CURVATURE ESTIMATES FOR A CLASS OF HESSIAN TYPE EQUATIONS

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Abstract. In this paper, we establish the curvature estimates for a class of Hessian type equations. Some applications are also discussed.

1. Introduction

Suppose that M is a hypersurface in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let $\kappa(X)$, $H(X)$ and $\nu(X)$ be the principal curvatures, mean curvature and unit outer normal at $X \in M$ respectively. Define the $(0,2)$-tensor field η on M by

$$\eta_{ij} = Hg_{ij} - h_{ij},$$

where g_{ij} and h_{ij} are the first and second fundamental forms of M respectively. In fact, η is the first Newton transformation of h with respect to g. Using $\lambda(\eta)$ to denote the eigenvalues of η (with respect to g), we see that

$$\lambda_i = H - \kappa_i = \sum_{j \neq i} \kappa_j, \quad \text{for } i = 1, 2 \cdots, n.$$

In this paper, we consider the k-Hessian equation of $\lambda(\eta)$, i.e.,

$$\sigma_k(\lambda(\eta)) = f(X, \nu(X)), \quad \text{for } X \in M, \tag{1.1}$$

where σ_k is the k-th elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \text{for } k = 1, 2, \cdots, n.$$

To study equation (1.1), we introduce the following elliptic condition.

Definition 1.1. *A C^2 regular hypersurface $M \subset \mathbb{R}^{n+1}$ is called (η, k)-convex if $\lambda(\eta) \in \Gamma_k$ for all $X \in M$, where Γ_k is the Gårding cone*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \ j = 1, 2, \cdots, k\}.$$

Our main result is the following curvature estimate for equation (1.1).

Theorem 1.2. *Let M be a closed star-shaped (η, k)-convex hypersurface satisfying the curvature equation (1.1) for some positive function $f \in C^2(\Gamma)$, where Γ is an open neighborhood of the unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$. Then there exists a constant C depending only n, k, $\|M\|_{C^1}$, $\inf f$ and $\|f\|_{C^2}$ such that*

$$\max_{X \in M, \ i = 1, \cdots, n} |\kappa_i(X)| \leqslant C. \tag{1.2}$$

If we replace $\lambda(\eta)$ by $\kappa(X)$ in (1.1), equation (1.1) becomes the classical prescribed curvature equation

$$\sigma_k(\kappa(X)) = f(X, \nu(X)), \quad \text{for } X \in M. \tag{1.3}$$

When $k = 1, 2$ and n, (1.3) are prescribed mean curvature, scalar curvature and Gauss curvature equation respectively. Establishing the global C^2 estimate for (1.3) is a longstanding problem. When $k = 1$, it is quasi-linear and the C^2 estimate follows from classical theory of quasi-linear PDEs. If $k = n$, (1.3) is Monge-Ampère type and the C^2 estimates for general $f(X, \nu)$ are established by Caffarelli-Nirenberg-Spruck [2]. When $k = 2$, the C^2 estimate for (1.3) was obtained by Guan-Ren-Wang [14] and their proof was simplified by Spruck-Xiao [26]. In [22, 23], Ren-Wang proved the C^2 estimate when $k = n-1$ and $n-2$.

When $2 < k < n$, Caffarelli-Nirenberg-Spruck [4] proved the C^2 estimate if f is independent of ν. Guan-Guan [10] obtained the C^2 estimate if f depends only on ν. Ivochkina [17, 18] considered the corresponding Dirichlet problem of (1.3) on Euclidean domain and established the C^2 estimate under some extra assumptions on the dependence of f on ν. For equations of the prescribing curvature measure problem, when $f(X, \nu) = \langle X, \nu\rangle \tilde{f}(X)$, the C^2 estimate was proved by Guan-Lin-Ma [13] and Guan-Li-Li [12]. For general $f(X, \nu)$, Li-Ren-Wang [19] established such estimates for $(k+1)$-convex hypersurfaces (i.e., $\kappa(X) \in \Gamma_{k+1}$ for all $X \in M$).

When $k = n = 2$, (1.1) is the same as (1.3), which is the prescribed Gauss curvature equation. Thus (1.1) can be regarded as a generalization of the classical prescribed curvature equation. When $k = n$, (1.1) becomes the following equation for (η, n)-convex hypersurface:

$$\det(\eta(X)) = f(X, \nu(X)), \quad \text{for } X \in M. \tag{1.4}$$

The (η, n)-convex hypersurface has been studied intensively by Sha [24, 25], Wu [32] and Harvey-Lawson [15]. We note that (η, n)-convexity was called $(n-1)$-convexity in [24, 25, 15] (Here $(n-1)$-convexity is different from the above) or $(n-1)$-positivity in [32]. On the other hand, the left hand side of (1.4) is a combination of Weingarten curvatures, which is a natural curvature function of (η, n)-convex hypersurfaces. So it is interesting to consider the curvature equation (1.4) and its generalization (1.1).

In the complex setting, the corresponding Hessian type equation of (1.1) has been studied extensively. In particular, when $k = n$, it is called $(n-1)$ Monge-Ampère equation, which is related to the Gauduchon conjecture (see [8, §IV.5]) in complex geometry. The Gauduchon conjecture was solved by Székelyhidi-Tosatti-Weinkove [28]. For more references, we refer the reader to [6, 7, 21, 27, 29, 30] and references therein.

Compared to the work of Guan-Ren-Wang [14], the curvature estimate in Theorem 1.2 can be established without the assumption of "strong" convexity of solution. Precisely, we prove the desired estimate for (η, k)-convex hypersurface. Clearly, the (η, k)-convexity is the natural elliptic condition for equation (1.1).

To obtain the existence of (η, k)-convex hypersurface satisfying the prescribed curvature equation (1.1), we need two additional conditions on f as in [1, 31, 4]. The first condition is that there exist two positive constants $r_1 < 1 < r_2$ such that

$$\begin{aligned} f\left(X, \frac{X}{|X|}\right) &\geqslant \frac{C_n^k (n-1)^k}{r_1^k}, \quad \text{for } |X| = r_1; \\ f\left(X, \frac{X}{|X|}\right) &\leqslant \frac{C_n^k (n-1)^k}{r_2^k}, \quad \text{for } |X| = r_2, \end{aligned} \tag{1.5}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. The second one is that for any fixed unit vector ν,

$$\frac{\partial}{\partial \rho}\left(\rho^k f(X, \nu)\right) \leqslant 0, \tag{1.6}$$

where $\rho = |X|$.

