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Sobolev space of functions valued in a monotone Banach family

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A R T I C L E   I N F O

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A B S T R A C T

We apply the metrical approach to Sobolev spaces, which arise in various evolution PDEs. Functions from those spaces are defined on an interval and take values in a nested family of Banach spaces. In this case we adapt the definition of Newtonian spaces. For a monotone family, we show the existence of weak derivative, obtain an isomorphism to the standard Sobolev space, and provide some scalar characteristics.

1. Introduction

Various applied problems in biology, materials science, mechanics, etc, involve PDEs with solution spaces with internal structure that changes over time. As examples, we review some of the recent research directions: [7] by M. L. Bernardi, G. A. Pozzi, and G. Savaré, [24] by F. Paronetto (equations on non-cylindrical domains), [28] by M. Vierling, [2] by A. Alphonse, C. M. Elliott, and B. Stinner (equations on evolving hypersurfaces), [23] by S. Meier and M. Böhm, [12] by J. Escher and D. Treutler (modeling of processes in a porous medium). It is common to all of the mentioned studies that solution spaces could be represented as sets of functions valued in a family of Banach spaces. However, different problems impose different requirements on the relations between spaces within families, for example, the existence of isomorphisms, embeddings, bounded operators and so on.

In this article, we consider Sobolev spaces associated with the above problems from the point of view of metric analysis. Although the family of Banach spaces cannot always be represented as a metric space, the metric definition of Sobolev classes remains meaningful. Such point of view on the studied spaces will make it possible to apply more universal and well-developed methods. In the 90s, several authors (L. Ambrosio [4], N.J. Korevaar and R.M. Schoen [22], Yu.G. Reshetnyak [26] and A. Ranjbar-Motlagh [25]) introduced and studied Sobolev spaces consisting of functions taking values in metric spaces. The case of functions defined

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on a non-Euclidean space is described by P. Hajłasz [14] and J. Heinonen, P. Koskela, N. Shanmugalingam, J.T. Tyson [19]. In [19] it was shown that all previously developed approaches are equivalent. For a detailed treatment and for references to the literature on the subject one may refer to the books [18] by J. Heinonen and [15] by P. Hajłasz and P. Koskela.

For our purposes, we adapt the following definition of Sobolev space (Newtonian spaces, for real-valued case see [27], and [19] for Banach-valued case). *Function $f : (\mathcal{M}, \rho) \to (\mathcal{N}, d)$ from the space $L^p(\mathcal{M}; \mathcal{N})$ belongs to $W^{1,p}(\mathcal{M}; \mathcal{N})$, if there exists scalar function $g \in L^p(\mathcal{M})$ such that*

$$d\big(f(\gamma(a)), f(\gamma(b))\big) \leq \sup_{\gamma} \int\limits_{\gamma} g \, d\sigma. \tag{1.1}$$

On the one hand, we have all the necessary objects to adapt this definition. On the other hand, in metric case, (1.1) allows us to introduce the reach theory of Sobolev-type spaces, including embedding theorems, Poincaré inequalities, and approximation technique (see [20]).

The evolution structure of a specific problem could be described with the help of the following objects. Let $\{X_t\}_{t\in(0,T)}$ be a family of Banach spaces, $(0, T) \subset \mathbb{R}$, and suppose that there is a set of operators $P(t, s) : X_s \to X_t$ for $t \geq s$. We consider functions with the property that $f(t) \in X_t$. Then, inequality (1.1) turns into

$$\|f(t) - P(t, s)f(s)\|_t \leq \int\limits_s^t g(\tau) \, d\tau,$$

and defines the space $W^{1,p}((0, T); \{X_t\})$.

The first natural question that arises from this definition is: what is the meaning of the function $g(t)$? In the case of a monotone family of reflexive spaces, the answer to this question is given by Theorem 4.5. Namely, under such assumptions, we can explicitly construct the weak derivative and show that its norm coincides with the smallest upper gradient of the original function.

In section 5 we establish the connection of the introduced space to the standard case. More precisely, suppose that there is a family of local isomorphisms $\Phi_t : X_t \to Y$. We are interested if there exists a global isomorphism between Sobolev spaces $W^{1,p}((0, T), \{X_t\})$ and $W^{1,p}((0, T), Y)$. Due to Theorem 5.9, the necessary and sufficient conditions for the existence of such an isomorphism are the close interconnection of Φ_t and the nesting operators $P(t, s)$.

In section 6, we discuss a possible scalar characterization of the introduced spaces. In particular, we make use of the approach by Yu. G. Reshetnyak, which has demonstrated its efficiency for functions valued in metric space. However, it seems that this method does not fully respond to our construction.

2. Sobolev space $W^{1,p}((0, T); \{X_t\})$

In this section, we give the definition of the main object – Sobolev functions valued in the family of Banach spaces. We also provide examples of families on which our methods can be applied.

2.1. Nested family of Banach spaces

Here, in bare outlines, we discuss the idea of defining the Sobolev space of functions valued in a nested family of Banach spaces. Let $(0, T) \subset \mathbb{R}$ be an interval, $\{X_t\}_{t\in(0,T)}$ be a family of Banach spaces, and $\{P(t, s)\}$ be a family of bounded operators $P(t, s) : X_s \to X_t$ (*nestings*) such that $P(t, r)P(r, s) = P(t, s)$, whenever $s \leq r \leq t$.

Consider a set $X_T := \bigcup_t X_t \times \{t\}$ and define an addition:

$$(x,s) + (y,t) := \begin{cases} (P(t,s)x + y, t), & \text{if } s \leq t, \\ (x + P(s,t)y, s), & \text{if } s > t, \end{cases} \tag{2.1}$$

and multiplication by scalars $\alpha \cdot (x,s) := (\alpha x, s)$. One can show that those operations satisfy the associativity, commutativity, and distributivity properties. So the pair $(X_T, +)$ is a vector space over $\mathbb{R}$.

Just for this moment we say that function $u : (0,T) \to X_T$ is measurable if $\tilde{u}(t) \in X_t$ a.e., where $u(t) = (\tilde{u}(t), t)$, and function $t \mapsto \|\tilde{u}(t)\|_t$ is measurable. Define $L^p((0,T);\{X_t\})$ as a set of all measurable functions such that $\|\tilde{u}(t)\|_t \in L^p((0,T))$. Then we define the Sobolev space $W^{1,p}((0,T);\{X_t\})$ as all functions from $L^p((0,T);\{X_t\})$ for which there exists a function $g \in L^p((0,T))$ so that

$$\|\tilde{u}(t) - P(t,s)\tilde{u}(s)\|_t \leq \int_s^t g(\tau)\, d\tau, \tag{2.2}$$

for almost all $s, t \in (0,T)$, $s \leq t$.

At this stage we can not say much about the introduced space. Indeed, this is even not necessarily a vector space. However, if an additional regularity on families $\{X_t\}$, $\{P(t,s)\}$ is involved one could derive a more meaningful construction.

In this work regularity assumptions are following. We consider a specific family of Banach spaces obtained as a norm-completion of quotients of the same vector space under different semi-norms. Moreover, these semi-norms are assumed to form a monotone family, which guarantees the existence of appropriate nestings (they are induced by natural embeddings). This allows us to define the Sobolev space as a subspace of L^p-direct integral.

2.2. Monotone family $\{X_t\}$

Let V be a vector space, $(0,T)$ be an interval (not necessarily bounded) equipped with the Lebesgue measure, and $\{||\cdot||_t\}_{t\in(0,T)}$ be a family of semi-norms on V. We will assume that for each $v \in V$ the function $\rho(t,v) = \|v\|_t$ is non-increasing:

$$\rho(t_1, v) \geq \rho(t_2, v), \text{ if } t_1 \leq t_2. \tag{2.3}$$

Define Banach space X_t to be a completion $V/\ker(\|\cdot\|_t)$ with respect to $\|\cdot\|_t$. Then, $\{X_t\}$ is said to be a monotone family of Banach spaces (or a monotone Banach family).

