C_{ij}\eta_i\eta_j+2\varepsilon\right)x_{ij}\Big|\\ &\ge \frac{1}{N}\sum_{j=1}^{N}\left(C_{Mj}\eta_M\eta_j+2\varepsilon\right)x_{Mj}\\ &\ge \frac{\min\limits_{1\le i,j\le N}\{C_{ij}\eta_i\eta_j+2\varepsilon\}}{N}\sum_{j=1}^{N}x_{Mj},\end{aligned}$$

which implies $\|\nabla f(\varphi)\|_\infty \ge \frac{2\min\limits_{1\le i,j\le N}\{C_{ij}\eta_i\eta_j+2\varepsilon\}}{N}\alpha$. It follows from Lemma 3.1 that we have

$$|f(\varphi)-f(\varphi^*)|\le \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big|C_{ij}\eta_i\eta_j+2\varepsilon\Big|\alpha^2.$$

As a consequence,

$$|f(\varphi^*)-f(\varphi)|^{\frac{1}{2}}\le C\|\nabla f(\varphi)\|_\infty.$$

By Lemma 2.1 (2), the convergence towards such an equilibrium is exponentially fast. □

In the next Proposition, we will construct a "maximum" set of mutually orthogonal binary patterns in $\{1,-1\}^N$ for special N, which is an example for the above theorem.

Proposition 3.1. *Let $N=2^n, n\in\mathbb{N}$. There exist N orthogonal binary patterns.*

Proof. We construct the orthogonal binary patterns using the idea of induction.

- For $n=1$, we construct a set of binary patterns:

$$\xi^{1,1}=[1\quad 1],\quad \xi^{1,2}=[1\quad -1].$$

- For $n=2$, we set

$$\xi^{2,1}=[\xi^{1,1}\quad \xi^{1,1}],\quad \xi^{2,2}=[-\xi^{1,1}\quad \xi^{1,1}],\quad \xi^{2,3}=[\xi^{1,2}\quad \xi^{1,2}],\quad \xi^{2,4}=[-\xi^{1,2}\quad \xi^{1,2}].$$

- Suppose the conclusion is true for $n-1$, i.e., there exist 2^{n-1} orthogonal binary patterns $\xi^{n-1,1},\xi^{n-1,2},\dots,\xi^{n-1,2^{n-1}}$. Then for n, we set

$$\begin{aligned}&\xi^{n,1}=[\xi^{n-1,1}\quad \xi^{n-1,1}],\quad \xi^{n,2}=[-\xi^{n-1,1}\quad \xi^{n-1,1}],\quad \xi^{n,3}=[\xi^{n-1,2}\quad \xi^{n-1,2}],\quad \xi^{n,4}=[-\xi^{n-1,2}\quad \xi^{n-1,2}],\\ &\xi^{n,5}=[\xi^{n-1,3}\quad \xi^{n-1,3}],\quad \xi^{n,6}=[-\xi^{n-1,3}\quad \xi^{n-1,3}],\quad \xi^{n,7}=[\xi^{n-1,4}\quad \xi^{n-1,4}],\quad \xi^{n,8}=[-\xi^{n-1,4}\quad \xi^{n-1,4}],\\ &\qquad\cdots\qquad\qquad\cdots\qquad\qquad\cdots\qquad\qquad\cdots\\ &\xi^{n,2^n-3}=[\xi^{n-1,2^{n-1}-1}\quad \xi^{n-1,2^{n-1}-1}],\quad \xi^{n,2^n-2}=[-\xi^{n-1,2^{n-1}-1}\quad \xi^{n-1,2^{n-1}-1}],\\ &\xi^{n,2^n-1}=[\xi^{n-1,2^{n-1}}\quad \xi^{n-1,2^{n-1}}],\quad \xi^{n,2^n}=[-\xi^{n-1,2^{n-1}}\quad \xi^{n-1,2^{n-1}}].\end{aligned}$$

We obtain the desired orthogonal binary patterns $\{\xi^{n,k}\}_{k=1}^N$ in $\{1,-1\}^N$. □

Remark 3.2. (1) Proposition 3.1 shows, there exist N orthogonal binary patterns in $\{1,-1\}^N$ for $N = 2^n, n \in \mathbb{N}$. Note that in space $\mathbb{R}^N$, an orthogonal vector set consists of at most N vectors. Therefore, this is a "maximum" set of mutually orthogonal binary patterns in $\{1,-1\}^N$.

(2) If $N = 2^n$, let us consider the case that the set of memorized patterns consists of $\{\xi^{n,k}\}_{k=1}^N$ constructed in Proposition 3.1. Then it is easy to see that

$$C_{ij} = \sum_{k=1}^{N} \xi_i^{n,k}\xi_j^{n,k} = 0,\ \forall i \neq j \in \{1,\dots,N\},$$

which leads to $\min\limits_{1\le i,j\le N}\{C_{ij}\eta_i\eta_j + 2\varepsilon\} = 2\varepsilon > 0$ for any $\varepsilon > 0$ and any $\eta \in \{1,-1\}^N$. Therefore, Theorem 3.2 is available.

(3) The conditions for stability in Section 3 (see Theorem 3.1) and that in [24] are based on the spectrum of some matrix. However, to calculate the eigenvalues of a matrix is a difficult problem if the matrix is large. Therefore, simple conditions for stability that are easy to verify, are highly desired. In the next section, we will study the special case when the memorized patterns are mutually orthogonal and derive simple conditions for stability.

4. Orthogonal memorized patterns

In this section, we will consider the system (2.1) with mutually orthogonal memorized patterns $\{\xi^k\}_{k=1}^M$, i.e., $\xi^k \cdot \xi^l = 0$ for any $l \neq k$. Simple conditions for stability/instability of a binary pattern will be derived in Subsection 4.1. Then in Subsection 4.2, we study the equilibrium property and stability/instability of those states on the straight lines connecting any pair of memorized patterns. Compared to the studies in Section 3 and [24], an important feature is that the conditions in this section are simple and easy to verify. In Subsection 4.3 we will demonstrate that a general case with nonorthogonal memorized patterns can be transformed to the case of orthogonal memorized patterns.

Throughout this section, we will assume the memorized patterns $\{\xi^k\}_{k=1}^M$ are *mutually orthogonal*, unless stated otherwise.

4.1. **Stability/instability of binary patterns.**

As we see in Remark 3.1, any binary pattern can be stable if a large ε is provided. This means the pattern retrieve process may give any binary pattern and the effect of memorized patterns is suppressed. So, our interest mainly lies in the case that the parameter ε is temperate. We will show that the phase-locked states corresponding to memorized patterns are asymptotically stable and isolated for any $\varepsilon > 0$. Furthermore, we will give a criterion to determine the stability/instability for any binary pattern.

Given memorized binary patterns $\{\xi^k\}_{k=1}^M$, the Jacobian matrix for linearization of (2.1) near $\varphi^*(\eta)$ is

$$A_\eta = \begin{pmatrix} -T_1 & \frac{1}{N}C_{12}\eta_1\eta_2 + \frac{2\varepsilon}{N} & \frac{1}{N}C_{13}\eta_1\eta_3 + \frac{2\varepsilon}{N} & \dots & \frac{1}{N}C_{1N}\eta_1\eta_N + \frac{2\varepsilon}{N} \\ \frac{1}{N}C_{21}\eta_2\eta_1 + \frac{2\varepsilon}{N} & -T_2 & \frac{1}{N}C_{23}\eta_2\eta_3 + \frac{2\varepsilon}{N} & \dots & \frac{1}{N}C_{2N}\eta_2\eta_N + \frac{2\varepsilon}{N} \\ \dots & \dots & \dots & \ddots & \dots \\ \frac{1}{N}C_{N1}\eta_N\eta_1 + \frac{2\varepsilon}{N} & \frac{1}{N}C_{N2}\eta_N\eta_2 + \frac{2\varepsilon}{N} & \frac{1}{N}C_{N3}\eta_N\eta_3 + \frac{2\varepsilon}{N} & \dots & -T_N \end{pmatrix},$$

where $T_i = \sum_{j=1,j\neq i}^N (\frac{1}{N}C_{ij}\eta_i\eta_j + \frac{2\varepsilon}{N})$. For convenience we recall some notations:

$$\mathbb{1} := (1,1,\dots,1)^{\mathrm{T}}, \quad E := \mathbb{1}\mathbb{1}^{\mathrm{T}}, \quad I = \mathrm{diag}(1,1,\dots,1).$$

Theorem 4.1. *For each $l \in \{1, 2, \ldots, M\}$, $\varphi^*(\xi^l)$ is an equilibrium of* (2.1) *with an eigenvalue spectrum of*

$$\underbrace{-1-2\varepsilon, \ldots, -1-2\varepsilon}_{N-M}, \underbrace{-2\varepsilon, \ldots, -2\varepsilon}_{M-1}, 0.$$

Therefore, $\varphi^*(\xi^l)$ is asymptotically stable for any $\varepsilon > 0$.

Proof. The matrix A_{ξ^l} can be rewritten as follows

$$A_{\xi^l} = J_{\xi^l} + \frac{2\varepsilon}{N} E - 2\varepsilon I,$$

where J_{ξ^l} denotes the Jacobian matrix corresponding to $\varepsilon = 0$, i.e., the part of A_{ξ^l} without ε. We recall from [14] that the eigenvalues and eigenvectors of J_{ξ^l} are given by

$$\begin{aligned} J_{\xi^l}\mathcal{X}_1 = -\mathcal{X}_1, \quad J_{\xi^l}\mathcal{X}_2 = -\mathcal{X}_2, \ldots, \ J_{\xi^l}\mathcal{X}_{N-M} = -\mathcal{X}_{N-M}, \\ J_{\xi^l}\mathcal{X}^{1l} = 0\mathcal{X}^{1l}, \quad J_{\xi^l}\mathcal{X}^{2l} = 0\mathcal{X}^{2l}, \ldots, \ J_{\xi^l}\mathcal{X}^{Ml} = 0\mathcal{X}^{Ml}, \end{aligned}$$

where $\mathcal{X}^{kl} = \left(\xi_1^k\xi_1^l, \xi_2^k\xi_2^l, \ldots, \xi_N^k\xi_N^l\right)^{\mathrm{T}}$, and $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{N-M}\}$ is a basis of the space

$$[\mathrm{span}(\mathcal{X}^{1l}, \mathcal{X}^{2l}, \ldots, \mathcal{X}^{Ml})]^{\perp}.$$

Note that

$$\mathcal{X}^{ll} \cdot \mathcal{X}^{kl} = \mathbb{1} \cdot \mathcal{X}^{kl} = \xi^k \cdot \xi^l = 0, \ \forall\, k \in \{1, 2, \ldots, M\} \backslash \{l\},$$

and

$$\mathcal{X}^{ll} \cdot \mathcal{X}_j = \mathbb{1} \cdot \mathcal{X}_j = 0, \ \forall\, j \in \{1, 2, \ldots, N-M\},$$

which implies that $E\mathcal{X}^{ll} = N\mathcal{X}^{ll}$ and $E\mathcal{X}_j = E\mathcal{X}^{kl} = \mathbf{0}\,(k \neq l)$. So we get

$$\begin{aligned} A_{\xi^l}\mathcal{X}^{ll} &= J_{\xi^l}\mathcal{X}^{ll} + \frac{2\varepsilon}{N}E\mathcal{X}^{ll} - 2\varepsilon\mathcal{X}^{ll} = 0\mathcal{X}^{ll} + 2\varepsilon\mathcal{X}^{ll} - 2\varepsilon\mathcal{X}^{ll} = 0\mathcal{X}^{ll}, \\ A_{\xi^l}\mathcal{X}_j &= J_{\xi^l}\mathcal{X}_j + \frac{2\varepsilon}{N}E\mathcal{X}_j - 2\varepsilon\mathcal{X}_j = -\mathcal{X}_j - 2\varepsilon\mathcal{X}_j = (-1-2\varepsilon)\mathcal{X}_j, \forall j \in \{1, 2, \ldots, N-M\}, \\ A_{\xi^l}\mathcal{X}^{kl} &= J_{\xi^l}\mathcal{X}^{kl} + \frac{2\varepsilon}{N}E\mathcal{X}^{kl} - 2\varepsilon\mathcal{X}^{kl} = 0\mathcal{X}^{kl}p - 2\varepsilon\mathcal{X}^{kl} = -2\varepsilon\mathcal{X}^{kl}, \forall k \in \{1, 2, \ldots, M\} \backslash \{l\}. \end{aligned}$$

Therefore, $\varphi^*(\xi^l)$ is an equilibrium with an eigenvalue spectrum of

$$\underbrace{-1-2\varepsilon, \ldots, -1-2\varepsilon}_{N-M}, \underbrace{-2\varepsilon, \ldots, -2\varepsilon}_{M-1}, 0.$$

The eigenvalue 0 is simple and it has an eigenvector $\mathbb{1}$, which is due to the global phase shift invariance, i.e, $\varphi^*(\xi^l) + c\mathbb{1}$ is still an equilibrium. So, $\varphi^*(\xi^l)$ is asymptotically stable. □

Theorem 4.1 tells that the memorized patterns $\{\xi^k\}_{k=1}^M$ are asymptotically stable for any $\varepsilon > 0$. This motivates the following definition.