Theorem 1.3. *Let $f \in C^2\left((\overline{B}_{r_2} \setminus B_{r_1}) \times \mathbb{S}^n\right)$ be a positive function satisfying conditions* (1.5) *and* (1.6)*. Then equation* (1.1) *has a unique $C^{3,\alpha}$ star-shaped (η, k)-convex solution M in $\{r_1 \leqslant |X| \leqslant r_2\}$ for any $\alpha \in (0, 1)$.*

We now discuss the proof of Theorem 1.2. To prove the curvature estimate, we apply the maximum principle to a quantity involving the logarithm of the largest principal curvature. Since the right hand side f depends on ν, there are more troublesome terms when we differentiate the equation (1.1). We overcome this difficulty by using some properties of the operator σ_k. The second difficulty is how to deal with bad third order terms. Our approach is to establish the partial curvature estimate which is very useful to analyze the concavity of the operator $\sigma_k^{1/k}$. This gives us more good third order terms, which is enough to control the bad third order terms.

Next, we give two applications of the above idea. The first application is the C^2 estimate for the corresponding Hessian type equation in Euclidean domains. Let Ω be a bounded domain in $\mathbb{R}^n$. For $\varphi \in C^2(\Omega)$, we define

$$\eta_{ij} = (\Delta \varphi)\delta_{ij} - \varphi_{ij}.$$

The function φ is called (η, k)-convex if the eigenvalues $\lambda(\eta)$ of η_{ij} is in Γ_k for all $x \in \Omega$. We consider the following equation

$$\sigma_k(\lambda(\eta)) = f(x, \varphi, \nabla \varphi), \quad \text{in } \Omega, \tag{1.7}$$

where f is a positive function defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

For equation (1.7), when f is independent of $\nabla \varphi$, the C^2 estimate was proved by Caffarelli-Nirenberg-Spruck [3], where they treated a general class of fully nonlinear equations. When f depends on $\nabla \varphi$, equation (1.7) falls into the setup of [9] (see also [11]), and C^2 estimate was obtained under the convexity assumption of f on $\nabla \varphi$. In the following theorem, we remove this assumption.

Theorem 1.4. *Let $\varphi \in C^4(\Omega)$ be a (η, k)-convex solution of* (1.7). *Then there exists a constant C depending only on n, k, $\|\varphi\|_{C^1}$, $\inf f$, $\|f\|_{C^2}$ and Ω such that*

$$\sup_{\Omega} |\nabla^2 \varphi| \leqslant C\big(1 + \sup_{\partial\Omega} |\nabla^2 \varphi|\big).$$

In this paper, we omit the proof of Theorem 1.4 since it is almost identical to that of Theorem 1.2.

The second application is an interior C^2 estimate for the following Dirichlet problem

$$\begin{cases} \sigma_k(\lambda(\eta)) = f(x, \varphi, \nabla\varphi) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Theorem 1.5. *For the Dirichlet problem (1.8), there exists a constant C and β depending only on n, k, $\|\varphi\|_{C^1}$, $\inf f$, $\|f\|_{C^2}$ and Ω such that*

$$\sup_{\Omega} \Big[(-\varphi)^{\beta} \Delta\varphi\Big] \leqslant C.$$

For k-Hessian equation, when f is independent of $\nabla\varphi$, the interior C^2 estimate was established by Pogorelov [20] for $k = n$ and Chou-Wang [5] for general k. When f depends on $\nabla\varphi$, Li-Ren-Wang [19] proved such estimate for $(k+1)$-convex solution (if $k = 2$, the 3-convexity condition can be replaced by 2-convexity condition).

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2. Preliminaries

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. A hypersurface $M \subset \mathbb{R}^{n+1}$ is called star-shaped if it is a radial graph of $\mathbb{S}^n$ for some positive function ρ. Thus, for $x \in \mathbb{S}^n$, $X(x) = \rho(x)x$ is the position vector. We have the following expressions of g_{ij}, h_{ij} and ν (see e.g. [13, p.1952])

$$g_{ij} = \rho^2 \hat{g}_{ij} + \rho_i \rho_j, \quad h_{ij} = \frac{\rho^2 \hat{g}_{ij} + 2\rho_i \rho_j - \rho\rho_{ij}}{\sqrt{\rho^2 + |\nabla\rho|^2}} \tag{2.1}$$

and

$$\nu = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}, \tag{2.2}$$

where $\hat{g}$ and ∇ denote the standard metric and the gradient on $\mathbb{S}^n$ respectively.

On the other hand, for $X_0 \in M$, let $\{e_1, e_2, \cdots, e_n\}$ be a local orthonormal frame near X_0. The following formulas are well-known:

$$\begin{aligned} \text{Guass formula}: &\quad X_{ij} = -h_{ij}\nu, \\ \text{Weingarten equation}: &\quad \nu_i = h_{ij}e_j, \\ \text{Codazzi formula}: &\quad h_{ijk} = h_{ikj}, \\ \text{Guass equation}: &\quad R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \end{aligned}$$

the derivatives here are covariant derivatives, and

$$h_{ijkl} = h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \tag{2.3}$$

where R_{ijkl} is the curvature tensor of M.

3. Curvature estimate

In this section, we give the proof of Theorem 1.2. When $k = 1$, Theorem 1.2 follows from classical theory of quasi-linear PDEs. So we assume that $k \geqslant 2$ in the following sections.

To prove Theorem 1.2, we define a function $u = \langle X, \nu\rangle$. By (2.2), it is clear that

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{3.1}$$

Then there exists a positive constant C depending on $\inf_M \rho$ and $\|\rho\|_{C^1}$ such that

$$\frac{1}{C} \leqslant \inf_M u \leqslant u \leqslant \sup_M u \leqslant C.$$

Let $\kappa_{\max}$ be the largest principal curvature. From $\eta \in \Gamma_k \subset \Gamma_1$, we see that the mean curvature is positive. It suffices to prove $\kappa_{\max}$ is uniformly bounded from above. Without loss of generality, we may assume that the set $D = \{\kappa_{\max} > 0\}$ is not empty. On D, we consider the following function

$$Q = \log\kappa_{\max} - \log(u - a) + \frac{A}{2}|X|^2,$$

where $a = \frac{1}{2}\inf_M u > 0$ and $A > 1$ is a constant to be determined later. Note that Q is continuous on D, and goes to $-\infty$ on ∂D. Hence Q achieves a maximum at a point X_0 with $\kappa_{\max}(X_0) > 0$. We choose a local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ near X_0 such that

$$h_{ij} = \delta_{ij}h_{ii} \quad \text{and} \quad h_{11} \geqslant h_{22} \geqslant \cdots \geqslant h_{nn} \quad \text{at } X_0.$$