For $t_1 \leq t_2$ there are the natural embeddings of normed spaces $V/\ker(\|\cdot\|_{t_1}) \to V/\ker(\|\cdot\|_{t_2})$, which are written as

$$P(t_2,t_1)v = \begin{cases} v, & \text{if } \|v\|_{t_2} \neq 0, \\ 0, & \text{if } \|v\|_{t_2} = 0. \end{cases}$$

So $\|P(t_2,t_1)v\|_{t_2} \leq \|v\|_{t_1}$, and, thus, there are extensions $P(t_2,t_1) : X_{t_1} \to X_{t_2}$. It is clear that the constructed operators are nestings (in sense of subsection 2.1). Then we treat $(\bigcup_t X_t, +)$ as a vector space while defining addition for $x_i \in X_{t_i}$

$$x_1 + x_2 := \begin{cases} P(t_2,t_1)x_1 + x_2, & \text{if } t_1 \leq t_2, \\ x_1 + P(t_1,t_2)x_2, & \text{if } t_1 > t_2. \end{cases} \tag{2.4}$$

Here we omit the notation of previous subsection and use x_1 instead of (x_1, t_1) implicitly assuming that x_1 'knows' the space it belongs to.

2.3. L^p-direct integral of Banach spaces

We deal with the L^p-spaces of mappings $f : (0,T) \to \bigcup_t X_t$ with the property that $f(t) \in X_t$ for each $t \in (0,T)$ (in other words, f is a section of $\{X_t\}$). To make this treatment rigorous, we apply the concept of direct integral of Banach spaces. A brief account of the theory of direct integral is given below (for detailed presentation see [17] and [10]).

Note that monotonicity condition (2.3) implies that $\{X_t\}$ is a measurable family of Banach spaces over $((0,T), dt, V)$ in the sense of [17, Section 6.1].

Definition 2.1. A *simple section* is a section f for which there exist $n \in \mathbb{N}$, $v_1, \dots, v_n \in V$, and measurable sets $A_1, \dots, A_n \subset (0,T)$ such that $f(t) = \sum_{k=1}^n \chi_{A_k} \cdot v_k$ for all $t \in (0,T)$.

Definition 2.2. A section f of $\{X_t\}_{t\in(0,T)}$ is said to be *measurable* if there exists a sequence of simple sections $\{f_k\}_{k\in\mathbb{N}}$ such that for almost all $t \in (0,T)$, $f_k(t) \to f(t)$ in X_t as $k \to \infty$.

The space of all equivalence classes of such measurable sections is a *direct integral* $\int_{(0,T)}^{\oplus} X_t\, dt$ of a monotone family of Banach spaces $\{X_t\}_{t\in(0,T)}$. We will denote this space as $L^0((0,T); \{X_t\})$.

Note that for a measurable section f the function $t \mapsto \|f(t)\|_t$ is measurable in the usual sense. For every $p \in [1,\infty]$ the space $L^p((0,T); \{X_t\}) = \left(\int_{(0,T)}^{\oplus} X_t\, dt\right)_{L^p}$ (L^p*-direct integral*) is defined as a space of all measurable sections f such that the function $t \mapsto \|f(t)\|_t$ belongs to $L^p((0,T))$. In this case

$$\|f\|_{L^p((0,T);\{X_t\})} := \begin{cases} \left(\int_0^T \|f(t)\|_t^p\, dt\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{(0,T)} |f(t)|, & \text{if } p = \infty \end{cases}$$

determines the norm on $L^p((0,T); \{X_t\})$.

Proposition 2.3 *([10, Proposition 3.2]). The space $L^p((0,T); \{X_t\})$ is a Banach space for all $1 \le p < \infty$.*

Define *sectional weak convergence* $f_n(t) \rightharpoonup f(t)$ for a.e. $t \in (0,T)$ as $\langle b'(t), f_n(t)\rangle_t \to \langle b'(t), f(t)\rangle_t$ a.e. for all $b'(t) \in X_t'$. Then applying a standard technique one can prove the following proposition.

Proposition 2.4. *Let $f_n \in L^p((0,T); \{X_t\})$, $\|f_n\|_{L^p((0,T);\{X_t\})} \le C < \infty$ and $f_n(t) \rightharpoonup f(t)$ for a.e. $t \in (0,T)$. Then $f \in L^p((0,T); \{X_t\})$ and $\|f\|_{L^p((0,T);\{X_t\})} \le C$.*

2.4. Sobolev space $W^{1,p}((0,T); \{X_t\})$

As it was pointed out in the introduction, we adapt (1.1) and obtain the definition of Newtonian space for functions valued in a monotone family.

Definition 2.5. A measurable section $u : (0,T) \to \bigcup_t X_t$ is said to be in the Sobolev space $W^{1,p}((0,T); \{X_t\})$ if $u \in L^p((0,T); \{X_t\})$, and if there exists a function $g \in L^p((0,T))$ so that

$$\|u(t) - P(t,s)u(s)\|_t \le \int_s^t g(\tau)\, d\tau, \tag{2.5}$$

for almost all $s, t \in (0, T)$, $s \le t$ (for all $s \le t$ from $(0, T) \setminus \Sigma$, where $|\Sigma| = 0$).

A function g satisfying (2.5) is called a *p-integrable upper gradient* of u (or just upper gradient). If u is a function in $W^{1,p}((0, T); \{X_t\})$, let

$$\|u\|_{W^{1,p}((0,T);\{X_t\})} := \|u\|_{L^p((0,T);\{X_t\})} + \inf_g \|g\|_{L^p((0,T))},$$

where the infimum is taken over all p-integrable upper gradients g of u. It is assumed that $W^{1,p}((0, T); \{X_t\})$ consists of equivalence classes of functions, where $f_1 \sim f_2$ means $\|f_1 - f_2\|_{W^{1,p}((0,T);\{X_t\})} = 0$. Thus $W^{1,p}((0, T); \{X_t\})$ is a normed space, and it is a subspace of L^p-direct integral of the measurable family $\{X_t\}$. In the case of reflexive spaces, we prove that it is a Banach space (see Theorem 4.6).

2.5. Examples

Here we will provide some more or less explicit examples of nested families of Banach spaces, which turn out to be monotone as well.

Example 2.1. Let $\{\Omega_t\}_{t\in(0,T)}$ be a non-increasing family of measurable sets: $\Omega_t \subset \Omega_s$ if $s < t$ (see Fig. 1). Let $\Omega_0 = \bigcup_t \Omega_t$. As a core vector space V, we choose the space of step functions on Ω_0 and define semi-norms $\rho(t, v) = \|v\|_{L^q(\Omega_t)}$. Then, family $\{L^q(\Omega_t)\}$ is monotone and operators $P(t, s) : L^q(\Omega_s) \to L^q(\Omega_t)$ are the restriction operators: $P(t, s)f = f_{|\Omega_t}$ for $f \in L^q(\Omega_s)$.

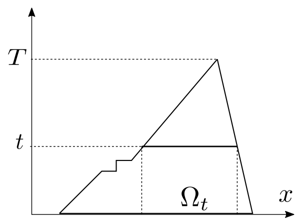

Fig. 1. Example 2.1.

Moreover, any element $u \in L^p((0, T); \{L^q(\Omega_t)\})$ could be represented as a function $u(t, x)$ belonging to mixed norm Lebesgue space $L^{p,q}(\Omega)$, where $\Omega = \bigcup_t t \times \Omega_t$.

Example 2.2 *(Evolving spaces).* In [3], an abstract framework has been developed for treating parabolic PDEs on evolving Hilbert spaces. Some applications of this method are in [1,2,11].