Definition 4.1. *An equilibrium of* (2.1) *is called* ε-independently stable *if it is stable for any $\varepsilon > 0$. We say a binary pattern η is ε-independently stable if $\varphi^*(\eta)$ is ε-independently stable.*

Next we discuss the stability of other binary patterns. We find that the quality

$$\sum_{k=1}^{M}(\xi^k\cdot\eta)^2$$

is important for this problem. We first gives a bound for this quality.

Proposition 4.1. *Let $\eta\in\{1,-1\}^N$ be a binary pattern, then we have $\sum_{k=1}^{M}(\xi^k\cdot\eta)^2\le N^2$.*

Proof. In space $\mathbb{R}^N$, the set of orthogonal vectors $\{\xi^k\}_{k=1}^M$ can be extended to an orthogonal basis $\{\xi^1,\xi^2,\dots,\xi^M,\ \xi^{M+1},\dots,\xi^N\}$ satisfying $\xi^l\cdot\xi^l=N, l\in\{1,2,\dots,N\}$. (For $l\in\{M+1,\dots,N\}$, the component of ξ^l is not necessarily 1 or -1.) Note that η is a linear combination of the basis, say

$$\eta=a_1\xi^1+a_2\xi^2+\cdots+a_N\xi^N.$$

Then we obtain

$$\begin{aligned}\xi^k\cdot\eta&=\sum_{l=1}^{N}a_l(\xi^k\cdot\xi^l)=a_k\xi^k\cdot\xi^k=a_kN,\quad k=1,2,\dots,M,\\ \eta\cdot\eta&=\sum_{l=1}^{N}a_l^2\xi^l\cdot\xi^l=N\sum_{l=1}^{N}a_l^2.\end{aligned}$$

Since $\eta\cdot\eta=N$, we have $\sum_{k=1}^{M}(\xi^k\cdot\eta)^2=N^2\sum_{k=1}^{M}a_k^2$ and $\sum_{l=1}^{N}a_l^2=1$. Hence,

$$\sum_{k=1}^{M}(\xi^k\cdot\eta)^2=N^2\sum_{k=1}^{M}a_k^2\le N^2\sum_{k=1}^{N}a_k^2=N^2.$$

□

Lemma 4.1. *Let $\eta\in\{1,-1\}^N$ be a binary pattern. If $\sum_{k=1}^{M}(\xi^k\cdot\eta)^2<N^2$, then the equilibrium $\varphi^*(\eta)$ is unstable for any $\varepsilon\in(0,\varepsilon_\eta)$, where ε_η is given by*

$$\varepsilon_\eta=\max_{l\in\{1,2,\dots,M\}}\frac{N^2-\sum_{k=1}^{M}(\xi^k\cdot\eta)^2}{2(N^2-(\xi^l\cdot\eta)^2)}.\tag{4.1}$$

Proof. For any $l\in\{1,2,\dots,M\}$, we let $y=(\xi_1^l\eta_1,\xi_2^l\eta_2,\dots,\xi_N^l\eta_N)^{\mathrm{T}}$. Then

$$\begin{aligned}&y^{\mathrm{T}}A_\eta y\\ =&-\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\Big(\frac{1}{N}C_{ij}\eta_i\eta_j+\frac{2\varepsilon}{N}\Big)\xi_i^l\eta_i\xi_i^l\eta_i+\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\Big(\frac{1}{N}C_{ij}\eta_i\eta_j+\frac{2\varepsilon}{N}\Big)\xi_i^l\eta_i\xi_j^l\eta_j\\ =&-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}C_{ij}\eta_i\eta_j-2\varepsilon(N-1)+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}C_{ij}\xi_i^l\xi_j^l+\frac{2\varepsilon}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\xi_i^l\eta_i\xi_j^l\eta_j\\ =&-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\eta_i\eta_j+\frac{1}{N}\sum_{i=1}^{N}C_{ii}\eta_i\eta_i-2\varepsilon(N-1)+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\xi_i^l\xi_j^l-\frac{1}{N}\sum_{i=1}^{N}C_{ii}\xi_i^l\xi_i^l\\ &+\frac{2\varepsilon}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i^l\eta_i\xi_j^l\eta_j-\frac{2\varepsilon}{N}\sum_{i=1}^{N}\xi_i^l\eta_i\xi_i^l\eta_i\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{N}\sum_{k=1}^{M}(\xi^k\cdot\eta)^2 + M - 2\varepsilon(N-1) + \frac{1}{N}\sum_{k=1}^{M}(\xi^k\cdot\xi^l)^2 - M + \frac{2\varepsilon}{N}(\xi^l\cdot\eta)^2 - 2\varepsilon\\
&= -\frac{1}{N}\sum_{k=1}^{M}(\xi^k\cdot\eta)^2 - 2\varepsilon N + \frac{1}{N}(\xi^l\cdot\xi^l)^2 + \frac{2\varepsilon}{N}(\xi^l\cdot\eta)^2\\
&= \frac{N^2 - \sum_{k=1}^{M}(\xi^k\cdot\eta)^2 - 2\varepsilon\Big(N^2 - (\xi^l\cdot\eta)^2\Big)}{N}.
\end{aligned}
$$

Here we used

$$
\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\eta_i\eta_j = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\eta_i\eta_j = \sum_{k=1}^{M}\Big(\sum_{i=1}^{N}\xi_i^k\eta_i\Big)\Big(\sum_{j=1}^{N}\xi_j^k\eta_j\Big) = \sum_{k=1}^{M}(\xi^k\cdot\eta)^2.
$$

If $\sum_{k=1}^{M}(\xi^k\cdot\eta)^2 < N^2$, we denote $\varepsilon_l^* := \frac{N^2-\sum_{k=1}^{M}(\xi^k\cdot\eta)^2}{2(N^2-(\xi^l\cdot\eta)^2)} > 0$. Then for any $\varepsilon\in(0,\varepsilon_l^*)$ we have $y^{\mathrm{T}}A_\eta y > 0$. By the minimax principle for eigenvalues [26], we find that A_η has a positive eigenvalue. Note that $\varepsilon_\eta := \max_{l\in\{1,2,\ldots,M\}}\varepsilon_l^*$, therefore, $\varphi^*(\eta)$ is unstable for any $\varepsilon\in(0,\varepsilon_\eta)$. □

According to Remark 3.1, any binary pattern η becomes stable if ε is sufficiently large. The following lemma tells that the ε leading to stability of η is a continuum.

Lemma 4.2. *Let $\eta\in\{1,-1\}^N$ be a binary pattern and $\varepsilon_2>\varepsilon_1>0$. If $\varphi^*(\eta)$ is stable for* (2.1) *with ε_1, then $\varphi^*(\eta)$ is stable for* (2.1) *with ε_2.*

Proof. For convenience, we denote the energy function in (2.2) by $f_\varepsilon(\varphi)$, i.e.,

$$
f_\varepsilon(\varphi) = -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\cos(\varphi_j-\varphi_i) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j-\varphi_i).
$$

It follows from Lemma 2.2, we have

$$
f_{\varepsilon_1}(\varphi) - f_{\varepsilon_1}(\varphi^*(\eta)) \ge 0, \quad \forall\varphi\in\mathcal{N}(\varphi^*(\eta)).
$$

For any $\varepsilon_2>\varepsilon_1$, we can get by Lemma 3.1

$$
f_{\varepsilon_2}(\varphi) - f_{\varepsilon_2}(\varphi^*(\eta)) \ge f_{\varepsilon_1}(\varphi) - f_{\varepsilon_1}(\varphi^*(\eta)).
$$

We apply Lemma 2.2 again to see that $\varphi^*(\eta)$ is stable for (2.1) with ε_2. □

Proposition 4.2. *For any $\eta\in\{-1,1\}^N$ and $\delta=(\delta_1,\delta_2,\ldots,\delta_N)^{\mathrm{T}}\in\mathbb{R}^N$, we have*

$$
\sum_{k=1}^{M}\Big[(\xi^k\cdot(\eta\circ\cos\delta))^2 + (\xi^k\cdot(\eta\circ\sin\delta))^2\Big] \le N^2,
$$

where $\eta\circ\cos\delta = (\eta_1\cos\delta_1,\eta_2\cos\delta_2,\ldots,\eta_N\cos\delta_N)^{\mathrm{T}}$ and $\eta\circ\sin\delta$ is similar.

Proof. We prove this estimate by a similar way as in Proposition 4.1. The orthogonal vectors $\{\xi^k\}_{k=1}^M$ can be extended to an orthogonal basis in $\mathbb{R}^N$

$$
\xi^1,\xi^2,\ldots,\xi^M,\xi^{M+1},\ldots,\xi^N
$$

satisfying $\xi^l\cdot\xi^l=N, l\in\{1,2,\ldots,N\}$. Suppose $\eta\circ\cos\delta$ and $\eta\circ\sin\delta$ are expressed as

$$
\eta\circ\cos\delta = b_1\xi^1 + b_2\xi^2 + \cdots + b_N\xi^N,
$$

$$\eta \circ \sin\delta = c_1\xi^1 + c_2\xi^2 + \cdots + c_N\xi^N.$$

Then we have

$$\xi^k \cdot (\eta \circ \cos\delta) = \sum_{l=1}^{N} b_l(\xi^k \cdot \xi^l) = b_k\xi^k \cdot \xi^k = b_k N, \quad k = 1, 2, \ldots, M,$$

$$\xi^k \cdot (\eta \circ \sin\delta) = \sum_{l=1}^{N} c_l(\xi^k \cdot \xi^l) = c_k\xi^k \cdot \xi^k = c_k N, \quad k = 1, 2, \ldots, M,$$

$$\|\eta \circ \cos\delta\|_2^2 = (\eta \circ \cos\delta) \cdot (\eta \circ \cos\delta) = \sum_{l=1}^{N} b_l^2\xi^l \cdot \xi^l = N\sum_{l=1}^{N} b_l^2,$$

$$\|\eta \circ \sin\delta\|_2^2 = (\eta \circ \sin\delta) \cdot (\eta \circ \sin\delta) = p\sum_{l=1}^{N} c_l^2\xi^l \cdot \xi^l = N\sum_{l=1}^{N} c_l^2.$$

Note that

$$\|\eta \circ \cos\delta\|_2^2 + \|\eta \circ \sin\delta\|_2^2 = \sum_{j=1}^{N}(\eta_j\cos\delta_j)^2 + \sum_{j=1}^{N}(\eta_j\sin\delta_j)^2 = \sum_{j=1}^{N}(\cos^2\delta_j + \sin^2\delta_j) = N,$$

we obtain $N\sum_{l=1}^{N} b_l^2 + N\sum_{l=1}^{N} c_l^2 = N$, i.e., $\sum_{l=1}^{N}(b_l^2 + c_l^2) = 1$. Hence,

$$\begin{aligned}\sum_{k=1}^{M}\left[(\xi^k \cdot (\eta \circ \cos\delta))^2 + (\xi^k \cdot (\eta \circ \sin\delta))^2\right] &= \sum_{k=1}^{M}(b_k^2N^2 + c_k^2N^2) = N^2\sum_{k=1}^{M}(b_k^2 + c_k^2) \\ &\le N^2\sum_{l=1}^{N}(b_l^2 + c_l^2) = N^2.\end{aligned}$$

□

Now we can give the main result for the stability of arbitrary binary patterns.