Recalling that $\eta_{ii} = \sum_{k\neq i} h_{kk}$, we have

$$\eta_{11} \leqslant \eta_{22} \leqslant \cdots \leqslant \eta_{nn}.$$

Near X_0, we define a new function $\hat{Q}$ by

$$\hat{Q} = \log h_{11} - \log(u - a) + \frac{A}{2}|X|^2.$$

Since $h_{11}(X_0) = \kappa_{\max}(X_0)$ and $h_{11} \leqslant \kappa_{\max}$ near X_0, $\hat{Q}$ achieves a maximum at X_0. From now on, all the calculations will be carried out at X_0. For convenience, we introduce the following notations:

$$G(\eta) = \sigma_k^{\frac{1}{k}}(\eta), \quad F(\kappa) = G(\eta(\kappa)),$$

$$G^{ij} = \frac{\partial G}{\partial \eta_{ij}}, \quad G^{ij,kl} = \frac{\partial^2 G}{\partial \eta_{ij} \partial \eta_{kl}} \text{ and } F^{ii} = \sum_{k \neq i} G^{kk}.$$

Thus,

$$G^{ii} = \frac{1}{k}[\sigma_k(\eta)]^{\frac{1}{k}-1}\sigma_{k-1}(\eta|i),$$

where $\sigma_{k-1}(\eta|i)$ denotes $(k-1)$-th elementary symmetric function with $\eta_{ii} = 0$. It then follows that

$$G^{11} \geqslant G^{22} \geqslant \cdots \geqslant G^{nn}, \quad F^{11} \leqslant F^{22} \leqslant \cdots \leqslant F^{nn}.$$

Applying the maximum principle, for any $1 \leqslant i \leqslant n$, we have

$$0 = \hat{Q}_i = \frac{h_{11i}}{h_{11}} - \frac{u_i}{u-a} + A\langle X, e_i \rangle \tag{3.2}$$

and

$$0 \geqslant F^{ii}\hat{Q}_{ii} = F^{ii}(\log h_{11})_{ii} - F^{ii}(\log(u-a))_{ii} + \frac{A}{2}F^{ii}(|X|^2)_{ii}. \tag{3.3}$$

We first need to estimate each term in (3.3).

Lemma 3.1. *We have at X_0,*

$$\begin{aligned} 0 \geqslant & -\frac{2}{h_{11}}\sum_{i \geqslant 2} G^{1i,i1}h_{11i}^2 - \frac{F^{ii}h_{11i}^2}{h_{11}^2} \\ & + \frac{aF^{ii}h_{ii}^2}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2} - Ch_{11} + A\sum_i F^{ii} - CA. \end{aligned}$$

Proof. We first deal with the term $\frac{A}{2}F^{ii}(|X|^2)_{ii}$ in (3.3). Since $\eta_{ii} = \sum_{j \neq i} h_{jj}$, we have

$$\sum_i \eta_{ii} = (n-1)\sum_i h_{ii}, \quad h_{ii} = \frac{1}{n-1}\sum_k \eta_{kk} - \eta_{ii}.$$

It then follows that

$$\begin{aligned} \sum_i F^{ii}h_{ii} &= \sum_i \left(\sum_k G^{kk} - G^{ii}\right)\left(\frac{1}{n-1}\sum_l \eta_{ll} - \eta_{ii}\right) \\ &= \sum_i G^{ii}\eta_{ii} = \frac{1}{k}[\sigma_k(\eta)]^{\frac{1}{k}-1}\sum_i \eta_{ii}\sigma_{k-1}(\eta|i) = f^{\frac{1}{k}}. \end{aligned} \tag{3.4}$$

Combining this with Gauss formula, we obtain

$$
\begin{aligned}
\frac{A}{2}F^{ii}(|X|^2)_{ii} &= A\sum_i F^{ii}(1+\langle X, X_{ii}\rangle) \\
&= A\sum_i F^{ii}(1-h_{ii}\langle X,\nu\rangle) \\
&= A\sum_i F^{ii} - Auf^{\frac{1}{k}}.
\end{aligned}
\tag{3.5}
$$

For the term $-F^{ii}(\log(u-a))_{ii}$ in (3.3), we compute

$$
-F^{ii}(\log(u-a))_{ii} = -\frac{F^{ii}u_{ii}}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2}.
$$

Using Guass formula, Weingarten equation and Codazzi formula,

$$
u_i = h_{ii}\langle X, e_i\rangle, \quad u_{ii} = \sum_k h_{iik}\langle X, e_k\rangle - uh_{ii}^2 + h_{ii}.
$$

It then follows that

$$
\begin{aligned}
&-F^{ii}(\log(u-a))_{ii} \\
=& -\frac{1}{u-a}\sum_k F^{ii}h_{iik}\langle X, e_k\rangle + \frac{uF^{ii}h_{ii}^2}{u-a} - \frac{F^{ii}h_{ii}}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2} \\
=& -\frac{1}{u-a}\sum_k F^{ii}h_{iik}\langle X, e_k\rangle + \frac{uF^{ii}h_{ii}^2}{u-a} - \frac{f^{\frac{1}{k}}}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2},
\end{aligned}
\tag{3.6}
$$

where we used (3.4) in the last line. By the definitions of F^{ii} and η_{ii}, we have

$$
\begin{aligned}
F^{ii}h_{iik} &= \left(\sum_j G^{jj} - G^{ii}\right)h_{iik} \\
&= \left(\sum_i G^{ii}\right)H_k - \sum_i G^{ii}h_{iik} = G^{ii}\eta_{iik}.
\end{aligned}
\tag{3.7}
$$

On the other hand, the curvature equation (1.1) can be written as

$$
G(\eta) = \tilde{f}, \tag{3.8}
$$

where $\tilde{f} = f^{\frac{1}{k}}$. Differentiating (3.8), we obtain

$$
G^{ii}\eta_{iik} = (d_X\tilde{f})(e_k) + h_{kk}(d_\nu\tilde{f})(e_k).
$$

Then (3.7) gives

$$
F^{ii}h_{iik} = (d_X\tilde{f})(e_k) + h_{kk}(d_\nu\tilde{f})(e_k). \tag{3.9}
$$

It then follows that

$$
-\frac{1}{u-a}\sum_k F^{ii}h_{iik}\langle X, e_k\rangle \geqslant -\frac{1}{u-a}\sum_k h_{kk}(d_\nu\tilde{f})(e_k)\langle X, e_k\rangle - C.
$$

Substituting this into (3.6), we have

$$
\begin{aligned}
-F^{ii}(\log(u-a))_{ii} \geqslant & -\frac{1}{u-a}\sum_k h_{kk}(d_\nu \tilde{f})(e_k)\langle X, e_k\rangle \\
& + \frac{uF^{ii}h_{ii}^2}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2} - C.
\end{aligned} \tag{3.10}
$$