Here we compare the compatibility property from [3] to our construction. As in [3], let $\{X(t)\}_{t\in[0,T]}$ be a family of Hilbert spaces and $\phi_t : X(0) \to X(t)$ a family of linear isomorphisms. For all $v \in X(0)$ and $u \in X(t)$ the following conditions are assumed

(C1) $\|\phi_t v\|_{X(t)} \le C\|v\|_{X(0)}$,
(C2) $\|\phi_t^{-1} u\|_{X(0)} \le C\|u\|_{X(t)}$,
(C3) $t \mapsto \|\phi_t v\|_{X(t)}$ is continuous.

On the one hand, we can not formulate this structure in our settings straightway. On the other hand, we can construct another family of Banach spaces such that L^2-direct integral of this family is isomorphic to space L^2_X form [3, Definition 2.7]. Set $V = X(0)$, $\rho(t, v) = \|\phi_t v\|_{X(t)}$, and $P(t, s) = \phi_t \phi_s^{-1}$. Then condition (C3) implies that $\{(X(t), \|\cdot\|_{X(t)})\}$ is a measurable family.

Example 2.3 *(Composition operator).* Let Ω_0 be a domain in $\mathbb{R}^n$. Let us consider the Sobolev space $W^{1,q}(\Omega_0;\mathbb{R})$ as a core vector space V. We are going to construct a monotone family of Banach spaces which is generated by a family of quasi-isometric mappings $\varphi(t,\cdot) : \Omega_0 \to \Omega_t$. Each of these mappings induces isomorphism $C_{\varphi(t,\cdot)} : W^{1,q}(\Omega_t) \to W^{1,q}(\Omega_0)$ by the composition rule ([29, Theorem 4]). Define $\rho(t,v) = \|C^{-1}_{\varphi(t,\cdot)}v\|_{W^{1,q}(\Omega_t)}$, and choose mappings $\varphi(t,\cdot)$ such that the family of norms is monotone. As a result, we obtain spaces X_t, which consist of functions from $W^{1,q}(\Omega_0;\mathbb{R})$ and endowed with the norm $\|C^{-1}_{\varphi(t,\cdot)}v\|_{W^{1,q}(\Omega_t)}$. Thus we can define the Sobolev space $W^{1,p}((0,T);\{X_t\})$ over this family in the sense of 2.5. In that case, operators $P(t,s) : W^{1,q}(\Omega_s) \to W^{1,q}(\Omega_t)$ are composition operators induced by $\varphi(s,\cdot)\circ\varphi^{-1}(t,\cdot) : \Omega_t \to \Omega_s$.

Example 2.4 *(Monotone family of Hilbert spaces).* The next example is taken from [7]. In that work, Cauchy-Dirichlet problems for linear Schrödinger-type equations in non-cylindrical domains are studied. Note that the monotonicity condition is important for their considerations. Let $Q \subset \mathbb{R}^n \times (0,T)$ be an open set and its sections $Q_t = \{x \in R^n : (x,t) \in Q\}$ be a non-decreasing family. Define $Q_T = \bigcup_t Q_t$ (see Fig. 2). Let $V = H^1_0(Q_T)$ and V_t be a completion of $\{v \in C^\infty_0(Q_T) : \operatorname{supp} v \subset Q_t\}$ with respect to the norm $\|\cdot\|_{H^1_0(Q_T)}$. Let $\pi(t) : H^1_0(\Omega_T) \to V_t$ be an orthogonal projector, then define $\rho(t,v) = \|\pi(t)v\|_{H^1_0(Q_T)}$. At the same time, V_t is a completion $V/\ker(\rho(t,\cdot))$ with respect to $\rho(t,\cdot)$. Therefore, operators $P(t,s)$ are just trivial extensions to Q_t.

Example 2.5 *(Nested Hilbert spaces).* In [13], A. Grossmann has introduced nested Hilbert spaces. A simple example is the following: for $t \in (0,T)$ let H_t be the Hilbert space of measurable functions such that

$$\|f\|^2_{H_t} = \int_{\mathbb{R}^n} |f(x)|^2 \exp(-t|x|)\, dx < 0.$$

Then, $\{H_t\}$ is a nested family with nestings $P(t,s)$ being the natural embeddings which associate to every $f \in H_s$ the same function considered as an element of H_t. By Grossmann a nested Hilbert space is an algebraic inductive limit H_T. Note that spaces H_t could be obtained as a norm-completion of the space of measurable functions. Thus, $\{H_t\}$ is a monotone family and we could proceed with our construction upon the nested space H_T.

There are nested families which do not originate from a monotone family of semi-norms. For example, take a family of Euclidean spaces $E_t = \mathbb{R}^{1+\left[\frac{1}{T-t}\right]}$.

3. Calculus of $\{X_t\}$-valued functions

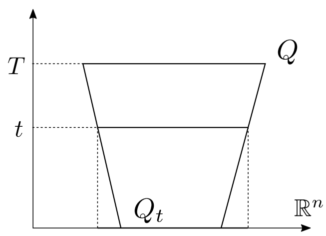

Fig. 2. Example 2.4.

In this section and below, we will denote as $t_1 \vee t_2$ the maximum of these numbers and as $t_1 \wedge t_2$ the minimum.

3.1. Limit and continuity

Here we introduce the concepts of limit, continuity and differentiability for $\{X_t\}$-valued functions. Due to the addition defined in (2.4), all basic properties are preserved for those notions.

Definition 3.1. An element $\xi \in X_{t_0}$ is said to be the limit of a section $f(t)$ for $t \to t_0$: $\lim\limits_{t\to t_0} f(t) = \xi$, if $\|f(t) - \xi\|_{t_0 \vee t} \to 0$ as $t \to t_0$.

Definition 3.2. A section $f(t)$ is continuous at the point $t_0 \in (0, T)$, if

$$\lim_{t\to t_0} f(t) = f(t_0) \in X_{t_0}.$$

By $C(J; \{X_t\})$ we will denote the set of continuous functions at every point of $J \subset (0, T)$.

Definition 3.3 *(Fréchet derivative).* A section $f(t)$ is differentiable at $t_0 \in (0, T)$ if there exists $l_{t_0} \in X_{t_0}$ and, for every $\varepsilon > 0$, exists $\delta > 0$ such that

$$\|f(t_0 + h) - f(t_0) - l_{t_0} h\|_{t_0 \vee (t_0+h)} \le \varepsilon |h|$$

for all $|h| \le \delta$. In what follows we denote $l_{t_0} = \dfrac{df}{dt}(t_0)$.

Definition 3.4. Let $[a, b] \subset (0, T)$ be a bounded interval. A function $f : [a, b] \to \bigcup_{t\in[a,b]} X_t$ is said to be *absolutely continuous*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\|_{b_i} \le \varepsilon$ for any collection of disjoint intervals $\{[a_i, b_i]\} \subset [a, b]$ such that $\sum_{i=1}^{n} (b_i - a_i) \le \delta$.

A function $f : J \to \bigcup_{t\in J} X_t$ is said to be *locally absolutely continuous* on a set J, if it is absolutely continuous for any interval $[a, b] \subset J$.

3.2. Local Bochner integral

Let there be given a simple function $s(t) = \sum_{i=1}^{m} v_i \chi_{E_i}$, where $v_i \in V$ and $\{E_i\} \subset (0, T)$ is a disjointed collection of measurable sets of finite measure. Then the integral is defined as

$$\int_0^T s(t)\, dt = \sum_{i=1}^{m} v_i |E_i|.$$

Now we introduce the notion of local integrability for a measurable section.

Definition 3.5. A measurable function $f \in L^0((0, T); \{X_t\})$ is called *locally integrable*, if for every compact set $J \subset (0, T)$ there exists a sequence of simple functions $\{s_k(t)\}$ such that

$$\int_0^T \|\chi_J(t) \cdot f(t) - s_k(t)\|_t \, dt \to 0 \quad \text{for } k \to \infty. \tag{3.1}$$

Note that if the function f is locally integrable, then for the sequence from Definition 3.5 we have $\lim\limits_{k\to\infty} \int_0^T \|s_k(t)\|_t \, dt = \int_0^T \|\chi_J(t) \cdot f(t)\|_t \, dt$.