Theorem 4.2. *Let $\eta \in \{1, -1\}^N$ be a binary pattern.*

(1) If $\sum_{k=1}^{M}(\xi^k \cdot \eta)^2 < N^2$, then there exists a critical strength $\varepsilon_\eta^* > 0$ such that $\varphi^*(\eta)$ is unstable for $\varepsilon \in (0, \varepsilon_\eta^*)$ and $\varphi^*(\eta)$ is stable for $\varepsilon \in (\varepsilon_\eta^*, +\infty)$. Moreover, $\varepsilon_\eta^* > \varepsilon_\eta$ where ε_η is given in (4.1).

(2) If $\sum_{k=1}^{M}(\xi^k \cdot \eta)^2 = N^2$, then η is ε-independently stable, i.e., it is stable for any $\varepsilon > 0$.

Proof. (1) This assertion follows from Remark 3.1, Lemma 4.1 together with Lemma 4.2.

(2) Suppose

$$\sum_{k=1}^{M}(\xi^k \cdot \eta)^2 = N^2.$$

Then we can calculate the value of the energy function f in (2.2) at $\varphi^*(\eta)$ (denoted by φ^* with component φ_j^*),

$$f(\varphi^*(\eta)) = -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\cos(\varphi_j^* - \varphi_i^*) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j^* - \varphi_i^*)$$

$$
\begin{aligned}
&= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\eta_i\eta_j - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}1 \\
&= -\frac{1}{2N}\sum_{k=1}^{M}(\xi^k\cdot\eta)^2 - \frac{\varepsilon}{4N}N^2 \\
&= -\frac{1}{2N}N^2 - \frac{\varepsilon N}{4} \\
&= -\frac{N}{2} - \frac{\varepsilon N}{4}.
\end{aligned}
$$

For $\varphi \in \mathcal{N}(\varphi^*)$, we denote

$$
\delta_j = \varphi_j - \varphi_j^*, \quad j = 1, 2, \ldots, N.
$$

Then

$$
\begin{aligned}
\cos(\varphi_j - \varphi_i) &= \cos(\varphi_j^* - \varphi_i^* + \delta_j - \delta_i) \\
&= \cos(\varphi_j^* - \varphi_i^*)\cos(\delta_j - \delta_i) - \sin(\varphi_j^* - \varphi_i^*)\sin(\delta_j - \delta_i) \\
&= \eta_i\eta_j\left(\cos\delta_j\cos\delta_i + \sin\delta_j\sin\delta_i\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
f(\varphi) &= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\cos(\varphi_j - \varphi_i) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i) \\
&= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\eta_i\eta_j\left(\cos\delta_j\cos\delta_i + \sin\delta_j\sin\delta_i\right) - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i) \\
&= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\eta_i\eta_j\cos\delta_j\cos\delta_i - \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{M}\xi_i^k\xi_j^k\eta_i\eta_j\sin\delta_j\sin\delta_i \\
&\quad - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i) \\
&= -\frac{1}{2N}\sum_{k=1}^{M}(\xi^k\cdot(\eta\circ\cos\delta))^2 - \frac{1}{2N}\sum_{k=1}^{M}(\xi^k\cdot(\eta\circ\sin\delta))^2 - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i) \\
&= -\frac{1}{2N}\sum_{k=1}^{M}\left[(\xi^k\cdot(\eta\circ\cos\delta)) + (\xi^k\cdot(\eta\circ\sin\delta))^2\right] - \frac{\varepsilon}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2(\varphi_j - \varphi_i) \\
&\geq -\frac{1}{2N}N^2 - \frac{\varepsilon}{4N}N^2 = f(\varphi^*(\eta)).
\end{aligned}
$$

Here we used Proposition 4.2 and $\cos 2(\varphi_j - \varphi_i) \leq 1$. By Lemma 2.2, $\varphi^*(\eta)$ is stable. □

Remark 4.1. *Theorem 4.2 tells that the binary patterns can be classified into two types: ε-independently stable or not. A question naturally arises: if there is any ε-independently stable binary pattern other than the memorized patterns ? We will give an example in Subsection 5.1 which shows that there can be some "extra" ε-independently stable binary pattern other than the memorized ones.*

Let $\mathcal{P}\{-1,1\}^N$ denote the power set of $\{-1,1\}^N$. We define a mapping $G_N : \mathcal{P}\{-1,1\}^N \to \mathcal{P}\{1,-1\}^N$ which carries a set of memorized binary patterns to the corresponding set of ε-independently stable binary patterns. Therefore, for a set of mutually orthogonal binary patterns $\{\xi^1,\dots,\xi^M\}$, we have

$$G_N\left(\{\xi^1,\dots,\xi^M\}\right) = \left\{\eta \in \{1,-1\}^N \;\middle|\; \sum_{k=1}^M (\xi^k\cdot\eta)^2 = N^2\right\}.$$

In the following context, we regard the patterns η and $-\eta$ as the same one. Next we study the monotonicity of this mapping and begin with the following proposition.

Proposition 4.3. *Let $\eta \in \{1,-1\}^N$ be a binary pattern. Then $\sum_{k=1}^M(\xi^k\cdot\eta)^2 = N^2$ if and only if there exists $(a_1,a_2,\dots,a_M)\in\mathbb{R}^M$ such that $\eta = \sum_{k=1}^M a_k\xi^k$.*

Proof. (*Necessity*) Let η be a binary pattern satisfying $\sum_{k=1}^M(\xi^k\cdot\eta)^2 = N^2$. For the set of memorized orthogonal patterns $\{\xi^k\}_{k=1}^M$, it can be extended to an orthogonal basis in $\mathbb{R}^N$:

$$\xi^1,\xi^2,\dots,\xi^M,\xi^{M+1},\dots,\xi^N$$

which satisfy $\xi^l\cdot\xi^l = N, l\in\{1,2,\dots,N\}$. Then there exists $(a_1,\dots,a_M,a_{M+1},\dots,a_N)\in\mathbb{R}^N$ such that

$$\eta = \sum_{k=1}^M a_k\xi^k + \sum_{k=M+1}^N a_k\xi^k.$$

According to $\sum_{k=1}^M(\xi^k\cdot\eta)^2 = N^2$ and $\xi^k\cdot\eta = a_kN, k=1,2,\dots,M$, we see $\sum_{k=1}^M a_k^2 = 1$. Note that

$$N = \eta\cdot\eta = \sum_{k=1}^M a_k^2N + \sum_{k=M+1}^N a_k^2N = \left(\sum_{k=1}^M a_k^2 + \sum_{k=M+1}^N a_k^2\right)N,$$

then we obtain $\sum_{k=M+1}^N a_k^2 = 0$, i.e., $a_k = 0, k = M+1,\dots,N$. Consequently, $\eta = \sum_{k=1}^M a_k\xi^k$.

(*Sufficiency*) Let η be a binary pattern satisfying $\eta = \sum_{k=1}^M a_k\xi^k, a_k\in\mathbb{R}, k=1,2,\dots,M$, then we have $N = \eta\cdot\eta = \sum_{k=1}^M a_k^2N$, and so $\sum_{k=1}^M a_k^2 = 1$. It is easy to see that $\sum_{k=1}^M(\xi^k\cdot\eta)^2 = \sum_{k=1}^M a_k^2N^2 = N^2$. □

Theorem 4.3. *Let $M_1 < M_2$ and let $\{\xi^1,\dots,\xi^{M_1},\dots,\xi^{M_2}\}$ be a set of mutually orthogonal patterns. Then*

$$G_N\left(\{\xi^1,\dots,\xi^{M_1}\}\right) \subset G_N\left(\{\xi^1,\dots,\xi^{M_1},\dots,\xi^{M_2}\}\right).$$

Proof. For any $\eta\in G_N\left(\{\xi^1,\dots,\xi^{M_1}\}\right)$, by Proposition 4.3 we see that there exists $(a_1,\dots,a_{M_1})$ such that $\eta = \sum_{k=1}^{M_1} a_k\xi^k$. Let $a_k = 0$ for $k = M_1+1,\dots,M_2$, then we have $\eta = \sum_{k=1}^{M_1} a_k\xi^k = \sum_{k=1}^{M_2} a_k\xi^k$. Applying Proposition 4.3 again we obtain $\eta\in G_N\left(\{\xi^1,\dots,\xi^{M_1},\dots,\xi^{M_2}\}\right)$. □

Theorem 4.4. *Let $N_1\in\mathbb{N}$ and let $\{\xi^k\}_{k=1}^M\subset\mathbb{R}^N$ and $\{\tilde\xi^k\}_{k=1}^M\subset\mathbb{R}^{N+N_1}$ be two sets of mutually orthogonal binary patterns such that*

$$\tilde\xi^k = [\xi^k,\hat\xi^k]$$

for some $\hat\xi^k\in\{1,-1\}^{N_1}$. Then

$$\left|G_{N+N_1}\Big(\{\tilde\xi^1,\dots,\tilde\xi^M\}\Big)\right| \le \left|G_N\Big(\{\xi^1,\dots,\xi^M\}\Big)\right|.$$

Proof. For any $\tilde{\eta} \in G_{N+N_1}\Big(\{\tilde{\xi}^1, \dots, \tilde{\xi}^M\}\Big)$, we let $\tilde{\eta} = [\eta, \hat{\eta}]$ with $\eta \in \{1, -1\}^N$ and $\hat{\eta} \in \{1, -1\}^{N_1}$. We define a mapping $T : \tilde{\eta} \mapsto \eta$. We prove the desired result by showing that T is an injection from $G_{N+N_1}\Big(\{\tilde{\xi}^1, \dots, \tilde{\xi}^M\}\Big)$ to $G_N\Big(\{\xi^1, \dots, \xi^M\}\Big)$. We first show that $T(\tilde{\eta}) \in G_N\Big(\{\xi^1, \dots, \xi^M\}\Big)$. As $\sum_{k=1}^M (\tilde{\xi}^k \cdot \tilde{\eta})^2 = (N + N_1)^2$, Proposition 4.3 tells us that there exists $(a_1, a_2, \dots, a_M) \in \mathbb{R}^M$ such that $\tilde{\eta} = \sum_{k=1}^M a_k \tilde{\xi}^k$. Note that η (ξ^k, resp.) is the vector consisting of the first N components of $\tilde{\eta}$ ($\tilde{\xi}^k$, resp.), therefore we have

$$\eta = \sum_{k=1}^M a_k \xi^k.$$

Applying Proposition 4.3 we obtain $T(\tilde{\eta}) = \eta \in G_N\Big(\{\xi^1, \dots, \xi^M\}\Big)$.

Next we show T is an injection. Suppose $\tilde{\eta}, \tilde{\varsigma} \in G_{N+N_1}\Big(\{\tilde{\xi}^1, \dots, \tilde{\xi}^M\}\Big)$ satisfy $T(\tilde{\eta}) = T(\tilde{\varsigma})$. Denote

$$\tilde{\eta} = [\eta, \hat{\eta}], \qquad \tilde{\varsigma} = [\varsigma, \hat{\varsigma}], \quad \text{with } \eta, \varsigma \in \{-1, 1\}^N.$$

By Proposition 4.3, there exist $(a_1, a_2, \dots, a_M) \in \mathbb{R}^M$ and $(b_1, b_2, \dots, b_M) \in \mathbb{R}^M$ such that

$$\tilde{\eta} = \sum_{k=1}^M a_k \tilde{\xi}^k, \quad \tilde{\varsigma} = \sum_{k=1}^M b_k \tilde{\xi}^k.$$

Then we have $T(\tilde{\eta}) = \sum_{k=1}^M a_k \xi^k$ and $T(\tilde{\varsigma}) = \sum_{k=1}^M b_k \xi^k$. Since $\xi^1, \dots, \xi^M$ are linearly independent, $T(\tilde{\eta}) = T(\tilde{\varsigma})$ implies that $a_k = b_k$ for $k = 1, 2, \dots, M$. This tells that $\tilde{\eta} = \tilde{\varsigma}$. $\square$

In the following context, a set of binary patterns $\{\tilde{\xi}^k\}_{k=1}^M \subset \mathbb{R}^{N+N_1}$ constructed by

$$\tilde{\xi}^k = [\xi^k, \hat{\xi}^k]$$

is called a *lift* of the set of binary patterns $\{\xi^k\}_{k=1}^M \subset \mathbb{R}^N$. Next we prove that we can construct a lift to avoid the "extra" ε-independently stable binary pattern.