For the term $F^{ii}(\log h_{11})_{ii}$ in (3.3), we compute

$$
F^{ii}(\log h_{11})_{ii} = \frac{F^{ii}h_{11ii}}{h_{11}} - \frac{F^{ii}h_{11i}^2}{h_{11}^2}. \tag{3.11}
$$

By (2.3) and (3.4), we have

$$
\begin{aligned}
F^{ii}h_{11ii} &= F^{ii}h_{ii11} + F^{ii}(h_{i1}^2 - h_{ii}h_{11})h_{ii} + F^{ii}(h_{ii}h_{11} - h_{i1}^2)h_{11} \\
&= F^{ii}h_{ii11} - F^{ii}h_{ii}^2 h_{11} + F^{ii}h_{ii}h_{11}^2 \\
&= F^{ii}h_{ii11} - F^{ii}h_{ii}^2 h_{11} + f^{\frac{1}{k}}h_{11}^2.
\end{aligned}
$$

Differentiating (3.8) twice and using the similar argument of (3.7), we obtain

$$
F^{ii}h_{ii11} = G^{ii}\eta_{ii11} \geqslant -G^{ij,kl}\eta_{ij1}\eta_{kl1} + \sum_k h_{k11}(d_\nu \tilde{f})(e_k) - Ch_{11}^2 - C.
$$

Applying the concavity of G and Codazzi formula, we have

$$
-G^{ij,kl}\eta_{ij1}\eta_{kl1} \geqslant -2\sum_{i\geqslant 2} G^{1i,i1}\eta_{1i1}^2 = -2\sum_{i\geqslant 2} G^{1i,i1}h_{1i1}^2 = -2\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2.
$$

It then follows that

$$
F^{ii}h_{11ii} \geqslant -2\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2 + \sum_k h_{k11}(d_\nu \tilde{f})(e_k) - F^{ii}h_{ii}^2 h_{11} - Ch_{11}^2 - C.
$$

Substituting this into (3.11),

$$
\begin{aligned}
F^{ii}(\log h_{11})_{ii} \geqslant & -\frac{2}{h_{11}}\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2 + \frac{1}{h_{11}}\sum_k h_{k11}(d_\nu \tilde{f})(e_k) \\
& - \frac{F^{ii}h_{11i}^2}{h_{11}^2} - F^{ii}h_{ii}^2 - Ch_{11}.
\end{aligned} \tag{3.12}
$$

Combining (3.3), (3.5), (3.10) and (3.12), we obtain

$$
\begin{aligned}
0 \geqslant F^{ii}\hat{Q}_{ii} \geqslant & -\frac{2}{h_{11}}\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2 - \frac{F^{ii}h_{11i}^2}{h_{11}^2} \\
& + \frac{1}{h_{11}}\sum_k h_{k11}(d_\nu \tilde{f})(e_k) - \frac{1}{u-a}\sum_k h_{kk}(d_\nu \tilde{f})(e_k)\langle X, e_k\rangle \\
& + \frac{aF^{ii}h_{ii}^2}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2} - Ch_{11} + A\sum_i F^{ii} - CA.
\end{aligned}
$$

By Codazzi formula, $u_k = h_{kk}\langle X, e_k\rangle$ and (3.2), we have

$$
\begin{aligned}
&\frac{1}{h_{11}}\sum_k h_{k11}(d_\nu \tilde{f})(e_k) - \frac{1}{u-a}\sum_k h_{kk}(d_\nu \tilde{f})(e_k)\langle e_k, X\rangle \\
={}& \sum_k \left(\frac{h_{11k}}{h_{11}} - \frac{u_k}{u-a}\right)(d_\nu \tilde{f})(e_k) = -A\sum_k (d_\nu \tilde{f})(e_k)\langle X, e_k\rangle \geqslant -CA.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 \geqslant{} & -\frac{2}{h_{11}}\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2 - \frac{F^{ii}h_{11i}^2}{h_{11}^2} \\
& + \frac{aF^{ii}h_{ii}^2}{u-a} + \frac{F^{ii}u_i^2}{(u-a)^2} - Ch_{11} + A\sum_i F^{ii} - CA,
\end{aligned}
$$

as required. □

Next, we deal with the bad term $-Ch_{11}$.

Lemma 3.2. *If h_{11} and A are sufficiently large, then we have*

$$
Ch_{11} \leqslant \frac{aF^{ii}h_{ii}^2}{2(u-a)} + \frac{A}{2}\sum_i F^{ii} \quad \text{at } X_0.
$$

Proof. The proof splits into two cases. The positive constant δ will be determined later.

Case 1. $|h_{ii}| \leqslant \delta h_{11}$ for all $i \geqslant 2$.

In this case, we have

$$
|\eta_{11}| \leqslant (n-1)\delta h_{11}
$$

and

$$
[1-(n-2)\delta]h_{11} \leqslant \eta_{22} \leqslant \cdots \leqslant \eta_{nn} \leqslant [1+(n-2)\delta]h_{11}.
$$

It then follows that

$$
\sigma_{k-1}(\eta) = \sigma_{k-1}(\eta|1) + \eta_{11}\sigma_{k-2}(\eta|1) \geqslant (1-C\delta)h_{11}^{k-1} - C\delta h_{11}^{k-1}.
$$

Choosing δ sufficiently small and using $k \geqslant 2$,

$$
\sigma_{k-1}(\eta) \geqslant \frac{h_{11}^{k-1}}{2} \geqslant \frac{h_{11}}{2}. \tag{3.13}
$$

By the definition of G^{ii} and F^{ii}, we obtain

$$
\sum_i F^{ii} = (n-1)\sum_i G^{ii} = \frac{(n-1)(n-k+1)}{k}[\sigma_k(\eta)]^{\frac{1}{k}-1}\sigma_{k-1}(\eta). \tag{3.14}
$$

Thanks to $\sigma_k(\eta) = f$, we have

$$
\sum_i F^{ii} = \frac{(n-1)(n-k+1)}{k} f^{\frac{1}{k}-1}\sigma_{k-1}(\eta) \geqslant \frac{\sigma_{k-1}(\eta)}{C}.
$$

Combining this with (3.13), and choosing A sufficiently large, we obtain

$$Ch_{11} \leqslant C\sigma_{k-1}(\eta) \leqslant \frac{A}{2}\sum_i F^{ii},$$

as required.

Case 2. $h_{22} > \delta h_{11}$ or $h_{nn} < -\delta h_{11}$.