Proposition 3.6. *Let $f \in L^0((0,T);\{X_t\})$ be locally integrable. Then, for every compact set $J \subset (0,T)$, there exists $x \in X_{t^*}$, $t^* = \sup J$, such that, for any sequence of simple functions $s_k(t)$ with the property $\int_0^T \|\chi_J(t) \cdot f(t) - s_k(t)\|_t \, dt \to 0$, the following convergence holds*

$$\Big\| \int_0^T s_k(t)\, dt - x \Big\|_{t^*} \to 0 \quad \textit{for } k \to \infty.$$

Proof. 1) Let $J \subset (0,T)$ be a compact set and $t^* = \sup J$. Next, we prove that the sequence $\int_0^T s_k(t)\, dt$ is fundamental in X_{t^*}.

$$\begin{aligned}\Big\| \int_0^T s_k(t)\, dt - \int_0^T s_m(t)\, dt \Big\|_{t^*} &\le \int_0^T \|s_k(t) - s_m(t)\|_{t^*}\, dt \\ &\le \int_0^T \|\chi_J(t) \cdot f(t) - s_m(t)\|_t \, dt + \int_0^T \|\chi_J(t) \cdot f(t) - s_k(t)\|_t \, dt \to 0\end{aligned}$$

for $k, m \to \infty$. Hence, there is $x \in X_{t^*}$ such that $\lim\limits_{k\to\infty} \int_0^T s_k(t)\, dt = x$ in X_{t^*}.

2) If, for another sequence of simple functions $r_k(t)$, it is true that $\int_0^T \|\chi_J(t) \cdot f(t) - r_k(t)\|_t \, dt \to 0$ for $k \to \infty$, then

$$\begin{aligned}\Big\| \int_0^T r_k(t)\, dt - x \Big\|_{t^*} &\le \int_0^T \|\chi_J(t) \cdot f(t) - r_k(t)\|_t \, dt \\ &+ \int_0^T \|\chi_J(t) \cdot f(t) - s_k(t)\|_t \, dt + \Big\| \int_0^T s_k(t)\, dt - x \Big\|_{t^*} \to 0. \quad \square\end{aligned}$$

Definition 3.7. The integral over compact set $J \subset (0,T)$ of a locally integrable function is an element x from Proposition 3.6, i.e.

$$\int_J f\, dt = \int_0^T \chi_J(t) \cdot f(t)\, dt := \lim_{k\to\infty} \int_0^T s_k(t)\, dt = x \in X_{t^*}, \tag{3.2}$$

where $t^* = \sup J$.

We say that $f \in L^1_{loc}((0,T);\{X_t\})$ if $\|f(t)\|_t \in L^1_{loc}((0,T))$. In essence, the introduced integral is a local version of the Bochner integral. Integral (3.2) has the usual additivity property. Thus, for two intervals $(a, b_1) \subset (a, b_2) \subset (0,T)$, we have the equality

$$\int_a^{b_1} f(t)\, dt - \int_a^{b_2} f(t)\, dt = -\int_{b_1}^{b_2} f(t)\, dt$$

Let us prove an analog of Bochner's theorem:

Theorem 3.8. *A measurable function $f \in L^0((0,T);\{X_t\})$ is locally integrable if and only if $f \in L^1_{loc}((0,T);\{X_t\})$. For any compact set $J \subset (0,T)$, the estimate holds*

$$\left\| \int_J f(t)\,dt \right\|_{t^*} \le \int_J \|f(t)\|_{t^*}\,dt \le \int_J \|f(t)\|_t\,dt,$$

where $t^* = \sup J$*.*

Proof. Let $f(t)$ be locally integrable. For arbitrary compact set $J \subset (0,T)$ there is a sequence of simple functions $s_k(t)$ such that convergence (3.1) holds. Then

$$\int_J \|f(t)\|_t\,dt \le \int_0^T \|\chi_J(t)\cdot f(t) - s_k(t)\|_t\,dt + \int_0^T \|s_k(t)\|_t\,dt.$$

The right hand of inequality is finite, thus $f \in L^1_{loc}((0,T);\{X_t\})$.

Now let $f \in L^1_{loc}((0,T);\{X_t\})$. Consider a sequence of simple functions $\{s_k(t)\}$ such that $\|f(t)-s_k(t)\|_t \to 0$ a.e. Let $J \subset (0,T)$ be a compact set. Define a new sequence of simple functions

$$r_k(t) = \begin{cases} s_k(t), & \text{if } \|s_k(t)\|_t \le 2\|f(t)\|_t \text{ and } t \in J,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\|\chi_J(t)\cdot f(t) - r_k(t)\|_t \to 0$ a.e. Further, $\|\chi_J(t)\cdot f(t) - r_k(t)\|_t \le \|r_k(t)\|_t + \|\chi_J(t)\cdot f(t)\|_t \le 3\chi_J(t)\cdot\|f(t)\|_t$ a.e. So $\|\chi_J(t)\cdot f(t) - r_k(t)\|_t$ has an integrable majorant, and by the Lebesgue theorem we obtain

$$\int_0^T \|\chi_J(t)\cdot f(t) - r_k(t)\|_t\,dt \to 0 \quad \text{for } k\to\infty.$$

Hence, the function $f(t)$ is locally integrable. □

Remark 3.9. Let $f(t)$ be locally integrable. Then, for any compact set $J \subset (0,T)$ a sequence of simple functions $\{s_k(t)\}$ as in Definition 3.5 can be chosen so that $\operatorname{supp} s_k(t) \subset J$ and $\|s_k(t)\|_t \le 2\|f(t)\|_t$.

Proposition 3.10 *(Dominated convergence theorem). Let* $f_n \in L^0((0,T);\{X_t\})$ *be a sequence such that* $f_n(t)$ *converges to* $f(t)$ *in* X_t *and* $\|f_n(t)\|_t \le g(t)$*,* $g(t) \in L^1((0,T))$*, for almost all* $t \in J$*,* J *is a compact subset of* $(0,T)$*. Then*

$$\lim_{n\to\infty} \int_J f_n\,dt = \int_J f\,dt.$$

Proof. Due to the Theorem 3.8, f_n is locally integrable. As a pointwise limit of measurable functions, f is also measurable, and $\|f(t)\|_t$ is dominated by $g(t)$, which implies locally integrability of f.

Applying Fatou lemma, we obtain

$$\limsup_{n\to\infty} \int_J \|f(t) - f_n(t)\|_t\,dt \le \int_J \limsup_{n\to\infty} \|f(t) - f_n(t)\|_t\,dt = 0,$$

which implies $\lim\limits_{n\to\infty} \int_J \|f(t) - f_n(t)\|_t\,dt = 0$.