Theorem 4.5. *Let $\{\xi^k\}_{k=1}^M \subset \mathbb{R}^N$ be a set of mutually orthogonal patterns. Then there exist $N_1 \in \mathbb{N}$ and a lift $\{\tilde{\xi}^k\}_{k=1}^M \subset \mathbb{R}^{N+N_1}$ such that*

$$G_{N+N_1}\Big(\{\tilde{\xi}^1, \dots, \tilde{\xi}^M\}\Big) = \{\tilde{\xi}^1, \dots, \tilde{\xi}^M\}.$$

Proof. By choosing suitable N_1 we can construct a set of mutually orthogonal patterns $\{\hat{\xi}^1, \dots, \hat{\xi}^M\} \subset \mathbb{R}^{N_1}$ whose first M components $\hat{\xi}^k_j$ $(j = 1, 2, \dots, M)$ satisfy

$$\hat{\xi}^k_j = \begin{cases} 1, & j \neq k, \\ -1, & j = k, \end{cases} \quad (k = 1, 2, \dots, M-1), \qquad \text{and} \quad \hat{\xi}^M_j = 1. \tag{4.2}$$

Here, (4.2) gives only the first M components of $\{\hat{\xi}^1, \dots, \hat{\xi}^M\}$ and the other $N_1 - M$ components should be constructed so that $\{\hat{\xi}^1, \dots, \hat{\xi}^M\}$ are mutually orthogonal. (The existence of such N_1 and $\{\hat{\xi}^1, \dots, \hat{\xi}^M\}$ is justified in Subsection 4.3.) We claim that any binary pattern in $G_{N_1}\Big(\{\hat{\xi}^1, \dots, \hat{\xi}^M\}\Big)$ must coincide with a pattern in $\{\hat{\xi}^1, \dots, \hat{\xi}^M\}$. Let

$\hat{\eta} \in G_{N_1}\left(\{\hat{\xi}^1, \ldots, \hat{\xi}^M\}\right)$, by Proposition 4.3 there exists $(a_1, \ldots, a_M) \in \mathbb{R}^M$ such that $\hat{\eta} = \sum_{k=1}^M a_k \hat{\xi}^k$. Then we have $\sum_{k=1}^M a_k^2 = 1$ and each component of $\hat{\eta}$ is -1 or 1, that is,

$$\left|a_1\hat{\xi}_j^1 + a_2\hat{\xi}_j^2 + \cdots + a_M\hat{\xi}_j^M\right| = 1, \quad \forall j \in \{1, 2, \ldots, N_1\}. \tag{4.3}$$

Substituting (4.2) into the first M equations of (4.3), we obtain

$$\begin{cases} |-a_1 + a_2 + a_3 + \cdots + a_{M-1} + a_M| = 1, \\ |a_1 - a_2 + a_3 + \cdots + a_{M-1} + a_M| = 1, \\ |a_1 + a_2 - a_3 + \cdots + a_{M-1} + a_M| = 1, \\ \qquad\vdots \\ |a_1 + a_2 + a_3 + \cdots - a_{M-1} + a_M| = 1, \\ |a_1 + a_2 + a_3 + \cdots + a_{M-1} + a_M| = 1. \end{cases} \tag{4.4}$$

Now we claim that there exists $k_0 \in \{1, 2, \ldots, M\}$ such that $a_{k_0}^2 = 1$. Suppose not, we have $a_k^2 \neq 1$ for all $k \in \{1, 2, \ldots, M\}$. We combine the first and last equations in (4.4) to see $a_1 = \pm 1$ or 0. In view of $a_k^2 \neq 1$, we have $a_1 = 0$. Similarly we combine the second and last equations in (4.4), together with $a_k^2 \neq 1$ again, to find that $a_2 = 0$. Note that we can repeat the same argument to obtain $a_3 = a_4 = \cdots = a_{M-1} = 0$. Then we recall (4.4) to find finally that $|a_M| = 1$, which contradicts to $a_M^2 \neq 1$. Therefore, there exists $k_0 \in \{1, 2, \ldots, M\}$ such that $a_{k_0}^2 = 1$. Since $\sum_{k=1}^M a_k^2 = 1$, we obtain $\hat{\eta} = \pm\hat{\xi}^{k_0}$. This proves the claim since we do not distinguish $\hat{\xi}^k$ and $-\hat{\xi}^k$.

We now construct a lift $\{\tilde{\xi}^k\}_{k=1}^M$ of $\{\xi^k\}_{k=1}^M$ by

$$\tilde{\xi}^k = [\xi^k, \hat{\xi}^k].$$

Note that $\{\xi^k\}_{k=1}^M$ and $\{\hat{\xi}^k\}_{k=1}^M$ are sets of mutually orthogonal binary patterns, then so does $\{\tilde{\xi}^k\}_{k=1}^M$. By Theorem 4.4, we find that

$$\left|G_{N+N_1}\left(\{\tilde{\xi}^1, \ldots, \tilde{\xi}^M\}\right)\right| \leq \left|G_{N_1}\left(\{\hat{\xi}^1, \ldots, \hat{\xi}^M\}\right)\right|.$$

Now the claim above tells that $G_{N_1}\left(\{\hat{\xi}^1, \ldots, \hat{\xi}^M\}\right) = \{\hat{\xi}^1, \ldots, \hat{\xi}^M\}$. On the other hand, by Theorem 4.1 we find $\{\tilde{\xi}^1, \ldots, \tilde{\xi}^M\} \subset G_{N+N_1}\left(\{\tilde{\xi}^1, \ldots, \tilde{\xi}^M\}\right)$. Therefore, we have

$$G_{N+N_1}\left(\{\tilde{\xi}^1, \ldots, \tilde{\xi}^M\}\right) = \{\tilde{\xi}^1, \ldots, \tilde{\xi}^M\}.$$

□

4.2. **Instability of middle states.** In [14], Hölzel et al. considered the case $\varepsilon = 0$ and proved that any point in the straight line connecting $\varphi^*(\xi^k)$ and $\varphi^*(\xi^l)$ is an equilibrium. Under some conditions they also claimed that $\varphi^*(\xi^k)$ is neutrally stable (see [14, Theorem 2.3]). To simplify the notations, in this subsection we use φ^{*k} to denote $\varphi^*(\xi^k)$ and use φ_j^{*k} to denote $\varphi_j^*(\xi^k)$. For (2.1) with $\varepsilon = 0$, Hölzel et al. proved the following result.

Lemma 4.3. [14] *When $\varepsilon = 0$, φ^{*k} are non-isolated and they are part of a single, connected set of degenerate stationary states which comprises all straight lines connecting any pair*

$\varphi^{*k}, \varphi^{*l}$ *in phase space defined by*

$$\varphi^{*k} + u(\varphi^{*l} - \varphi^{*k}), \quad \forall u \in \mathbb{R}. \tag{4.5}$$

In this subsection, we will study the system (2.1) with $\varepsilon > 0$. Let

$$\varphi_u := \varphi^{*k} + u(\varphi^{*l} - \varphi^{*k}),$$

and we use $(\varphi_u)_j$ to denote the j-th component of φ_u. For two binary patterns ξ^k and ξ^l, we set

$$\begin{aligned}
J_1 &:= \{j \in \{1, 2, \ldots, N\} \,|\, \xi^l_j = \xi^l_1, \xi^k_j = \xi^k_1\},\\
J_2 &:= \{j \in \{1, 2, \ldots, N\} \,|\, \xi^l_j \neq \xi^l_1, \xi^k_j = \xi^k_1\},\\
J_3 &:= \{j \in \{1, 2, \ldots, N\} \,|\, \xi^l_j = \xi^l_1, \xi^k_j \neq \xi^k_1\},\\
J_4 &:= \{j \in \{1, 2, \ldots, N\} \,|\, \xi^l_j \neq \xi^l_1, \xi^k_j \neq \xi^k_1\},
\end{aligned}$$

then $J_1 \neq \emptyset$ since $1 \in J_1$. We use $|J|$ to denote the cardinality of a set J, i.e., the number of elements in J.

Proposition 4.4. *The patterns ξ^k and ξ^l are orthogonal if and only if $|J_1|+|J_4|= |J_2|+|J_3|= \frac{N}{2}$.*

Proof. Since ξ^k and ξ^l are orthogonal, we see

$$|J_1|+|J_2|+|J_3|+|J_4|= N, \quad \text{and} \quad |J_1|-|J_2|-|J_3|+|J_4|= 0.$$

The result immediately follows. □

According to Proposition 4.4, at least one of J_2 and J_3 is not empty. Without loss of generality we may assume that J_3 is not empty in the following.

Proposition 4.5. *φ_u is not an equilibrium if $u \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$.*

Proof. It follows from Lemma 4.3 that for any $i = 1, 2, \ldots, N$,

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{M} \xi^k_i \xi^k_j \sin((\varphi_u)_j - (\varphi_u)_i) = 0.$$

Therefore, it suffices to show that there exists $i_0 \in \{1, 2, \ldots, N\}$ such that

$$\sum_{j=1}^{N} \sin 2((\varphi_u)_j - (\varphi_u)_{i_0}) \neq 0. \tag{4.6}$$

Since

$$\left|\varphi^{*k}_j - \varphi^{*k}_i\right| = \begin{cases} 0, & \xi^k_j = \xi^k_i;\\ \pi, & \xi^k_j \neq \xi^k_i, \end{cases}$$

According to the global phase shift invariance, we can choose the representations of φ^{*k}:

$$\varphi^{*k} = \left(0, \frac{\pi}{2}\left(1 - \xi^k_1\xi^k_2\right), \frac{\pi}{2}\left(1 - \xi^k_1\xi^k_3\right), \ldots, \frac{\pi}{2}\left(1 - \xi^k_1\xi^k_N\right)\right),$$

and without loss of generality we can make a similar choice for φ^{*l}. We claim that φ_u is not equilibrium point if $u \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. It is easy to see

$$\varphi^{*k}_j = \frac{\pi}{2}\left(1 - \xi^k_1\xi^k_j\right), \quad \varphi^{*l}_j = \frac{\pi}{2}\left(1 - \xi^l_1\xi^l_j\right), \quad \forall j = 1, 2, \ldots, N.$$

This implies $2(\varphi_j^{*k}-\varphi_i^{*k})=\pi\xi_1^k(\xi_i^k-\xi_j^k)=0$ or $\pm 2\pi$, and therefore,

$$\begin{aligned}
&\sum_{j=1}^N \sin 2\left((\varphi_u)_j-(\varphi_u)_i\right)\\
&=\sum_{j=1}^N \sin 2\left(\left[\varphi_j^{*k}+u(\varphi_j^{*l}-\varphi_j^{*k})\right]-\left[\varphi_i^{*k}+u(\varphi_i^{*l}-\varphi_i^{*k})\right]\right)\\
&=\sum_{j=1}^N \sin\left[2u(\varphi_j^{*l}-\varphi_j^{*k}-\varphi_i^{*l}+\varphi_i^{*k})\right].
\end{aligned}\tag{4.7}$$

Let $i_0\in J_1$, we can find

$$\begin{cases}
\varphi_j^{*l}-\varphi_j^{*k}=\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}, \quad \forall j\in J_1;\\
\varphi_j^{*l}-\varphi_j^{*k}=(\varphi_{i_0}^{*l}+\pi)-\varphi_{i_0}^{*k}=\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}+\pi, \quad \forall j\in J_2;\\
\varphi_j^{*l}-\varphi_j^{*k}=\varphi_{i_0}^{*l}-(\varphi_{i_0}^{*k}+\pi)=\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}-\pi, \quad \forall j\in J_3;\\
\varphi_j^{*l}-\varphi_j^{*k}=(\varphi_{i_0}^{*l}+\pi)-(\varphi_{i_0}^{*k}+\pi)=\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}, \quad \forall j\in J_4.
\end{cases}\tag{4.8}$$

We substitute (4.8) into (4.7) to find

$$\begin{aligned}
&\sum_{j=1}^N \sin 2\left((\varphi_u)_j-(\varphi_u)_{i_0}\right)\\
&=\sum_{j=1}^N \sin\left[2u(\varphi_j^{*l}-\varphi_j^{*k}-\varphi_{i_0}^{*l}+\varphi_{i_0}^{*k}\right]\\
&=\sum_{j\in J_1}\sin 2u0+\sum_{j\in J_2}\sin 2u\pi-\sum_{j\in J_3}\sin 2u\pi+\sum_{j\in J_4}\sin 2u0\\
&=(|J_2|-|J_3|)\sin 2u\pi.
\end{aligned}$$