In this case, we have

$$\frac{aF^{ii}h_{ii}^2}{2(u-a)} \geqslant \frac{1}{C}(F^{22}h_{22}^2 + F^{nn}h_{nn}^2) \geqslant \frac{\delta^2}{C}F^{22}h_{11}^2.$$

By the definitions of F^{ii} and G^{ii},

$$F^{22} = \sum_{i\neq 2} G^{ii} \geqslant G^{11} \geqslant \frac{1}{n}\sum_i G^{ii} = \frac{1}{n(n-1)}\sum_i F^{ii}. \tag{3.15}$$

It then follows that

$$\frac{aF^{ii}h_{ii}^2}{2(u-a)} \geqslant \frac{\delta^2 h_{11}^2}{C}\sum_i F^{ii}. \tag{3.16}$$

Using (3.14) and Maclaurin's inequality,

$$\sum_i F^{ii} = \frac{(n-1)(n-k+1)}{k}[\sigma_k(\eta)]^{\frac{1}{k}-1}\sigma_{k-1}(\eta) \geqslant \frac{1}{C}. \tag{3.17}$$

Combining this with (3.16), if $h_{11} \geqslant \frac{C}{\delta^2}$, we obtain

$$Ch_{11} \leqslant \frac{\delta^2 h_{11}^2}{C}\sum_i F^{ii} \leqslant \frac{aF^{ii}h_{ii}^2}{2(u-a)},$$

as required. □

Combining Lemma 3.1 and 3.2, we obtain

$$\begin{aligned} 0 \geqslant & -\frac{2}{h_{11}}\sum_{i\geqslant 2} G^{1i,i1}h_{11i}^2 - \frac{F^{ii}h_{11i}^2}{h_{11}^2} \\ & + \frac{aF^{ii}h_{ii}^2}{2(u-a)} + \frac{F^{ii}u_i^2}{(u-a)^2} + \frac{A}{2}\sum_i F^{ii} - CA. \end{aligned} \tag{3.18}$$

The following lemma can be regarded as the partial curvature estimate.

Lemma 3.3. *If h_{11} and A are sufficiently large, then we have*

$$|h_{ii}| \leqslant CA \ \text{at}\ X_0, \quad \text{for } i \geqslant 2.$$

Proof. Using (3.2) and the Cauchy-Schwarz inequality,

$$\frac{F^{ii}h_{11i}^2}{h_{11}^2} \leqslant \frac{1+\varepsilon}{(u-a)^2}F^{ii}u_i^2 + \left(1+\frac{1}{\varepsilon}\right)A^2F^{ii}\langle X, e_i\rangle^2.$$

Substituting this into (3.18) and dropping the positive term $-\frac{2}{h_{11}}\sum_{i\geqslant 2}G^{1i,i1}h_{11i}^2$, we have

$$\begin{aligned} 0 &\geqslant \frac{aF^{ii}h_{ii}^2}{2(u-a)} - \frac{\varepsilon F^{ii}u_i^2}{(u-a)^2} - \frac{CA^2}{\varepsilon}\sum_i F^{ii} - CA \\ &\geqslant \left(\frac{a}{2(u-a)} - \frac{C\varepsilon}{(u-a)^2}\right)F^{ii}h_{ii}^2 - \frac{CA^2}{\varepsilon}\sum_i F^{ii} - CA, \end{aligned}$$

where we used $u_i = h_{ii}\langle X, e_i\rangle$ in the second inequality. Choosing ε sufficiently small and recalling (3.17), we obtain

$$0 \geqslant \frac{F^{ii}h_{ii}^2}{C} - \frac{CA^2}{\varepsilon}\sum_i F^{ii}. \tag{3.19}$$

By (3.15), for $i \geqslant 2$, we have

$$F^{ii} \geqslant F^{22} \geqslant \frac{1}{n(n-1)}\sum_k F^{kk}.$$

Then (3.19) gives

$$0 \geqslant \frac{1}{C}\left(\sum_i F^{ii}\right)\left(\sum_{k\geqslant 2} h_{kk}^2\right) - \frac{CA^2}{\varepsilon}\sum_i F^{ii}.$$

which implies

$$\sum_{k\geqslant 2} h_{kk}^2 \leqslant CA^2,$$

as required. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By (3.2) and the Cauchy-Schwarz inequality, we have

$$\frac{F^{11}h_{111}^2}{h_{11}^2} \leqslant \frac{1+\varepsilon}{(u-a)^2}F^{11}u_1^2 + \left(1+\frac{1}{\varepsilon}\right)A^2F^{11}\langle X, e_1\rangle^2.$$

Recalling $u_1 = h_{11}\langle \partial_1, X\rangle$ and choosing ε sufficiently small,

$$\begin{aligned} \frac{F^{11}h_{111}^2}{h_{11}^2} &\leqslant \frac{F^{11}u_1^2}{(u-a)^2} + \frac{\varepsilon F^{11}u_1^2}{(u-a)^2} + \frac{CA^2F^{11}}{\varepsilon} \\ &\leqslant \frac{F^{11}u_1^2}{(u-a)^2} + \frac{C\varepsilon F^{11}h_{11}^2}{(u-a)^2} + \frac{CA^2F^{11}}{\varepsilon} \\ &\leqslant \frac{F^{11}u_1^2}{(u-a)^2} + \frac{aF^{ii}h_{ii}^2}{16(u-a)} + \frac{CA^2F^{11}}{\varepsilon}. \end{aligned}$$

Without loss of generality, we assume that $h_{11}^2 \geqslant \frac{C_1 A^2}{\varepsilon}$ for some $C_1 > 0$ sufficiently large, which implies

$$\frac{CA^2F^{11}}{\varepsilon} \leqslant \frac{aF^{ii}h_{ii}^2}{16(u-a)}.$$

It then follows that

$$\frac{F^{11}h_{111}^2}{h_{11}^2} \leqslant \frac{F^{11}u_1^2}{(u-a)^2} + \frac{aF^{ii}h_{ii}^2}{8(u-a)}. \tag{3.20}$$

Thanks to Lemma 3.3, we assume that $|h_{ii}| \leqslant \delta h_{11}$ for $i \geqslant 2$, where δ is a constant to be determined later. Thus,

$$\frac{1}{h_{11}} \leqslant \frac{1+\delta}{h_{11}-h_{ii}}.$$

Combining this with $-G^{1i,i1} = \frac{G^{11}-G^{ii}}{\eta_{ii}-\eta_{11}}$ for $i \geqslant 2$, we obtain

$$\begin{aligned}
\sum_{i\geqslant 2} \frac{F^{ii}h_{11i}^2}{h_{11}^2} &= \sum_{i\geqslant 2} \frac{F^{ii}-F^{11}}{h_{11}^2} h_{11i}^2 + \sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \\
&\leqslant \frac{1+\delta}{h_{11}} \sum_{i\geqslant 2} \frac{F^{ii}-F^{11}}{h_{11}-h_{ii}} h_{11i}^2 + \sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \\
&= \frac{1+\delta}{h_{11}} \sum_{i\geqslant 2} \frac{G^{11}-G^{ii}}{\eta_{ii}-\eta_{11}} h_{11i}^2 + \sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \\
&= -\frac{1+\delta}{h_{11}} \sum_{i\geqslant 2} G^{1i,i1} h_{11i}^2 + \sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2}.
\end{aligned} \tag{3.21}$$