Therefore,

$$\lim_{n\to\infty} \left\| \int_J f(t)\,dt - \int_J f_n(t)\,dt \right\|_{t^*} = \lim_{n\to\infty} \left\| \int_J (f(t) - f_n(t))\,dt \right\|_{t^*} \le \lim_{n\to\infty} \int_J \|f(t) - f_n(t)\|_t\,dt.$$

Then, we conclude that $\lim\limits_{n\to\infty} \int_J f_n\, dt = \int_J f\, dt$. $\square$

Proposition 3.11 *(Lebesgue's differentiation theorem). Let $f \in L^1_{loc}((0,T);\{X_t\})$. Then, for $h > 0$,*

$$\lim_{h\to 0} \frac{1}{h} \int\limits_{t-h}^{t} \|f(s) - f(t)\|_t\, ds = 0. \tag{3.3}$$

In particular,

$$f(t) = \lim_{h\to 0} \frac{1}{h} \int\limits_{t-h}^{t} f(s)\, ds. \tag{3.4}$$

Proof. Let us choose a sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ that is dense in X_t. For every $n \in \mathbb{N}$, we consider real-valued function $\|f(t) - x_n(t)\|_t$. Applying the real-valued Lebesgue's differentiation theorem, we can find a set $E_n \subset \mathbb{R}$ for all $n \in \mathbb{N}$ such that

$$\|f(t) - x_n(t)\|_t = \lim_{h\to 0} \frac{1}{h} \int\limits_{t-h}^{t} \|f(s) - x_n(t)\|_t\, ds$$

for all $t \notin E_n$. Further, for arbitrary $\varepsilon > 0$ and $t \notin \bigcup_{n\in\mathbb{N}} E_n$, there is a number n such that $\|f(t)-x_n(t)\|_t < \frac{\varepsilon}{2}$. Using the inequality from Remark 3.9, we have

$$\begin{aligned} 0 &\le \lim_{h\to 0} \frac{1}{h} \int\limits_{t-h}^{t} \|f(s) - f(t)\|_t\, ds \\ &\le \lim_{h\to 0} \frac{1}{h} \int\limits_{t-h}^{t} \|f(s) - x_n(t)\|_t + \|x_n(t) - f(t)\|_t\, ds = 2\|f(t) - x_n(t)\|_t < \varepsilon. \end{aligned}$$

Due to arbitrariness of choosing ε, statement (3.3) of the theorem is proven. The second assertion follows from the first and from Theorem 3.8. $\square$

Proposition 3.12. *Let f belong to $L^p(\mathbb{R};\{X_t\})$, $1 \le p < \infty$. For every $h > 0$ we define a new function $M_h f$ as*

$$M_h f(t) = \frac{1}{h} \int\limits_{t-h}^{t} f(s)\, ds.$$

Then $M_h f$ belongs to $L^p(\mathbb{R};\{X_t\}) \cap C(\mathbb{R};\{X_t\})$ and $\lim\limits_{h\to 0} M_h f = f$ a.e. and in $L^p(\mathbb{R};\{X_t\})$.

Proposition 3.13. *Let $g \in L^1_{loc}((0,T);\{X_t\})$ and $t_0 \in (0,T)$. Define a function $f(t) = \int_{t_0}^{t} g(s)\, ds$ for $t \ge t_0$. Then*

1) $f \in C(\{t \ge t_0\} \cap (0,T);\{X_t\})$,
2) f is locally absolutely continuous on $\{t \ge t_0\} \cap (0,T)$,
3) $\int_0^T \varphi'(t) f(t)\, dt = \int_0^T \varphi(t) g(t)\, dt$ for all $\varphi \in C_0^\infty(\{t \ge t_0\} \cap (0,T))$,
4) f is differentiable a.e. on $\{t \ge t_0\} \cap (0,T)$ and $\dfrac{df}{dt}(t) = g(t)$.

Proof. 1) According to Definition 3.2 of continuity for any $t_1 \in \{t \geq t_0\} \cap (0, T)$, we obtain

$$\left\| \int_{t_0}^{t} g(s)\, ds - \int_{t_0}^{t_1} g(s)\, ds \right\|_{t_1 \vee t} = \left\| \int_{t \wedge t_1}^{t \vee t_1} g(s)\, ds \right\|_{t_1 \vee t} \leq \int_{t \wedge t_1}^{t \vee t_1} \|g(s)\|_s\, ds \to 0 \quad \text{for } t \to t_1.$$

2) This assertion is also verified by definition

$$\sum_{i=1}^{n} \|f(b_i) - f(a_i)\|_{b_i} = \sum_{i=1}^{n} \left\| \int_{a_i}^{b_i} g(s)\, ds \right\|_{b_i} \leq \sum_{i=1}^{n} \int_{a_i}^{b_i} \|g(s)\|_s\, ds.$$

Thus, the statement follows from the absolute continuity of the Lebesgue integral.

3) Let $\varphi \in C_0^\infty(\{t \geq t_0\} \cap (0, T))$. We choose $h_* > 0$ so that $\operatorname{supp} \varphi(t + h_*) \subset \{t \geq t_0\} \cap (0, T)$. By Proposition 3.12

$$\begin{aligned}
\int_0^T \varphi'(t) f(t)\, dt &= \int_0^T \lim_{h \to 0, h < h_*} \frac{\varphi(t+h) - \varphi(t)}{h} f(t)\, dt \\
&= \lim_{h \to 0, h < h_*} \left(\int_0^T \frac{\varphi(t+h)}{h} f(t)\, dt - \int_0^T \frac{\varphi(t)}{h} f(t)\, dt \right) \\
&= \lim_{h \to 0, h < h_*} \left(\int_0^T \frac{\varphi(t)}{h} f(t-h)\, dt - \int_0^T \frac{\varphi(t)}{h} f(t)\, dt \right) \\
&= - \lim_{h \to 0, h < h_*} \left(\int_0^T \varphi(t) \frac{f(t-h) - f(t)}{h}\, dt \right) \\
&= - \lim_{h \to 0, h < h_*} \left(\int_0^T \varphi(t) M_h g(t)\, dt \right) \\
&= - \int_0^T \varphi(t) g(t)\, dt.
\end{aligned}$$

4) Let us verify differentiability by Definition 3.3. Fix $\varepsilon > 0$ and $t_1 \in \{t \geq t_0\} \cap (0, T)$. By Lebesgue Theorem 3.11, there is $\delta > 0$ such that, for all $|h| < \delta$,

$$\frac{1}{h} \int_{t_1}^{t_1+h} \|g(s) - g(t_1)\|_{t_1+h}\, ds < \varepsilon. \tag{3.5}$$

Then, for $h > 0$ (the case $h < 0$ can be considered similarly), we obtain

$$\begin{aligned}
\|f(t_1 + h) - f(t_1) - hg(t_1)\|_{t_1 \vee (t_1+h)} &= \left\| \int_{t_1}^{t_1+h} g(s)\, ds - hg(t_1) \right\|_{t_1+h} \left\| \int_{t_1}^{t_1+h} g(s) - g(t_1)\, ds \right\|_{t_1+h} \\
&\leq |h| \frac{1}{|h|} \int_{t_1}^{t_1+h} \|g(s) - g(t_1)\|_{t_1+h}\, ds \leq |h| \varepsilon. \quad \square
\end{aligned}$$

4. Sobolev space $W^{1,p}((0,T);\{X_t\})$ via weak derivative

Here we show that for a Sobolev function from $W^{1,p}((0,T);\{X_t\})$ there exists a weak derivative, which is a section of $\{X_t\}$ and belongs to $L^p((0,T);\{X_t\})$. To do that, we adapt the classical scheme for Banach valued functions by using the concept of local Bochner integral, for example, see [9].

4.1. Weak derivatives

Definition 4.1. Let $f \in L^1_{loc}((0,T);\{X_t\})$. A function $g \in L^1_{loc}((0,T);\{X_t\})$ is called *a weak derivative of f* (the usual notation $g = f'$), if for all $\varphi \in C_0^\infty((0,T))$ the next equality holds

$$\left\| \int\limits_0^T \varphi'(t)f(t)\,dt + \int\limits_0^T \varphi(t)g(t)\,dt \right\|_{t^*} = 0, \tag{4.1}$$

where $t^* = \sup\{\operatorname{supp}\varphi\}$.

Proposition 4.2. *Let $f \in L^1_{loc}((0,T);\{X_t\})$ and a weak derivative $f'(t) = 0$ a.e. on interval $J \subset (0,T)$. Then, there exists an element $x_0 \in \bigcap\limits_{t\in J} X_t$ such that*

$$\int\limits_0^T \varphi(t)f(t)\,dt = x_0 \int\limits_0^T \varphi(t)\,dt$$

for all $\varphi \in C_0^\infty(J)$. In other words, $f(t) = x_0$ a.e. on J.