• Case 1: If $|J_2|-|J_3|\neq 0$, we obtain

$$\sum_{j=1}^N \sin 2\left((\varphi_u)_j-(\varphi_u)_{i_0}\right)\neq 0,$$

in view of $u\in(-\frac{1}{2},0)\cup(0,\frac{1}{2})$. In other words, i_0 justifies (4.6).
• Case 2: If $|J_2|-|J_3|=0$, we obtain $|J_2|=|J_3|=\frac{N}{4}$ by Proposition 4.4. In this case we need to find another i_0' to justify the desired estimate (4.6). Let $i_0'\in J_3$, we deduce that

$$\varphi_{i_0'}^{*l}-\varphi_{i_0'}^{*k}=\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}-\pi,\tag{4.9}$$

We substitute (4.8) into (4.7) and use (4.9) to find that for $u\in(-\frac{1}{2},0)\cup(0,\frac{1}{2})$

$$\begin{aligned}
&\sum_{j=1}^N \sin 2\left((\varphi_u)_j-(\varphi_u)_{i_0'}\right)\\
&=\sum_{j=1}^N \sin\left[2u(\varphi_j^{*l}-\varphi_j^{*k}-\varphi_{i_0'}^{*l}+\varphi_{i_0'}^{*k})\right]
\end{aligned}$$

$$
\begin{aligned}
=&\sum_{j\in J_1}\sin\left[2u\left(\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}-\varphi_{i_0'}^{*l}+\varphi_{i_0'}^{*k}\right)\right]+\sum_{j\in J_2}\sin\left[2u\left(\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}+\pi-\varphi_{i_0'}^{*l}+\varphi_{i_0'}^{*k}\right)\right]\\
&+\sum_{j\in J_3}\sin\left[2u\left(\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}-\pi-\varphi_{i_0'}^{*l}+\varphi_{i_0'}^{*k}\right)\right]+\sum_{j\in J_4}\sin\left[2u\left(\varphi_{i_0}^{*l}-\varphi_{i_0}^{*k}-\varphi_{i_0'}^{*l}+\varphi_{i_0'}^{*k}\right)\right]\\
=&|J_1|\sin 2u\pi+|J_2|\sin 4u\pi+0+|J_4|\sin 2u\pi\\
=&\sin 2u\pi\left(|J_1|+2|J_2|\cos 2u\pi+|J_4|\right)\\
=&\sin 2u\pi\left(\frac{N}{2}+\frac{N}{2}\cos 2u\pi\right)\\
=&\frac{N}{2}\sin 2u\pi\left(1+\cos 2u\pi\right)\neq 0.
\end{aligned}
$$

We now Combine Case 1 and Case 2 to see that φ_u is not an equilibrium if $u\in(-\frac{1}{2},0)\cup(0,\frac{1}{2})$. □

Theorem 4.6. *φ_u $(u\in\mathbb{R})$ is an equilibrium if and only if $u=\frac{m}{2}, m\in\mathbb{Z}$.*

Proof. By Lemma 4.3 again, it suffices to show that for any $i\in\{1,2,\dots,N\}$ we have

$$\sum_{j=1}^{N}\sin 2((\varphi_u)_j-(\varphi_u)_{i_0})=0.$$

The proof is divided into three steps.

• Step 1: For any $u\in\mathbb{R}$, there exist $v\in[-\frac{1}{2},\frac{1}{2})$ and $z\in\mathbb{Z}$ such that $u=v+z$. We claim that φ_u is an equilibrium if and only if φ_v is too; this means that we only need to consider φ_u with $u\in[-\frac{1}{2},\frac{1}{2})$ instead of $u\in\mathbb{R}$. Indeed, we have

$$
\begin{aligned}
&\sin 2((\varphi_u)_j-(\varphi_u)_i)\\
=&\sin 2\left(\left[\varphi_j^{*k}+u(\varphi_j^{*l}-\varphi_j^{*k})\right]-\left[\varphi_i^{*k}+u(\varphi_i^{*l}-\varphi_i^{*k})\right]\right)\\
=&\sin 2\left(\left[\varphi_j^{*k}+v(\varphi_j^{*l}-\varphi_j^{*k})+z(\varphi_j^{*l}-\varphi_j^{*k})\right]-\left[\varphi_i^{*k}+v(\varphi_i^{*l}-\varphi_i^{*k})+z(\varphi_i^{*l}-\varphi_i^{*k})\right]\right)\\
=&\sin\left(2\left[\left(\varphi_j^{*k}+v(\varphi_j^{*l}-\varphi_j^{*k})\right)-\left(\varphi_i^{*k}+v(\varphi_i^{*l}-\varphi_i^{*k})\right)\right]+2z(\varphi_j^{*l}-\varphi_j^{*k}-\varphi_i^{*l}+\varphi_i^{*k})\right)\\
=&\sin 2((\varphi_v)_j-(\varphi_v)_i).
\end{aligned}
$$

This implies that φ_u is an equilibrium if and only if φ_v is too.

• Step 2: We show that φ_u is an equilibrium if $u\in\{-\frac{1}{2},0\}$. For $u=0$, it's obviously true. If $u=-\frac{1}{2}$, we use the relation (1.2), i.e.,

$$\left|\varphi_j^{*p}-\varphi_i^{*p}\right|=\begin{cases}0, & \xi_j^p=\xi_i^p\\ \pi, & \xi_j^p\neq\xi_i^p\end{cases},\quad p=1,2,\dots,M,$$

to derive

$$
\begin{aligned}
\sin 2\left((\varphi_u)_j-(\varphi_u)_i\right)=&\sin 2\left[\left(\varphi_j^{*k}-\frac{1}{2}\left(\varphi_j^{*l}-\varphi_j^{*k}\right)\right)-\left(\varphi_i^{*k}-\frac{1}{2}\left(\varphi_i^{*l}-\varphi_i^{*k}\right)\right)\right]\\
=&\sin\left[3\left(\varphi_j^{*k}-\varphi_i^{*k}\right)-\left(\varphi_j^{*l}-\varphi_i^{*l}\right)\right]=0,
\end{aligned}
$$

which shows that φ_u is an equilibrium.

• Step 3: In Proposition 4.5 we proved that φ_u is not an equilibrium if $u\in[-\frac{1}{2},\frac{1}{2})\setminus\{-\frac{1}{2},0\}$. Combining Steps 1-3 yields the desired results. □

Next we study the stability of equilibriums φ_u in (4.5) with $u = \frac{m}{2}$, $m \in \mathbb{Z}$. In order to classify these equilibriums, we define an equivalence class as follows:

$$[\varphi] := \{\tilde{\varphi} | \, \exists \, q = (q_1, q_2, \dots, q_N) \in \mathbb{Z}^N, \text{ such that } \tilde{\varphi} = \varphi + 2\pi q\}.$$

Proposition 4.6. *Given* $\varphi_u = \varphi^{*k} + u(\varphi^{*l} - \varphi^{*k}), u = \frac{m}{2}, m \in \mathbb{Z}$. *We have*

(1) if $m = 0 \, (\mathrm{mod}\, 4)$, *then* $\varphi_u \in [\varphi^{*k}]$,
(2) if $m = 1 \, (\mathrm{mod}\, 4)$, *then* $\varphi_u \in [\varphi_{\frac{1}{2}}]$,
(3) if $m = 2 \, (\mathrm{mod}\, 4)$, *then* $\varphi_u \in [\varphi^{*l}]$,
(4) if $m = 3 \, (\mathrm{mod}\, 4)$, *then* $\varphi_u \in [\varphi_{\frac{3}{2}}]$.

Furthermore, if φ_u *and* $\varphi_{u'}$ *are taken from the same equivalent class, then* φ_u *is stable if and only if* $\varphi_{u'}$ *is stable.*

Proof. The assertions (1)-(4) are obviously true and we omit the proof. The last statement holds true due to the 2π periodicity of the system. □

If $m = 0 \, (\mathrm{mod}\, 4)$ or $m = 2 \, (\mathrm{mod}\, 4)$, φ_u coincides with the stable equilibrium φ^{*l} or φ^{*k}. For the state φ_u with $m = 1 \, (\mathrm{mod}\, 4)$ or $m = 3 \, (\mathrm{mod}\, 4)$, in this context we call a *middle state* of φ^{*l} and φ^{*k}. Next, we prove the instability of middle states. Therefore, the existence of such equilibriums does not matter much in applications.

Theorem 4.7. *If the memorized patterns* $\{\xi^k\}_{k=1}^M$ *are mutually orthogonal, then for any* $k, l \in \{1, 2, \dots, M\}$ *with* $k \neq l$, *the equilibriums* $\varphi_{\frac{1}{2}}$ *and* $\varphi_{\frac{3}{2}}$ *are unstable.*

Proof. We only show the proof for the instability of $\varphi_{\frac{1}{2}}$, and the proof for $\varphi_{\frac{3}{2}}$ is the same. The linearization matrix $J = (J_{ij})$ of (2.1) at $\varphi_{\frac{1}{2}}$ is given by

$$\begin{aligned}
J_{ii} &= -\frac{1}{N} \sum_{j=1, j\neq i}^{N} C_{ij} \cos\left((\varphi_{\frac{1}{2}})_j - (\varphi_{\frac{1}{2}})_i\right) - \frac{2\varepsilon}{N} \sum_{j=1, j\neq i}^{N} \cos 2\left((\varphi_{\frac{1}{2}})_j - (\varphi_{\frac{1}{2}})_i\right), \\
J_{ij} &= \frac{1}{N} C_{ij} \cos\left((\varphi_{\frac{1}{2}})_j - (\varphi_{\frac{1}{2}})_i\right) + \frac{2\varepsilon}{N} \cos 2\left((\varphi_{\frac{1}{2}})_j - (\varphi_{\frac{1}{2}})_i\right), \quad j \neq i.
\end{aligned}$$

We will show $J\mathcal{X}^{kl} = 2\varepsilon \mathcal{X}^{kl}$ to see that 2ε is an eigenvalue of J, which implies the desired result. Note that for $p = 1, 2, \dots, M$ we have

$$\cos\frac{\varphi_j^{*p} - \varphi_i^{*p}}{2} = \frac{1}{2}\xi_j^p \xi_i^p + \frac{1}{2}, \qquad \sin\frac{\varphi_j^{*p} - \varphi_i^{*p}}{2} = \begin{cases} \frac{1}{2}\xi_j^p \xi_i^p - \frac{1}{2}, & \varphi_j^{*p} = \varphi_i^{*p} - \pi, \\ -\frac{1}{2}\xi_j^p \xi_i^p + \frac{1}{2}, & \varphi_j^{*p} = \varphi_i^{*p} + \pi, \\ 0, & \varphi_j^{*p} = \varphi_i^{*p}. \end{cases}$$

Let

$$\begin{aligned}
I_1 &:= \left\{ (i,j) \middle| \; \left\{ \begin{array}{l} \varphi_j^{*l} = \varphi_i^{*l} - \pi \\ \varphi_j^{*k} = \varphi_i^{*k} - \pi \end{array} \right. \text{ or } \left\{ \begin{array}{l} \varphi_j^{*l} = \varphi_i^{*l} + \pi \\ \varphi_j^{*k} = \varphi_i^{*k} + \pi \end{array} \right. \right\}, \\
I_2 &:= \left\{ (i,j) \middle| \; \left\{ \begin{array}{l} \varphi_j^{*l} = \varphi_i^{*l} + \pi \\ \varphi_j^{*k} = \varphi_i^{*k} - \pi \end{array} \right. \text{ or } \left\{ \begin{array}{l} \varphi_j^{*l} = \varphi_i^{*l} - \pi \\ \varphi_j^{*k} = \varphi_i^{*k} + \pi \end{array} \right. \right\}, \\
I_3 &:= \left\{ (i,j) \middle| \; \varphi_j^{*l} = \varphi_i^{*l} \text{ or } \varphi_j^{*k} = \varphi_i^{*k} \right\}.
\end{aligned}$$