Using (3.2), the Cauchy-Schwarz inequality and $u_i = h_{ii}\langle X, e_i\rangle$, we have

$$\begin{aligned}
\sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} &\leqslant 2\sum_{i\geqslant 2} \frac{F^{11}u_i^2}{(u-a)^2} + 2A^2 \sum_{i\geqslant 2} F^{11}\langle X, e_i\rangle^2 \\
&\leqslant C\sum_{i\geqslant 2} F^{11}h_{ii}^2 + CA^2F^{11}.
\end{aligned}$$

Recalling that $|h_{ii}| \leqslant \delta h_{11}$ for $i \geqslant 2$ and choosing δ sufficiently small, we see that

$$C\sum_{i\geqslant 2} F^{11}h_{ii}^2 + CA^2F^{11} \leqslant \frac{aF^{11}h_{11}^2}{8(u-a)}.$$

It then follows that

$$\sum_{i\geqslant 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \leqslant \frac{aF^{11}h_{11}^2}{8(u-a)}.$$

Combining this with (3.21),

$$\begin{aligned}\sum_{i\geqslant 2}\frac{F^{ii}h_{11i}^2}{h_{11}^2} &\leqslant -\frac{1+\delta}{h_{11}}\sum_{i\geqslant 2}G^{1i,i1}h_{11i}^2+\frac{aF^{11}h_{11}^2}{8(u-a)}\\ &\leqslant -\frac{2}{h_{11}}\sum_{i\geqslant 2}G^{1i,i1}h_{11i}^2+\frac{aF^{11}h_{11}^2}{8(u-a)}.\end{aligned}\tag{3.22}$$

Substituting (3.20) and (3.22) into (3.18), we obtain

$$\begin{aligned}0 &\geqslant \frac{aF^{ii}h_{ii}^2}{4(u-a)}+\frac{A}{2}\sum_i F^{ii}-CA\\ &= \frac{aF^{ii}h_{ii}^2}{4(u-a)}+\frac{(n-1)A}{2}\sum_i G^{ii}-CA.\end{aligned}\tag{3.23}$$

It then follows that

$$\sum_i G^{ii}\leqslant C.$$

Combining this with [16, Lemma 2.2], we obtain

$$F^{11}=\sum_{i\neq 1}G^{ii}\geqslant \frac{1}{C}.$$

Substituting this into (3.23), we obtain

$$h_{11}\leqslant C,$$

as required. □

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. Since M is star-shaped, we assume that M is the radial graph of positive function ρ on $\mathbb{S}^n$. So $X(x)=\rho(x)x$ is the position vector of M. We first prove the gradient estimate.

Lemma 4.1. *Suppose that f satisfies* (1.6) *and ρ has positive lower and upper bounds. Then there exists a constant C depending only on n, k, $\inf_M\rho$, $\sup_M\rho$, $\inf f$ and $\|f\|_{C^1}$ such that*

$$|\nabla\rho|\leqslant C,\tag{4.1}$$

where ∇ denotes the gradient on $\mathbb{S}^n$.

Proof. By (3.1), it suffices to obtain a positive lower bound of u. As in [13], we consider the following quantity

$$w=-\log u+\gamma(|X|^2),$$

where γ is a single-variable function to be determined later. Suppose that the maximum of w is achieved at $X_0\in M$. If the vector X_0 is not parallel to the outer normal vector of M at X_0, we can choose the local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ near X_0 such that

$$\langle X, e_1\rangle \neq 0 \ \text{ and } \ \langle X, e_i\rangle = 0 \ \text{ for } i \geqslant 2. \tag{4.2}$$

Using Weingarten equation, we obtain

$$u_i = \sum_j h_{ij}\langle X, e_j\rangle = h_{i1}\langle X, e_1\rangle.$$

Therefore, at X_0, we have

$$0 = w_i = -\frac{u_i}{u} + 2\gamma'\langle X, e_i\rangle = -\frac{h_{i1}\langle X, e_1\rangle}{u} + 2\gamma'\langle X, e_i\rangle, \tag{4.3}$$

so that $h_{11} = 2\gamma' u$ and $h_{1i} = 0$ for $i \geqslant 2$. Furthermore, after rotating $\{e_2, e_3, \cdots, e_n\}$, we assume that $(h_{ij}(X_0))$ is diagonal. For convenience, we use the same notations as in Section 3. By (4.3), we obtain $\frac{u_i^2}{u^2} = 4(\gamma')^2\langle X, e_i\rangle^2$. Using the maximum principle,

$$\begin{aligned}
0 \geqslant\ & F^{ii}w_{ii} \\
=\ & F^{ii}\left(-\frac{u_{ii}}{u} + \frac{u_i^2}{u^2} + \gamma''(|X|^2)_i^2 + \gamma'(|X|^2)_{ii}\right) \\
=\ & -\frac{F^{ii}u_{ii}}{u} + 4(\gamma')^2F^{ii}\langle X, e_i\rangle^2 + 4\gamma''F^{ii}\langle X, e_i\rangle^2 + \gamma'F^{ii}(|X|^2)_{ii} \\
=\ & -\frac{F^{ii}u_{ii}}{u} + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + \gamma'F^{ii}(|X|^2)_{ii}.
\end{aligned} \tag{4.4}$$

To deal with terms $-\frac{F^{ii}u_{ii}}{u}$ and $\gamma'F^{ii}(|X|^2)_{ii}$, we apply the similar argument of Lemma 3.1 and obtain

$$F^{ii}u_{ii} = \langle X, e_1\rangle(d_X\tilde{f})(e_1) + h_{11}\langle X, e_1\rangle(d_\nu\tilde{f})(e_1) - uF^{ii}h_{ii}^2 + \tilde{f}$$

and

$$F^{ii}(|X|^2)_{ii} = 2\sum_i F^{ii} - 2u\tilde{f},$$

where $\tilde{f} = f^{\frac{1}{k}}$. Substituting these into (4.4) and using $h_{11} = 2\gamma' u$,

$$\begin{aligned}
0 \geqslant\ & -\frac{1}{u}\left(\langle X, e_1\rangle(d_X\tilde{f})(e_1) + \tilde{f}\right) - 2\gamma'\langle X, e_1\rangle(d_\nu\tilde{f})(e_1) \\
& + F^{ii}h_{ii}^2 + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + 2\gamma'\sum_i F^{ii} - 2u\gamma'\tilde{f}.
\end{aligned} \tag{4.5}$$