Proof. Let $b \in J$. Let us choose $\vartheta \in C_0^\infty(J)$ such that $b = \sup\{\operatorname{supp}\varphi(t)\}$ and $\int_J \vartheta(t)\,dt = 1$. Let $x_0 = \int_J \vartheta(t)f(t)\,dt \in X_b$.

For an arbitrary function $\varphi \in C_0^\infty(J)$, we define a new function

$$\psi(t) = \int\limits_{t_0}^t \left(\varphi(s) - \vartheta(s)\int\limits_0^T \varphi(\sigma)\,d\sigma\right) ds,$$

where $t_0 = \inf\{\operatorname{supp}\varphi\}$. Then, $\psi \in C_0^\infty(J)$ and $\psi'(t) = \varphi(t) - \vartheta(t)\int_0^T \varphi(\sigma)\,d\sigma$. By the hypothesis of the theorem

$$\begin{aligned} 0 &= \int\limits_0^T \psi(t)f'(t)\,dt = \int\limits_0^T \psi'(t)f(t)\,dt \\ &= \int\limits_0^T \varphi(t)f(t)\,dt - \int\limits_0^T \varphi(\sigma)\,d\sigma \cdot \int\limits_0^T \vartheta(t)f(t)\,dt. \end{aligned}$$

Therefore $\int_0^T \varphi(t)f(t)\,dt = x_0 \int_0^T \varphi(\sigma)\,d\sigma$. Due to arbitrariness of b, we conclude that $x_0 \in \bigcap\limits_{t\in J} X_t$. $\square$

Proposition 4.3. *Let $1 \le p \le \infty$ and let $u \in L^p((0,T);\{X_t\})$ have a weak derivative $u' \in L^p((0,T);\{X_t\})$. Then, the equality*

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\, ds$$

holds for almost all $t_0, t \in (0, T)$, $t_0 \le t$. Moreover

$$\|u(t) - u(t_0)\|_t \le \int_{t_0}^{t} \|u'(s)\|_t\, ds \le \int_{t_0}^{t} \|u'(s)\|_s\, ds, \tag{4.2}$$

and particularly $u \in W^{1,p}((0, T); \{X_t\})$.

Proof. Let $t_1 \in (0, T)$. Define functions $f(t) = \int_{t_1}^{t} u'(s)\, ds$ and $r(t) = u(t) - f(t)$ for all $t > t_1$. By Proposition 3.13

$$\int_0^T \varphi'(t)r(t)\, dt = \int_0^T \varphi'(t)u(t)\, dt - \int_0^T \varphi'(t)f(t)\, dt$$
$$= -\int_0^T \varphi(t)u'(t)\, dt + \int_0^T \varphi'(t)u(t)\, dt = 0$$

for all $\varphi \in C_0^\infty(\{t_0 \le t\})$.

Due to Proposition 4.2 there exists $x_0 \in X_{t_1}$ such that $w(t) = x_0$ for almost all $t \ge t_1$. Hence $u(t) = u(t_0) + \int_{t_1}^{t} u'(s)\, ds$ for almost all $t \ge t_1$. Choose such a point $t_0 \ge t_1$ which satisfies the equation. $\square$

Proposition 4.4. *Let $u \in L^p((0, T); \{X_t\})$ for some $1 \le p \le \infty$, then the following statements are equivalent*

(i) There exists a weak derivative $u' \in L^p((0, T); \{X_t\})$.

(ii) Function u locally absolutely continuous on $(0, T)$, differentiable a.e. and the derivative $\dfrac{du}{dt} \in L^p((0, T); \{X_t\})$.

(iii) There exists a function $v \in L^p((0, T); \{X_t\})$ such that for a.e. $b \in (0, T)$ and each $x'(b) \in X'_b$ the function $\psi_b(t) = \langle x'(b), u(t)\rangle_b$ is locally absolutely continuous and $\psi'_b(t) = \langle x'(b), v(t)\rangle_b$ for a. e. $t \le b$.

(iv) There exists a function $v \in L^p((0, T); \{X_t\})$ such that for a.e. $b \in (0, T)$ and each $e'(b) \in E'_b \subset X'_b$ (E'_b is a countable dense subset of X'_b) the function $\psi_b(t) = \langle e'(b), u(t)\rangle_b$ is locally absolutely continuous and $\psi'_b(t) = \langle e'(b), v(t)\rangle_b$ for a.e. $t \le b$.

Proof. (i) ⇒ (ii) Thanks to Proposition 4.3 $u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\, ds$ for almost all $t, t_0 \in (0, T)$, $t_0 \le t$. Then it follows from Proposition 3.13 that function u locally absolutely continuous and differentiable a.e. on $(0, T)$. As well $\dfrac{du}{dt} = u' \in L^p((0, T); \{X_t\})$.

(ii) ⇒ (iii) Note that

$$|\langle x'(b), u(t_2)\rangle_b - \langle x'(b), u(t_1)\rangle_b| \le \|x'(b)\| \cdot \|u(t_2) - u(t_1)\|$$

holds for $t_1 \le t_2 \le b$. This implies that function $\psi_b(t)$ is locally absolutely continuous. Compute the derivative:

$$\psi'_b(t) = \lim_{h\to 0+} \frac{\psi_b(t) - \psi_b(t-h)}{h}$$
$$= \lim_{h\to 0+} \langle x'(b), \frac{u(t) - u(t-h)}{h}\rangle_b = \langle x'(b), \frac{du}{dt}(t)\rangle_b.$$

Take $v = \frac{du}{dt} \in L^p((0,T);\{X_t\})$.

(iii) $\Rightarrow$ (iv) clear.

(iv) $\Rightarrow$ (i) We show that $v = u'$. Let $\varphi \in C_0^\infty((0,T))$ and $b = \sup\{\operatorname{supp}\varphi\}$. Then for any $e'(b) \in E_b' \subset X_b'$

$$\langle e'(b), \int\limits_0^T \varphi' u \, dt\rangle_b = \int\limits_0^T \varphi' \langle e'(b), u(t)\rangle_b \, dt = -\int\limits_0^T \varphi \langle e'(b), v(t)\rangle_b \, dt.$$

Consequently, $\int_0^T \varphi' u \, dt = \int_0^T \varphi v \, dt$ in X_b as desired. $\square$

We are now ready to formulate and prove our first main result.

Theorem 4.5. *Let $\{X_t\}$ be a monotone family of reflexive Banach spaces. If $u \in W^{1,p}((0,T);\{X_t\})$, then there exists a section $v(t) \in L^p((0,T);\{X_t\})$ such that $\|v(t)\|_t$ is an upper gradient of u and $\|v\|_{L^p((0,T);\{X_t\})} = \inf\limits_g \|g\|_{L^p((0,T))}$.*

Proof. 1. Prove that we can choose a continuous representative u. Let $\Sigma \subset (0,T)$ be the null set where (2.5) fails. Observe that thanks to inequality (2.5) for any convergence sequence $\{s_n\} \subset (0,T) \setminus \Sigma$ we have

$$\|u(s_n) - u(s_m)\|_{s_n \vee s_m} \to 0 \quad \text{when } n, m \to \infty. \tag{4.3}$$

For each point $t \in \Sigma$ we choose an increasing sequence $\{s_n(t)\} \subset (0,T) \setminus \Sigma$ such that $\lim\limits_{n\to\infty} s_n(t) = t$. Due to (4.3) $P(t, s_n(t))u(s_n(t))$ is a Cauchy sequence in X_t, and, hence, there is a limit $\lim\limits_{n\to\infty} P(t, s_n(t))u(s_n(t)) \in X_t$. Note that for any other increasing sequence $\{\zeta_n\} \subset (0,T) \setminus \Sigma$ converging to t it is true that $\lim\limits_{n\to\infty} u(\zeta_n) = \lim\limits_{n\to\infty} u(s_n(t))$. The function

$$\bar{u}(t) := \begin{cases} u(t), & \text{if } t \in (0,T) \setminus \Sigma, \\ \lim\limits_{n\to\infty} P(t, s_n(t))u(s_n(t)), & \text{if } t \in \Sigma \end{cases}$$

coincides with $u(t)$ a.e. on $(0,T)$.