Then, for any $j \neq i$, we obtain

$$\cos\left((\varphi_{\frac{1}{2}})_j - (\varphi_{\frac{1}{2}})_i\right) = \cos\left(\frac{\varphi_j^{*l} + \varphi_j^{*k}}{2} - \frac{\varphi_i^{*l} + \varphi_i^{*k}}{2}\right)$$

$$
\begin{aligned}
&= \cos\left(\frac{\varphi_j^{*l}-\varphi_i^{*l}}{2}+\frac{\varphi_j^{*k}-\varphi_i^{*k}}{2}\right)\\
&= \cos\frac{\varphi_j^{*l}-\varphi_i^{*l}}{2}\cos\frac{\varphi_j^{*k}-\varphi_i^{*k}}{2}-\sin\frac{\varphi_j^{*l}-\varphi_i^{*l}}{2}\sin\frac{\varphi_j^{*k}-\varphi_i^{*k}}{2}\\
&= \begin{cases} (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2})-(\frac{1}{2}\xi_j^l\xi_i^l-\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k-\frac{1}{2}), & (i,j)\in I_1\\ (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2})-(-\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k-\frac{1}{2}), & (i,j)\in I_2\\ (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2}), & (i,j)\in I_3 \end{cases}\\
&= \begin{cases} \frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2}\xi_j^k\xi_i^k, & (i,j)\in I_1\\ \frac{1}{2}\xi_j^l\xi_i^l\xi_j^k\xi_i^k+\frac{1}{2}, & (i,j)\in I_2.\\ (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2}), & (i,j)\in I_3. \end{cases}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_j^k\xi_j^l &= \begin{cases} (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2}\xi_j^k\xi_i^k)\xi_j^k\xi_j^l, & (i,j)\in I_1\\ (\frac{1}{2}\xi_j^l\xi_i^l\xi_j^k\xi_i^k+\frac{1}{2})\xi_j^k\xi_j^l, & (i,j)\in I_2\\ (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2})\xi_j^k\xi_j^l, & (i,j)\in I_3 \end{cases}\\
&= \begin{cases} \frac{1}{2}(\xi_i^l\xi_j^k+\xi_i^k\xi_j^l), & (i,j)\in I_1\\ \frac{1}{2}(\xi_i^l\xi_i^k+\xi_j^k\xi_j^l), & (i,j)\in I_2\\ \frac{1}{4}(\xi_j^l\xi_j^k+\xi_j^k\xi_i^l+\xi_i^k\xi_j^l+\xi_i^k\xi_i^l), & (i,j)\in I_3 \end{cases}\\
&= \begin{cases} (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2}\xi_j^k\xi_i^k)\xi_i^k\xi_i^l, & (i,j)\in I_1\\ (\frac{1}{2}\xi_j^l\xi_i^l\xi_j^k\xi_i^k+\frac{1}{2})\xi_i^k\xi_i^l, & (i,j)\in I_2\\ (\frac{1}{2}\xi_j^l\xi_i^l+\frac{1}{2})(\frac{1}{2}\xi_j^k\xi_i^k+\frac{1}{2})\xi_i^k\xi_i^l, & (i,j)\in I_3p \end{cases}\\
&= \cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_i^k\xi_i^l,
\end{aligned}
\tag{4.10}
$$

and

$$
\cos 2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)=2\cos^2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)-1=\xi_j^l\xi_i^l\xi_j^k\xi_i^k,\quad \forall j\neq i. \tag{4.11}
$$

Therefore, the ith component of $J\mathcal{X}^{kl}$ is given by

$$
\begin{aligned}
&\left[-\frac{1}{N}\sum_{j=1,j\neq i}^{N}C_{ij}\cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)-\frac{2\varepsilon}{N}\sum_{j=1,j\neq i}^{N}\cos 2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\right]\xi_i^k\xi_i^l\\
&\quad+\sum_{j=1,j\neq i}^{N}\left[\frac{1}{N}C_{ij}\cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)+\frac{2\varepsilon}{N}\cos 2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\right]\xi_j^k\xi_j^l\\
&=-\frac{1}{N}\sum_{j=1,j\neq i}^{N}C_{ij}\cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_i^k\xi_i^l-\frac{2\varepsilon}{N}\sum_{j=1,j\neq i}^{N}\cos 2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_i^k\xi_i^l\\
&\quad+\frac{1}{N}\sum_{j=1,j\neq i}^{N}C_{ij}\cos\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_j^k\xi_j^l+\frac{2\varepsilon}{N}\sum_{j=1,j\neq i}^{N}\cos 2\left((\varphi_{\frac{1}{2}})_j-(\varphi_{\frac{1}{2}})_i\right)\xi_j^k\xi_j^l
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2\varepsilon}{N}\sum_{j=1}^{N}\xi_j^l\xi_i^l\xi_j^k\xi_i^k\xi_i^k\xi_i^l + \frac{2\varepsilon}{N}\sum_{j=1}^{N}\xi_j^l\xi_i^l\xi_j^k\xi_i^k\xi_j^k\xi_j^l \\
&= -\frac{2\varepsilon}{N}\sum_{j=1}^{N}\xi_j^k\xi_j^l + \frac{2\varepsilon}{N}\sum_{j=1}^{N}\xi_i^k\xi_i^l \\
&= 2\varepsilon\xi_i^k\xi_i^l, \quad \forall i,
\end{aligned}
$$

where we used (4.10), (4.11) and $\xi^k\cdot\xi^l = 0$. This is $J\mathcal{X}^{kl} = 2\varepsilon\mathcal{X}^{kl}$. Hence, $\varphi_{\frac{1}{2}}$ is unstable. □

Remark 4.2. *In this section, we have studied the stability/instability of equilibriums corresponding to binary patterns or states in the straight lines connecting a pair of memorized patterns. However, it is still possible for a non-binary stable equilibriums to emerge (we call non-binary if it does not correspond to any binary pattern). Therefore, an open question is: can we avoid the non-binary stable equilibriums by introducing a lift ?*

4.3. **Nonorthogonality and orthogonality.** In this subsection, we discuss the case that the memorized patterns are not mutually orthogonal.

If the M memorized patterns are mutually orthogonal, Theorem 4.1 tells that each of the memorized patterns $\{\xi^k\}_{k=1}^M$ is ε-independently stable. Then a question is, whether the memorized patterns are still ε-independently stable if they are not mutually orthogonal ? The following example gives a negative answer.

Example 4.1. *We set the memorized patterns $\{\xi^1, \xi^2, \xi^3\}$ as*

$$
\xi^1 = [1 \quad 1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1], \quad \xi^2 = [1 \quad 1 \quad -1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1],
$$
$$
\xi^3 = [1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad -1 \quad -1],
$$

and we consider the stability of ξ^1 by Theorem 3.1. Calculating the eigenvalues of $D - 2\varepsilon E + 2\varepsilon NI$, we obtain

$$
\begin{gathered}
16\varepsilon+4, \quad 16\varepsilon+8, \quad 16\varepsilon+8, \quad 16\varepsilon+8, \quad 16\varepsilon+8, \\
9\varepsilon + \frac{\sqrt{196\varepsilon^2 - 68\varepsilon + 17}}{2} - \frac{1}{2}, \quad 9\varepsilon - \frac{\sqrt{196\varepsilon^2 - 68\varepsilon + 17}}{2} - \frac{1}{2}.
\end{gathered}
$$

There exists a positive $\varepsilon_* > 0$ such that the last eigenvalue in the above list is negative for $\varepsilon \in (0, \varepsilon_*)$ and positive for $\varepsilon > \varepsilon_*$. Therefore, $\varphi^*(\xi^1)$ is unstable if $\varepsilon \in (0, \varepsilon_*)$, and the memorized pattern ξ^1 is not ε-independently stable.

Example 4.1 shows that the general case is indeed different with the ideal case with orthogonal memorized patterns. This means that the nice theory in this section is not available when we consider a general case. Next, we will explain that a pattern retrieve problem with nonorthogonal memorized binary patterns can be transformed into a related problem with orthogonal memorized binary patterns by introducing a *lift*.

Let $\{\xi^k\}_{k=1}^M \subset \{1, -1\}^{N_1}, N_1 \in \mathbb{N}$ be a set of binary patterns which are not mutually orthogonal. The basic idea is as follows: First, we construct a set of mutually orthogonal patterns $\{\tilde{\xi}^k\}_{k=1}^M$ which is a lift of $\{\xi^k\}_{k=1}^M$ by adding some components for each pattern. Let us denote the dimension of patterns $\{\tilde{\xi}^k\}$ by N_2 ($N_2 > N_1$). We now use $\{\tilde{\xi}^k\}_{k=1}^M$ as the memorized patterns to produce a system (2.1) with N_2 oscillators for pattern retrieve. For a defective pattern η with dimension N_1, we also lift the dimension to produce a "larger" defective pattern $\tilde{\eta}$. After a pattern is retrieved using system (2.1) with memorized patterns

$\{\tilde{\xi}^k\}_{k=1}^M$ and defective pattern $\tilde{\eta}$, we remove the $N_2 - N_1$ elements to obtain the recognized pattern. In Subsection 5.4, we will use a simulation to illustrate the above framework.

In the following, we just show that the required lift (with mutually orthogonal binary patterns) does exist. Let $\{\tilde{\xi}^k\}_{k=1}^M \subset \{1,-1\}^{N_2}$ be a lift of $\{\xi^k\}_{k=1}^M$. For $a_1a_2\ldots a_M$ with $a_k \in \{0,1\}, k \in \{1,2,\ldots,M\}$, we define

$$J^1_{a_1a_2\ldots a_M} := \left\{ j \in \{1,2,\ldots,N_1\} \,\middle|\, \tilde{\xi}^1_j = (-1)^{a_1}\tilde{\xi}^1_1, \tilde{\xi}^2_j = (-1)^{a_2}\tilde{\xi}^2_1, \ldots, \tilde{\xi}^M_j = (-1)^{a_M}\tilde{\xi}^M_1 \right\},$$
$$J^2_{a_1a_2\ldots a_M} := \left\{ j \in \{N_1+1,N_1+2,\ldots,N_2\} \,\middle|\, \tilde{\xi}^1_j = (-1)^{a_1}\tilde{\xi}^1_1, \tilde{\xi}^2_j = (-1)^{a_2}\tilde{\xi}^2_1, \ldots, \tilde{\xi}^M_j = (-1)^{a_M}\tilde{\xi}^M_1 \right\}.$$

Then we have

$$\sum_{\substack{a_1a_2\ldots a_M\\ a_i\in\{0,1\}}} |J^1_{a_1a_2\ldots a_M}| = N_1, \qquad \sum_{\substack{a_1a_2\ldots a_M\\ a_i\in\{0,1\}}} |J^2_{a_1a_2\ldots a_M}| = N_2 - N_1.$$

To simplify the notations, in the following context we will omit the subscript "$a_i \in \{0,1\}$". For any $k,l \in \{1,2,\ldots,M\}$, by Proposition 4.4 we see that $\tilde{\xi}^k \cdot \tilde{\xi}^l = 0$ holds if and only if

$$\begin{aligned}
&\sum_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=0}} \left(|J^1_{a_1a_2\ldots a_M}| + |J^2_{a_1a_2\ldots a_M}|\right) + \sum_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=1}} \left(|J^1_{a_1a_2\ldots a_M}| + |J^2_{a_1a_2\ldots a_M}|\right)\\
=&\sum_{\substack{a_1a_2\ldots a_M\\ a_k=0,a_l=1}} \left(|J^1_{a_1a_2\ldots a_M}| + |J^2_{a_1a_2\ldots a_M}|\right) + \sum_{\substack{a_1a_2\ldots a_M\\ a_k=1,a_l=0}} \left(|J^1_{a_1a_2\ldots a_M}| + |J^2_{a_1a_2\ldots a_M}|\right)\\
=&\frac{N_2}{2},
\end{aligned}$$

that is,

$$\begin{cases}
\sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=0}} |J^2_{a_1a_2\ldots a_M}| + \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=1}} |J^2_{a_1a_2\ldots a_M}| = \frac{N_2}{2} - \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=0}} |J^1_{a_1a_2\ldots a_M}| - \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=a_l=1}} |J^1_{a_1a_2\ldots a_M}|,\\
\sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=0,a_l=1}} |J^2_{a_1a_2\ldots a_M}| + \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=1,a_l=0}} |J^2_{a_1a_2\ldots a_M}| = \frac{N_2}{2} - \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=0,a_l=1}} |J^1_{a_1a_2\ldots a_M}| - \sum\limits_{\substack{a_1a_2\ldots a_M\\ a_k=1,a_l=0}} |J^1_{a_1a_2\ldots a_M}|.
\end{cases}$$

For each pair (k,l), there are two linear equations as above in which $|J^2_{a_1a_2\ldots a_M}|$'s are the unknowns. So, there are $2C^2_M = M(M-1)$ linear equations with 2^M unknowns. It is easy to see that taking an appropriate N_2, the equations have a solution. So nonorthogonal binary patterns can be transformed into orthogonal binary patterns.