By (4.2), at X_0, we have

$$X = \langle X, e_1\rangle e_1 + \langle X, \nu\rangle\nu,$$

which implies

$$(d_X\tilde{f})(X) = \langle X, e_1\rangle(d_X\tilde{f})(e_1) + \langle X, \nu\rangle(d_X\tilde{f})(\nu). \tag{4.6}$$

From (1.6), (4.6) and $X(x) = \rho(x)x$, we see that

$$
\begin{aligned}
0 &\geqslant \frac{\partial}{\partial\rho}\left(\rho^k f(X,\nu)\right) = \frac{\partial}{\partial\rho}\left(\rho^k \tilde{f}^k(X,\nu)\right) \\
&= k(\rho\tilde{f})^{k-1}\left(\tilde{f} + (d_X\tilde{f})(X)\right) \\
&= k(\rho\tilde{f})^{k-1}\left(\tilde{f} + \langle X, e_1\rangle(d_X\tilde{f})(e_1) + \langle X,\nu\rangle(d_X\tilde{f})(\nu)\right).
\end{aligned}
$$

It then follows that

$$
-\left(\langle X, e_1\rangle(d_X\tilde{f})(e_1) + \tilde{f}\right) \geqslant \langle X,\nu\rangle(d_X\tilde{f})(\nu) = u(d_X\tilde{f})(\nu).
$$

Substituting this into (4.5), we obtain

$$
\begin{aligned}
0 \geqslant\; & (d_X\tilde{f})(\nu) - 2\gamma'\langle X, e_1\rangle(d_\nu\tilde{f})(e_1) \\
& + F^{ii}h_{ii}^2 + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + 2\gamma'\sum_i F^{ii} - 2u\gamma'\tilde{f}.
\end{aligned}
\tag{4.7}
$$

It is clear that

$$
|X|^2 = \rho^2 \geqslant \inf_M \rho^2 > 0.
$$

Without loss of generality, we assume that

$$
\langle X, e_1\rangle^2 \geqslant \frac{1}{2}\inf_M \rho^2. \tag{4.8}
$$

Otherwise we obtain the positive lower bound of u at X_0 directly:

$$
u^2 = \langle X,\nu\rangle^2 = |X|^2 - \langle X, e_1\rangle^2 \geqslant \frac{1}{2}\inf_M \rho^2.
$$

Now we choose $\gamma(t) = \frac{\alpha}{t}$, where α is a constant to be determined later. Recalling that $h_{11} = 2\gamma' u$ at X_0, we have $h_{11}(X_0) < 0$. Using $H > 0$, there exists $2 \leqslant k \leqslant n$ such that $h_{kk}(X_0) > 0$. Combining this with the definitions of η_{kk} and G^{kk},

$$
\eta_{kk} < \eta_{11} \text{ and } G^{kk} \geqslant G^{11}.
$$

It then follows that

$$
F^{11} = \sum_{j\neq 1} G^{jj} \geqslant \frac{1}{2}\sum_i G^{ii} = \frac{1}{2(n-1)}\sum_i F^{ii}. \tag{4.9}
$$

Combining (4.7), (4.8) and (4.9),

$$
0 \geqslant \left(\frac{\alpha^2}{C} - C\alpha\right)\sum_i F^{ii} - C\alpha,
$$

where C is a constant depending only on n, k, $\inf_M \rho$, $\sup_M \rho$ and $\|\tilde{f}\|_{C^1}$. Using (3.17) and choosing α sufficiently large, we obtain a contradiction. This shows that X is parallel to ν at X_0. Hence, at X_0, we obtain

$$
u = \langle X,\nu\rangle = \rho \geqslant \inf_M \rho,
$$

as required. $\square$

Now we use the continuity method as in [4] to prove Theorem 1.3.

Proof of Theorem 1.3. For $t \in [0,1]$, we consider the following family of functions

$$f^t(X,\nu) = tf(X,\nu) + (1-t)C_n^k(n-1)^k\left[\frac{1}{|X|^k} + \varepsilon\left(\frac{1}{|X|^k} - 1\right)\right],$$

where the constant ε is small sufficiently such that

$$\min_{r_1 \leqslant \rho \leqslant r_2}\left[\frac{1}{\rho^k} + \varepsilon\left(\frac{1}{\rho^k} - 1\right)\right] \geqslant c_0 > 0$$

for some positive constant c_0. It is easy to see that $f^t(X,\nu)$ satisfies (1.5) and (1.6) with strict inequalities for $0 \leqslant t < 1$.

Let M_t be the solution of the equation

$$\sigma_k(\lambda(\eta)) = f^t(X_t, \nu_t),$$

where X_t and ν_t are position vector and unit outer normal of M_t respectively. Clearly, when $t = 0$, we have $M_0 = \mathbb{S}^n$ and $X_0 = x$. For $t \in (0,1)$, suppose that x_0 is the maximum point of the function $\rho_t = |X_t|$. Thus at x_0, by (2.1), we have

$$g_{ij} = \rho_t^2\hat{g}_{ij}, \quad h_{ij} = -(\rho_t)_{ij} + \rho_t\hat{g}_{ij}.$$

It then follows that

$$\eta_{ij} = Hg_{ij} - h_{ij} \geqslant (n-1)\rho_t\hat{g}_{ij},$$

which implies

$$\sigma_k(\lambda(\eta)) \geqslant \sigma_k\left(\frac{n-1}{\rho_t}(1,\cdots,1)\right) = \frac{C_n^k(n-1)^k}{\rho_t^k}.$$

On the other hand, at x_0, the unit outer normal ν_t is parallel to X_t, i.e., $\nu_t = \frac{X_t}{|X_t|}$. If $\rho_t(x_0) = r_2$, then we obtain

$$\frac{C_n^k(n-1)^k}{r_2^k} > f\left(X_t, \frac{X_t}{|X_t|}\right) = f(X_t,\nu_t) = \sigma_k(\lambda(\eta)) \geqslant \frac{C_n^k(n-1)^k}{r_2^k},$$

which is a contradiction. This shows $\sup_{M_t}\rho_t \leqslant r_2$. Similarly, we get $\inf_{M_t}\rho_t \geqslant r_1$. Using Lemma 4.1 and Theorem 1.2, we obtain the C^1 and C^2 estimates. Higher order estimates follows from the Evans-Krylov theory. Applying the similar argument of [4], we get the existence and uniqueness of solution to the equation (1.1). $\square$

5. Proof of Theorem 1.5

Theorem 1.5 can be proved by the similar argument of Theorem 1.2. For the reader's convenience, we give a sketch here.