Now prove that (2.5) holds for the function $\bar{u}(t)$ everywhere on $(0,T)$. Suppose that $t, t_0 \in (0,T)$ and $t_0 < t$. Then $s_n(t_0) < s_n(t)$ for large n and

$$\begin{aligned} \|\bar{u}(t) - \bar{u}(t_0)\|_t &\le \|\bar{u}(t) - \bar{u}(s_n(t))\|_t + \|\bar{u}(t_0) - \bar{u}(s_n(t_0))\|_t + \|\bar{u}(s(t)) - \bar{u}(s_n(t_0))\|_t \\ &\le \|\bar{u}(t) - \bar{u}(s_n(t))\|_t + \|\bar{u}(t_0) - \bar{u}(s_n(t_0))\|_t + \int\limits_{s_n(t_0)}^{s_n(t)} g(s)\, ds \to \int\limits_{t_0}^{t} g(s)\, ds, \quad \text{as } n \to \infty. \end{aligned}$$

This also implies the continuity of the function $\bar{u}(t)$ on $(0,T)$. Thus, we can assume that the function $u(t)$ is continuous, and the inequality (2.5) is valid everywhere on the interval $(0,T)$. Due to the continuity, $u((0,T)) \cap X_{t_1} = u(\{t \le t_1\} \cap (0,T))$ is a separable space for any $t_1 \in (0,T)$. From now, we will deal with $\tilde{X}_{t_1} = u(\{t \le t_1\} \cap (0,T))$. Note that thanks to the reflexivity of X_{t_1}, $\tilde{X}_{t_1}'$ is separable.

2. Here we will show that the family of difference quotients is bounded in L^p. Moreover, it is bounded uniformly.

For $h > 0$, define the function $u_h(t) = \frac{u(t) - u(t-h)}{h} \in X_t$. Let $J \subset (0,T)$ and $\operatorname{dist}(J, \{0,T\}) > h$. Applying (2.5) and Hölder's inequality, derive

$$\|u_h(t)\|_t^p = \frac{1}{h^p}\|u(t)-u(t-h)\|_t^p \le \frac{1}{h^p}\bigg(\int\limits_{t-h}^{t} g(s)\,ds\bigg)^p \le \frac{1}{h}\int\limits_{t-h}^{t} |g(s)|^p\,ds$$

for all $t \in J$. Next, with the help of Fubini's theorem we obtain $\|u_h\|_{L^p(J;\{X_t\})} \le \|g\|_{L^p((0,T))}$, which means that $\{u_h\}$ is a bounded family in $L^p(J;\{X_t\})$.

Let $F \subset (0,T)$ be such a set of null measure that

$$g(t) = \lim_{h\to 0}\frac{1}{h}\int\limits_{t-h}^{t} g(s)\,ds, \quad \text{for all } t \in (0,T)\setminus F.$$

Then, the inequality $\|u_h(t)\|_t \le \frac{1}{h}\int_{t-h}^{t} g(s)\,ds$ guarantees the uniform estimate $\|u_h(t)\|_t \le K_t$ for all $t \in (0,T)\setminus F$ and small h.

3. To apply Proposition 4.4 (i $\Leftrightarrow$ iv), we will show that sequence $u_h(t)$ has a weak limit in L^p, which is the desired derivative.

Fix $b \in (0,T)$. Let $\{x_n'(b)\}_{n\in\mathbb{N}}$ be a dense sequence in $\tilde{X}_b'$ (it is possible due to step 1). For $x_n'(b)$ and $t \le b$, define the function $\psi_{n,b}(t) = \langle x_n'(b), u(t)\rangle_b$. Note that

$$|\psi_{n,b}(t)-\psi_{n,b}(t_0)| \le \|x_n'(b)\|\int\limits_{t_0}^{t} g(s)\,ds.$$

Therefore, the function $\psi_{n,b}(t)$ is locally absolutely continuous on $t \le b$.

Since $\tilde{X}_b$ is reflexive, there exists a sequence $h_k \to 0$ and an element $w(t) \in X_t$ such that $u_{h_k}(t) \rightharpoonup w(t)$. Particularly,

$$\langle x_n'(b), w(t)\rangle_b = \lim_{k\to\infty}\langle x_n'(b), u_{h_k}(t)\rangle_b = \psi_{n,b}'(t) \quad \text{for all } t \in (0,T)\setminus F,$$

namely $\psi_{n,b}'(t) = \langle x_n'(b), w(t)\rangle_b$ a.e. on $t \le b$.

It only remains to verify that $w \in L^p((0,T);\{X_t\})$. For any sequence $h_m \to 0$ we have

$$\lim_{m\to\infty}\langle x_n'(b), u_{h_m}(t)\rangle_b = \psi_{n,b}'(t) = \langle x_n'(b), w(t)\rangle_b.$$

Let $x'(b) \in X_b'$ and $\varepsilon > 0$. Choose $x_n'(b)$ such that $\|x'(b)-x_n'(b)\| \le \varepsilon$. For small $h>0$ infer

$$\begin{aligned}|\langle x'(b), u_h(b)-w(b)\rangle_b| &\le |\langle x'(b)-x_n'(b), u_h(b)-w(b)\rangle_b| \\ &\quad + |\langle x_n'(b), u_h(b)-w(b)\rangle_b| \le \varepsilon(K_b+\|w(b)\|_b)+\varepsilon.\end{aligned}$$

Consequently, $u_h(b) \rightharpoonup w(b)$, and, as b was chosen arbitrarily, $u_h(t) \rightharpoonup w(t)$ for all $t \in (0,T)$. By Proposition 2.4, $w \in L^p((0,T);\{X_t\})$. And by Proposition 4.4, there exists a weak derivative $u' = w$.

4. From step 2 $\|\|u'(t)\|_t\|_{L^p((0,T))} = \|u'\|_{L^p((0,T);\{X_t\})} \le \|g\|_{L^p((0,T))}$ for any function g which satisfy (2.5). On the other hand, by inequality (4.2) we have

$$\|u(t)-u(t_0)\|_t \le \int\limits_{t_0}^{t} \|u'(s)\|_s\,ds.$$

Thus, $\|u'\|_{L^p((0,T);\{X_t\})} = \inf\limits_{g}\|g\|_{L^p((0,T))}$. $\square$

From the above proof, we obtain

$$\|u\|_{W^{1,p}((0,T);\{X_t\})} = \|u\|_{L^p((0,T);\{X_t\})} + \|u'\|_{L^p((0,T);\{X_t\})}.$$

Therefore, in the case of reflexive Banach spaces, our Definition 2.5 is equivalent to the definition in the standard form: space $W^{1,p}((0,T);\{X_t\})$ is a set of all functions in $L^p((0,T);\{X_t\})$ with weak derivatives which are also in $L^p((0,T);\{X_t\})$. Now, in the usual manner one can prove the following.

Theorem 4.6. *Let $\{X_t\}$ be a monotone family of reflexive Banach spaces. The space $W^{1,p}((0,T);\{X_t\})$ is a Banach space for all $1 \leq p < \infty$. In the case of Hilbert spaces X_t, $W^{1,2}((0,T);\{X_t\})$ is also a Hilbert space.*

4.2. Difference quotient criterion

We are going to make use of the so-called difference quotient criterion. This property is well known in the real-valued case [8, Proposition 9.3] and for vector case (see [21, Proposition 2.5.7] and [5, Theorem 2.2]). In our settings, we should additionally assume that Banach space $L^p((0,T);\{X_t\})$ has the Radon–Nikodým property. We use the following definition of this property. A Banach space Y has the Radon–Nikodým property if each Lipschitz continuous function $f : I \to Y$ is differentiable almost everywhere (a good account on this notion see in [6, Section 1.2] and in [21, Section 1.3]).