5. Numerical simulations

5.1. **Orthogonal memorized patterns.** We consider a network with $N = 16$ oscillators and $M = 3$ memorized orthogonal patterns $\{\xi^1, \xi^2, \xi^3\}$ with $\varepsilon = 0.03$

$$\dot{\varphi}_i = \frac{1}{16}\sum_{j=1}^{16}\sum_{k=1}^{3} \xi^k_i\xi^k_j \sin(\varphi_j - \varphi_i) + \frac{0.03}{16}\sum_{j=1}^{16} \sin 2(\varphi_j - \varphi_i),\ i = 1,2,\ldots,16.$$

The memorized patterns $\{\xi^1, \xi^2, \xi^3\}$ and the defective pattern η are shown in Figure 1. We apply the two-step process and after the initialization stage the phase shifts will evolve towards a distribution reflecting the pattern η. We see from Figure 1 that η is a slightly defective copy of ξ^1, and Figure 2 shows that the final state reflects ξ^1 since $m^1 = 1$ and $m^2 = m^3 = 0$, where m^k denotes $m(\xi^k)$. We notice that there is no ε-independently stable binary patterns except the memorized patterns $\{\xi^1, \xi^2, \xi^3\}$ by examining the condition in Theorem 4.2.

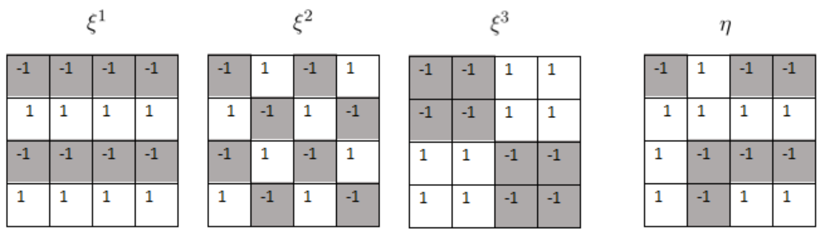


FIGURE 1. The memorized patterns ξ^1, ξ^2, ξ^3, and the defective pattern η.

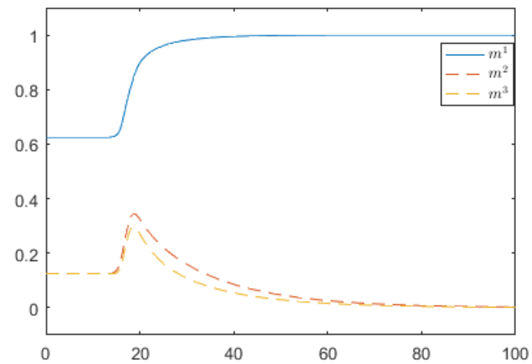


FIGURE 2. η correctly identified ξ^1.

Next, we do a simulation with $N = 16$ oscillators and $M = 7$ memorized mutually orthogonal patterns $\{\xi^k\}_{k=1}^7$, shown in Figure 1 and Figure 3. We choose $\varepsilon = 0.03$ and initial data $\varphi(0) \in [0, \pi]^{16}$

$$(0.74,\ 0.25,\ 3.76,\ 3.80,\ 3.24,\ 0.05,\ 4.33,\ 5.94,\ 5.49,\ 0.71,\ 2.23,\ 1.52,\ 3.52,\ 3.85,\ 1.89,\ 5.01).$$

In Figure 4, we can see that $\varphi(t)$ converges to $\varphi^*(\tilde{\eta})$ and it is easy to verify that this $\tilde{\eta}$ is ε-independently stable by Theorem 4.2 since $\sum_{k=1}^7 (\xi^k \cdot \tilde{\eta})^2 = 16^2$. This example shows that an extra ε-independently stable binary pattern $\tilde{\eta}$ emerges with the memorized patterns $\{\varphi^*(\xi^k)\}_{k=1}^7$. This gives an answer for the question in Remark 4.1.

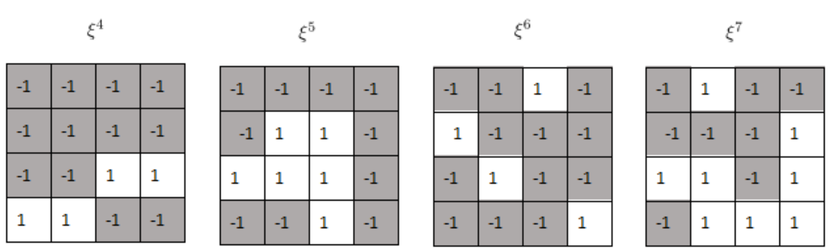
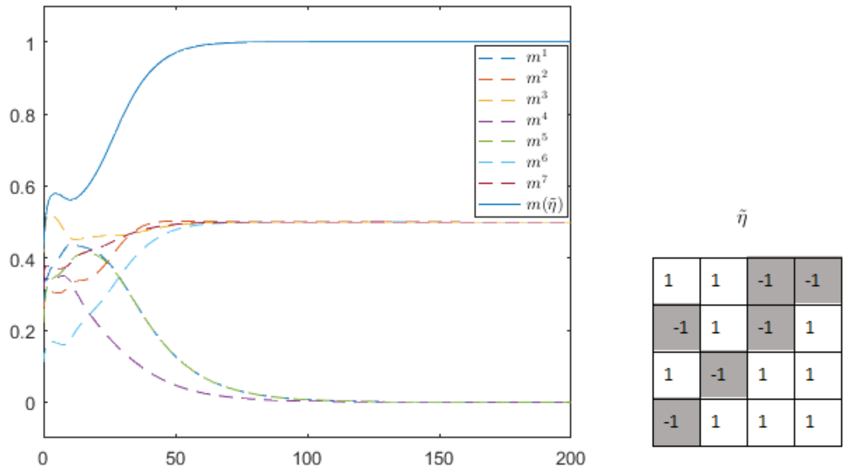


FIGURE 3. Mutually orthogonal patterns $\xi^4, \xi^5, \xi^6, \xi^7$.



(a) (b)

FIGURE 4. $\varphi(t)$ converges to $\varphi^*(\tilde{\eta})$.

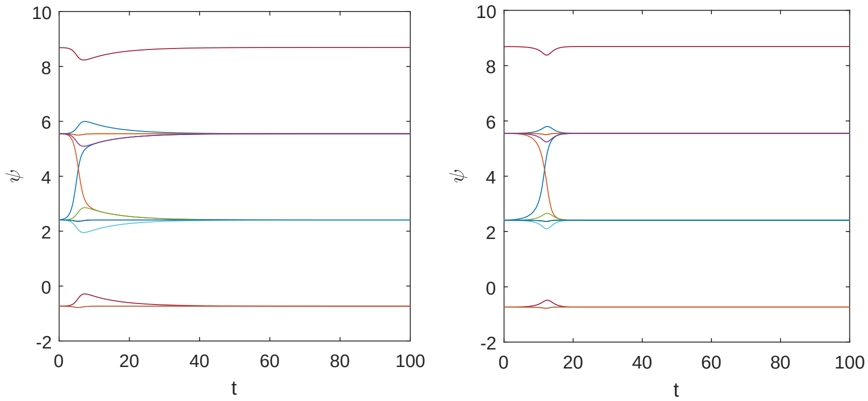
ε	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	...	0.75
number	6	6	6	6	14	14	14	14	110	15776	...	65536

TABLE 1. The number of stable binary patterns for different values of ε. Here, the six stable patterns for small ε are $\pm\xi^1, \pm\xi^2$ and $\pm\xi^3$.

5.2. **Varying ε.** In this subsection, we use simulations to examine how the values of ε influence the dynamics.

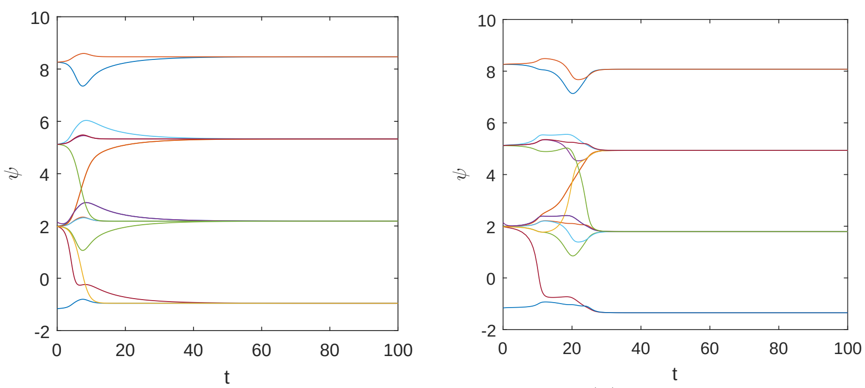
First, we consider again the mutually orthogonal memorized patterns $\{\xi^1, \xi^2, \xi^3\}$ shown in Figure 1. We examine the stability of binary patterns by computing the eigenvalues of Jacobian matrix. For given ε, we count the number of binary patterns at which the Jacobian matrix has $(N-1)$ negative eigenvalues (0 is an eigenvalue due to global phase shift invariance). Table 1 shows, as the strength ε increases, the number of (asymptotically) stable binary patterns is gradually increased. This is consistent with Lemma 4.2. On the other hand, this means that for different binary patterns η_1 and η_2, the critical strengths $\varepsilon^*_{\eta_1}$ and $\varepsilon^*_{\eta_2}$ can be different.

Next, we do simulations to show how the strength ε influences the dynamics. In Figures 5-6, we illustrate the trajectories of $\varphi(t)$ with respect to time t for different ε and different retrieved patterns ξ^k. They show that the convergence typically become faster as the strength ε increase. For the solutions in Figure 6, we make a table for the phase differences $\varphi_j(t) - \varphi_1(t)$ at $t = 100$, see Table 2. Note that for different ε, the values $\hat{\varphi}_{1j}$ are equal or differ by 2π; this means that the solutions converge to the same binary pattern. Then an interesting future problem is how does the basin of a stable binary pattern changes according to different choices of ε ?



(a) $\varepsilon = 0.05$ (b) $\varepsilon = 0.35$

FIGURE 5. $\varphi(t)$ converges to $\varphi^*(\xi^2)$.



(a) $\varepsilon = 0.05$ (b) $\varepsilon = 0.35$

FIGURE 6. $\varphi(t)$ converges to $\varphi^*(\xi^3)$.

ε	$\hat{\varphi}_{12}$	$\hat{\varphi}_{13}$	$\hat{\varphi}_{14}$	$\hat{\varphi}_{15}$	$\hat{\varphi}_{16}$	$\hat{\varphi}_{17}$	$\hat{\varphi}_{18}$	$\hat{\varphi}_{19}$
0.05	0	3.1416	3.1416	0	0	-3.1416	-3.1416	3.1416
0.35	0	3.1416	3.1416	0	0	-3.1416	-3.1416	3.1416
ε	$\hat{\varphi}_{1,10}$	$\hat{\varphi}_{1,11}$	$\hat{\varphi}_{1,12}$	$\hat{\varphi}_{1,13}$	$\hat{\varphi}_{1,14}$	$\hat{\varphi}_{1,15}$	$\hat{\varphi}_{1,16}$	
0.05	-3.1416	0	0	3.1416	3.1416	6.2832	6.2832	
0.35	3.1416	0	6.2832	3.1416	3.1416	6.2832	6.2832	

TABLE 2. $\hat{\varphi}_{1j} := \varphi_j(t) - \varphi_1(t)$ at $t = 100$ for the solutions in Figure 6.

5.3. **Recognition with non-binary initial patterns.** In this subsection, we do simulations to show how our model (2.1) can be used to recognize a binary pattern (black-white image) from a given non-binary "pattern" (gray scale image). The idea was briefly introduced in Section 1.