Let $\nabla^2\varphi = \{\varphi_{ij}\}$ denote the Hessian of φ. For convenience we shall use similar notations of Section 3:

$$\eta_{ij} = (\Delta\varphi)\delta_{ij} - \varphi_{ij}, \quad G(\eta) = \sigma_k^{1/k}(\lambda(\eta)), \quad F(\nabla^2\varphi) = G(\eta(\nabla^2\varphi)), \text{ etc.}$$

Proof of Theorem 1.5. By the maximum principle, we assume without loss of generality that $\varphi < 0$ in Ω. For $x \in \Omega$ and unit vector ξ, we consider the function

$$Q(x,\xi) = \log \varphi_{\xi\xi} + \frac{a}{2}|\nabla\varphi|^2 + \frac{A}{2}|x|^2 + \beta\log(-\varphi),$$

where a, A and β are constants to be determined later. Suppose that Q achieves its maximum at (x_0, ξ_0). Near x_0, we choose coordinate system such that

$$\xi_0 = (1, 0, \cdots, 0), \quad \varphi_{ij} = \delta_{ij}\varphi_{ii}, \quad \varphi_{11} \geqslant \varphi_{22} \geqslant \cdots \geqslant \varphi_{nn} \ \text{ at } x_0.$$

Thus the new function defined by

$$\hat{Q}(x) = \log \varphi_{11} + \frac{a}{2}|\nabla\varphi|^2 + \frac{A}{2}|x|^2 + \beta\log(-\varphi)$$

has a maximum at x_0. Thus,

$$0 = \frac{\varphi_{11i}}{\varphi_{11}} + a\varphi_i\varphi_{ii} + Ax_i + \frac{\beta\varphi_i}{\varphi}. \tag{5.1}$$

Using the similar notations in Section 3, at x_0, we compute (cf. Lemma 3.1 and 3.2)

$$\begin{aligned} 0 \geqslant & -\frac{2}{\varphi_{11}}\sum_{i\geqslant 2} G^{1i,i1}\varphi_{11i}^2 - \frac{F^{ii}\varphi_{11i}^2}{\varphi_{11}^2} + \frac{aF^{ii}\varphi_{ii}^2}{2} \\ & + \frac{A}{2}\sum_i F^{ii} - CA - \frac{\beta F^{ii}\varphi_i^2}{\varphi^2} + \frac{C\beta}{\varphi}. \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), and choosing the constant a sufficiently small, we obtain (cf. Lemma 3.3)

$$|\varphi_{ii}| \leqslant -\frac{C_{a,A,\beta}}{\varphi} \ \text{ for } i \geqslant 2,$$

where $C_{a,A}$ is a uniform constant depending on a, A and β. Without loss of generality, we assume that

$$|\varphi_{ii}| \leqslant \delta\varphi_{11} \ \text{ for } i \geqslant 2, \tag{5.3}$$

where δ is a constant to be determined later.

On the other hand, by (5.1) and the Cauchy-Schwarz inequality, we have

$$-\frac{F^{11}\varphi_{111}^2}{\varphi_{11}^2} \geqslant -Ca^2F^{11}\varphi_{11}^2 - CA^2F^{11} - \frac{C\beta^2F^{11}}{\varphi^2}$$

and

$$-\sum_{i\geqslant 2}\frac{\beta F^{ii}\varphi_i^2}{\varphi^2} \geqslant -\frac{3}{\beta}\sum_{i\geqslant 2}\frac{F^{ii}\varphi_{11i}^2}{\varphi_{11}^2} - \frac{Ca^2}{\beta}\sum_{i\geqslant 2}F^{ii}\varphi_{ii}^2 - \frac{CA^2}{\beta}\sum_{i\geqslant 2}F^{ii}.$$

Substituting these into (5.2),

$$
\begin{aligned}
0 \geqslant & -\frac{2}{\varphi_{11}} \sum_{i \geqslant 2} G^{1i,i1} \varphi_{11i}^2 - \left(1 + \frac{3}{\beta}\right) \sum_{i \geqslant 2} \frac{F^{ii} \varphi_{11i}^2}{\varphi_{11}^2} \\
& + \left(\frac{a}{2} - \frac{Ca^2}{\beta}\right) F^{ii} \varphi_{ii}^2 + \left(\frac{A}{2} - \frac{CA^2}{\beta}\right) \sum_i F^{ii} \\
& - Ca^2 F^{11} \varphi_{11}^2 - CA^2 F^{11} - \frac{C\beta^2 F^{11}}{\varphi^2} \\
& - CA - \frac{\beta F^{11} \varphi_1^2}{\varphi^2} + \frac{C\beta}{\varphi}.
\end{aligned}
$$

Choosing β sufficiently large and sufficiently small a if needed, we see that

$$
\begin{aligned}
0 \geqslant & -\frac{2}{\varphi_{11}} \sum_{i \geqslant 2} G^{1i,i1} \varphi_{11i}^2 - \left(1 + \frac{3}{\beta}\right) \sum_{i \geqslant 2} \frac{F^{ii} \varphi_{11i}^2}{\varphi_{11}^2} \\
& + \frac{aF^{ii} \varphi_{ii}^2}{4} - \left(CA^2 + \frac{C\beta^2}{\varphi^2}\right) F^{11} + \frac{C\beta}{\varphi} - CA.
\end{aligned}
$$

Using (5.3) and choosing δ sufficiently small, we obtain (cf. the first inequality of (3.22))

$$
\left(1 + \frac{3}{\beta}\right) \sum_{i \geqslant 2} \frac{F^{ii} \varphi_{11i}^2}{\varphi_{11}^2} \leqslant -\frac{2}{\varphi_{11}} \sum_{i \geqslant 2} G^{1i,i1} \varphi_{11i}^2 + \frac{a}{6} F^{ii} \varphi_{ii}^2 + \frac{C\beta^2 F^{11}}{\varphi^2}.
$$

It then follows that

$$
0 \geqslant \frac{F^{11} \varphi_{11}^2}{C} - \frac{CF^{11}}{\varphi^2} + \frac{C}{\varphi} - C. \tag{5.4}
$$

By (5.3) and [16, Lemma 2.2],

$$
F^{11} \varphi_{11}^2 \geqslant G^{nn} \varphi_{11}^2 \geqslant \frac{G^{nn} \eta_{nn} \varphi_{11}}{C} \geqslant \frac{\varphi_{11}}{C}.
$$

Combining this with (5.4), we obtain $(-\varphi)^\beta \varphi_{11} \leqslant C$, as required. □

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