As a typical image file format, the bitmap image is a grid made of rows and columns where a specific cell (pixel) is given a value that measures its color. For a typical gray scale image, the value is called the gray scale which is set in $[0,255]_{\mathbb{Z}}$ (the set of integers in $[0,255]$) where 0 stands for "black" and 255 for "white". So a gray scale image is a vector $P=(p_1,p_2,\dots,p_N)$ in $[0,255]_{\mathbb{Z}}^N$ where N is the number of pixels. For each cell we identify an oscillator so that we can make a system (2.1) with N oscillators, where the memorized patterns $\{\xi^1,\xi^2,\dots,\xi^M\}$ should be given according to the real situation. In order to use a gray scale image as an initial data for (2.1), a reasonable way is to perform a transformation from $[0,255]_{\mathbb{Z}}$ to $[0,\pi]$, which is equivalent to extend the binary patterns in $\{-1,1\}^N$ to non-binary patterns in $[-1,1]^N$. For this aim we employ two continuous and monotone maps g and f

$$[0,255]_{\mathbb{Z}} \xrightarrow{g} [-1,1] \xrightarrow{f} [0,\pi].$$

Then we produce an initial phase data $\varphi^0\in[0,\pi]^N$ by $\varphi_i^0=(f\circ g)(p_i)$.

In our simulation, we use the simple maps

$$g(x) := \frac{2}{255}x-1, \qquad f(x) := \frac{\pi}{2}(x+1).$$

For simplicity we take $N=16$, and use the orthogonal memorized patterns $\{\xi^1,\xi^2,\xi^3\}$ shown in Figure 1 again. Figure 7 gives three non-binary patterns (gray scale images) η_1,η_2 and η_3 where the gray scales are denoted by values in $[-1,1]$. We will identify each of them a memorized pattern by (2.1) and the initial phases are

$$\begin{aligned}
\varphi^0(\eta_1) &= (0\quad \tfrac{200\pi}{255}\quad 0\quad 0\quad \pi\quad \pi\quad \pi\quad \pi\quad 0\quad 0\quad \tfrac{125\pi}{255}\quad 0\quad \pi\quad \pi\quad \pi\quad \pi),\\
\varphi^0(\eta_2) &= (0\quad \pi\quad 0\quad \tfrac{15\pi}{255}\quad \pi\quad 0\quad \pi\quad 0\quad \tfrac{200\pi}{255}\quad \pi\quad 0\quad \pi\quad \pi\quad 0\quad \tfrac{15\pi}{255}\quad 0),\\
\varphi^0(\eta_3) &= (\tfrac{200\pi}{255}\quad 0\quad \pi\quad \pi\quad 0\quad \tfrac{200\pi}{255}\quad \pi\quad \pi\quad \pi\quad \pi\quad \tfrac{170\pi}{255}\quad 0\quad \pi\quad \pi\quad 0\quad \tfrac{183\pi}{255}).
\end{aligned}$$

In Figure 8, we see that the three gray scale images have been successfully identified.

η_1

-1	g(200)	-1	-1
1	1	1	1
-1	-1	g(125)	-1
1	1	1	1

η_2

-1	1	-1	g(15)
1	-1	1	-1
g(200)	1	-1	1
1	-1	g(15)	-1

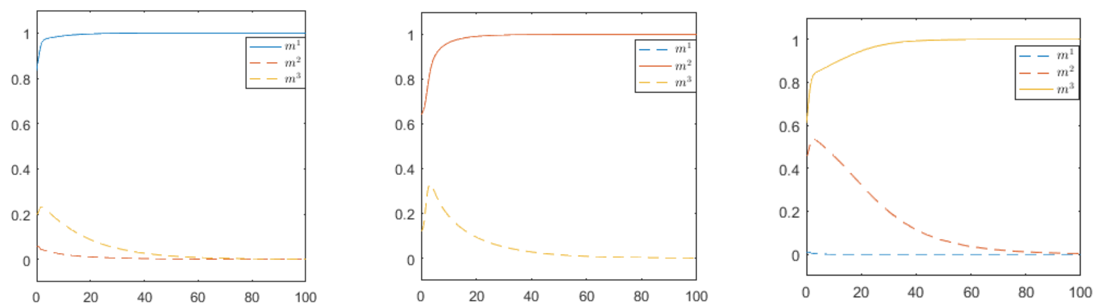
η_3

g(200)	-1	1	1
-1	g(200)	1	1
1	1	g(170)	-1
1	1	-1	g(183)

FIGURE 7. Gray images

5.4. **Extending nonorthogonal memorized patterns.** In this subsection, we give an example to illustrate that the case of nonorthogonal binary patterns can be transformed to the case of orthogonal patterns by extending the dimensions, as proposed in Subsection 4.3.

Consider $N_1=63$ and $M=3$. The memorized nonorthogonal patterns $\{\xi^1,\xi^2,\xi^3\}$ are shown in Figure 9 and two defective patterns $\{\varsigma_1,\varsigma_2\}$ are given in Figure 10. Notice that



(a) (b) (c)

FIGURE 8. ξ^1, ξ^2, ξ^3 are retrieved with defective patterns η_1, η_2, η_3, respectively.

the defective patterns ς_1 and ς_2 are obtained by wiping the black cells in two rows of the pattern ξ^3.

Next we will use the framework in Subsection 4.3 to retrieve one of the memorized patterns that is closet to the defective ones. We extend the memorized patterns $\{\xi^1, \xi^2, \xi^3\}$ to $\{\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3\}$ shown in Figure 11. Note that the dimension of each pattern is extended to $N_2 = 180$ and the new patterns $\{\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3\}$ are mutually orthogonal. We now develop a network with 180 oscillators using $\{\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3\}$ as the memorized patterns. According to Theorem 4.2, we know that the patterns $\{\tilde{\xi}^k\}_{k=1}^3$ are asymptotically stable for any $\varepsilon > 0$. In order to perform the pattern retrieve with defective patterns $\{\varsigma_1, \varsigma_2\}$, we extend them to new patterns $\{\tilde{\varsigma}_1, \tilde{\varsigma}_2\}$ with 180 cells, shown in Figure 12. Here we extend the defective patterns by adding $N_2 - N_1$ cells and each of these extra cells is given a "gray scale" in $[-1, 1]$ which is the average of that in the extended memorized patterns $\{\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3\}$. To retrieve a binary pattern we are using the recognition process with non-binary initial patterns as in Subsection 5.3. Figure 13 show that $\tilde{\xi}^3$ is retrieved which gives the pattern ξ^3 after removing the extra cells.

ξ^1

1	1	1	1	1	1	1
1	1	1	-1	1	1	1
1	1	-1	-1	1	1	1
1	1	1	-1	1	1	1
1	1	1	-1	1	1	1
1	1	1	-1	1	1	1
1	1	1	-1	1	1	1
1	1	-1	-1	-1	1	1
1	1	1	1	1	1	1

ξ^2

1	1	1	1	1	1	1
1	1	-1	-1	-1	1	1
1	-1	1	1	1	-1	1
1	1	1	1	1	-1	1
1	1	1	1	-1	1	1
1	1	1	-1	1	1	1
1	1	-1	1	1	1	1
1	-1	-1	-1	-1	-1	1
1	1	1	1	1	1	1

ξ^3

1	1	1	1	1	1	1
1	1	-1	-1	-1	1	1
1	-1	1	1	1	-1	1
1	1	1	1	1	-1	1
1	1	-1	-1	-1	1	1
1	1	1	1	1	-1	1
1	-1	1	1	1	-1	1
1	1	-1	-1	-1	1	1
1	1	1	1	1	1	1

FIGURE 9. Memorized nonorthogonal patterns ξ^1, ξ^2, ξ^3.

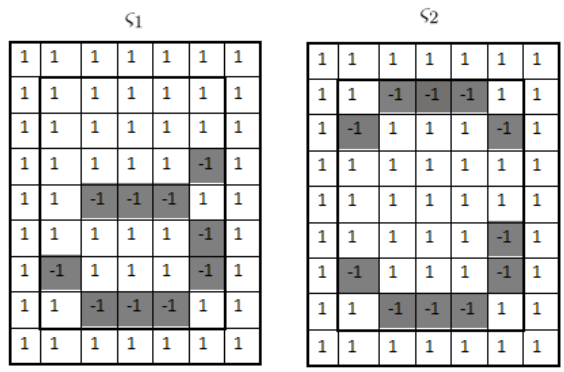


FIGURE 10. ς_1, ς_2 differ from ξ^3 by two rows, respectively.

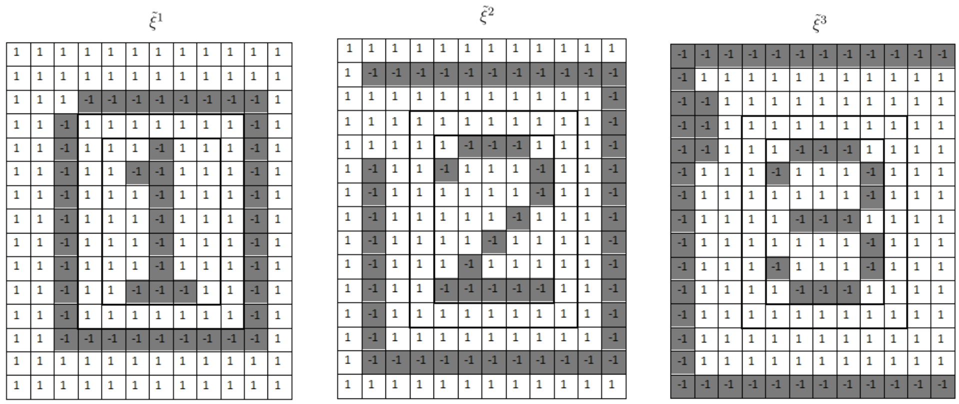


FIGURE 11. Nonorthogonal binary patterns ξ^1, ξ^2, ξ^3 are extended to orthogonal modes $\tilde{\xi}^1, \tilde{\xi}^2, \tilde{\xi}^3$.

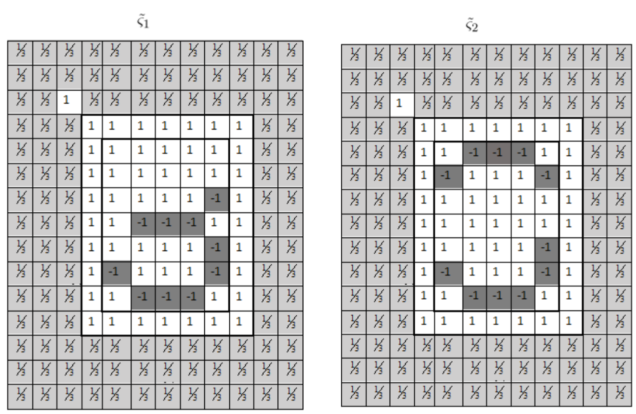
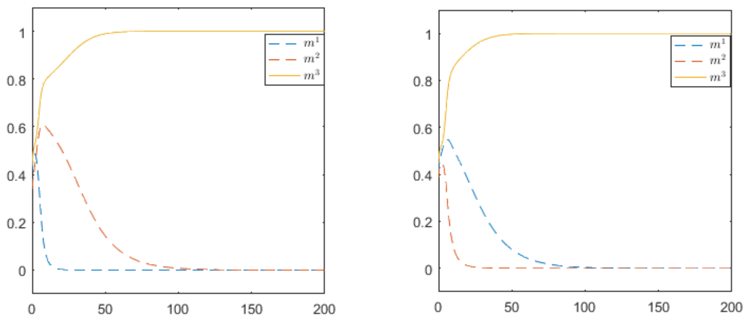


FIGURE 12. ς_1, ς_2 are extended to $\tilde{\varsigma}_1, \tilde{\varsigma}_2$.

6. Conclusions

In this paper, we study the dynamic properties of oscillators networks for binary pattern retrieve. We first give sufficient conditions for the stability of binary patterns when the



(a) (b)

FIGURE 13. $\tilde{\xi}^3$ is retrieved with defective patterns $\tilde{\varsigma}_1$ or $\tilde{\varsigma}_2$.

memorized patterns are general. Then we focus on the special case that the memorized patterns are mutually orthogonal. Several results are given for the stability of binary patterns for which a significant advantage is they are simple and easy to verify. Finally, we give a new idea that the case with nonorthogonal memorized patterns can be transformed into the case of orthogonal binary patterns by extending the dimension. We also suggest to use this model to recognize a binary pattern (or a black-white image) from a non-binary pattern (typically a gray scale image).

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