Define operator $u \mapsto \tau_h u$ by the rule $\tau_h u(t) = u(t+h)$.

Proposition 4.7. *1) If $u \in W^{1,p}((0,T);\{X_t\})$, then for all $J \subset (0,T)$ and $h \in \mathbb{R}$, $|h| < \operatorname{dist}(J,\{0,T\})$*

$$\|\tau_h u - u\|_{L^p(J;\{X_{t\vee t+h}\})} \leq Ch.$$

2) Let $L^p((0,T);\{X_t\})$ has the Radon–Nikodým property. If $u \in L^p((0,T);\{X_t\})$ and there exists a constant C such that for all $J \subset (0,T)$ and $h \in \mathbb{R}$, $|h| < \operatorname{dist}(J,\{0,T\})$

$$\|\tau_h u - u\|_{L^p(J;\{X_{t\vee t+h}\})} \leq Ch$$

then $u \in W^{1,p}((0,T);\{X_t\})$.

Although the proof of this proposition is a fairly straightforward adaptation of the proof of [5, Theorem 2.2], we will examine whether it goes through our setting or not.

Proof. 1) This part follows from step 2 of the proof of Theorem 4.5.

2) Let $J \subset J' \subset (0,T)$ and $\delta > 0$ s.t. $\delta < \operatorname{dist}(J,\partial J')$ define function $G : (0,\delta) \to L^p(J;\{X_t\})$ by $t \mapsto u(\cdot - t)$. Then, for $t,s \in (0,\delta)$

$$\|G(t)-G(s)\|_{L^p(J;\{X_t\})} = \Bigg(\int\limits_J \|u(\xi-t)-u(\xi-s)\|_\xi^p \, d\xi\Bigg)^{\frac{1}{p}}$$

$$= \Bigg(\int\limits_{J+s} \|u(\xi-t+s)-u(\xi)\|_{\xi+s}^p \, d\xi\Bigg)^{\frac{1}{p}} \leq \Bigg(\int\limits_{J'} \|u(\xi-t+s)-u(\xi)\|_{\xi\vee(\xi+s-t)}^p \, d\xi\Bigg)^{\frac{1}{p}}$$

$$= \|\tau_{s-t}u - u\|_{L^p(J';\{X_{\xi\vee(\xi+s-t)}\})} \leq C|s-t|.$$

Thus, function G is Lipschitz continuous, and due to the Radon–Nikodým property of $L^p((0,T);\{X_t\})$, there exists a derivative for almost all $s \in (0,\delta)$:

$$G'(s) = \lim_{h\to 0} \frac{u(\cdot - s + h) - u(\cdot - s)}{h} \in L^p(J; \{X_t\}). \tag{4.4}$$

Then, there exists a sequence of negative numbers $h_n \to 0$ such that $\frac{u(\xi+h_n)-u(\xi)}{h_n}$ converges to $g_J(\xi)$ in X_ξ for almost all $\xi \in J$. Here we use the fact that almost all $\xi \in J$ can be written as $\xi = \xi' - s$, for some $\xi' \in J$ and $s \in (0, \delta)$. Thanks to (4.4), g_J belongs to $L^p(J; \{X_t\})$ and $\|g_J\|_{L^p(J;\{X_t\})} \le C$. Given $\varphi \in C_0^\infty(J)$, Lebesgue's dominated convergence theorem (Proposition 3.10)

$$\begin{aligned}\int\limits_J \varphi(t) g_J(t)\, dt &= \lim_{n\to\infty} \int\limits_J \varphi(t) \frac{u(t+h_n) - u(t)}{h_n}\, dt \\ &= -\lim_{n\to\infty} \int\limits_J u(t) \frac{\varphi(t-h_n) - \varphi(t)}{h_n}\, dt = \int\limits_J \varphi'(t) u(t)\, dt. \end{aligned}\tag{4.5}$$

Now let $J_n \subset J_{n+1} \subset (0, T)$ be such that $\bigcup_n J_n = (0, T)$ and let g_{J_n} correspond to J_n as in the previous step. For any m, n functions g_{J_m}, g_{J_n} agree on $J_m \cap J_n$, and this allows us define $g(t) = g_{J_n}(t)$ whenever t in some J_n. Note that g is a measurable section and by the Fatou lemma $\|g\|_{L^p((0,T);\{X_t\})} \le C$. With the help of (4.5) we conclude that g is a weak derivative of u. Finally, by Proposition 4.3 $u \in W^{1,p}((0, T); \{X_t\})$.

For $p = \infty$ use the first part and $p \to \infty$. □

5. Isomorphism between $W^{1,p}((0,T); \{X_t\})$ and $W^{1,p}((0,T); Y)$

Here we establish requirements to the regularity of the family $\{X_t\}$, which allows us to construct the isomorphism $W^{1,p}((0, T); \{X_t\})$ onto a standard Sobolev space. To demonstrate these conditions, we give positive and negative examples.

5.1. Measurable family of operators

To introduce operators on $\{X_t\}$-valued functions, we need a notion of measurability for a family of operators. Let $\{Y_t\}$ be another measurable family of Banach spaces, originating from the measurable family of semi-norms $\{\|\cdot\|_{Y_t}\}_{t\in(0,T)}$ on a vector space $\widetilde{V}$.

Definition 5.1. A map $t \mapsto \Phi_t \in \mathcal{L}(X_t, Y_t)$ defines *a measurable family of bounded linear operators* if

1. For any $v \in V$, a map $t \mapsto \Phi_t v$ is a measurable section of $\{Y_t\}_{t\in(0,T)}$;
2. $\|\Phi_t\|_{\mathcal{B}(X_t,Y_t)} < \infty$ almost everywhere on $(0, T)$.

Lemma 5.2. *Let $\Phi_t : X_t \to Y_t$ be a measurable family of bounded linear operators, then for any measurable section $\zeta(t)$ of $\{X_t\}$ the image $\Phi_t\zeta(t)$ is a measurable section of $\{Y_t\}$.*

Proof. Let $\zeta(t)$ be a measurable section, then there is a sequence of simple sections $f_k(t) \to \zeta(t)$ in X_t a.e. on $(0, T)$. From Definition 5.1, we have $\Phi_t f_k(t)$ is a measurable section, and $\Phi_t f_k(t) \to \Phi_t\zeta(t)$ in Y_t a.e. on $(0, T)$. Thus, $\Phi_t\zeta(t)$ is a measurable section of $\{Y_t\}_{t\in(0,T)}$. □

In particular, Lemma 5.2 guarantees that the operator $\zeta \mapsto \Phi\zeta$, acting by the rule $\Phi\zeta(t) = \Phi_t\zeta(t)$, is well defined in the sense that the image of some measurable section is again a measurable section.

To define an appropriate norm of measurable operator family, we apply the concept of lattice supremum (this technique we have learned from [16, Section 2]).

Definition 5.3. Let $\mathcal{F}$ be a family of measurable functions on $(0,T)$. The lattice supremum $\bigvee \mathcal{F}$ is a function that satisfies the following two properties:

1. For any $f \in \mathcal{F}$, we have $f \leq \bigvee \mathcal{F}$ a.e. on $(0,T)$;
2. If g is a measurable function such that for all $f \in \mathcal{F}$ $f \leq g$ a.e. on $(0,T)$, then $\bigvee \mathcal{F} \leq g$ a.e. on $(0,T)$.

The set $\mathcal{M}$ of all measurable families of bounded linear operators as in Definition 5.1 is an L^∞-module. Define a mapping $N : \mathcal{M} \to L^0((0,T))^+$ which maps a measurable family to a nonnegative measurable function by the rule

$$N(\Phi)(t) := \bigvee \left\{ \frac{\|\Phi_t v\|_{Y_t}}{\|v\|_{X_t}} : v \in V \right\},$$

with the convention that $\frac{\|\Phi_t v\|_{Y_t}}{\|v\|_{X_t}} = 0$ whenever $\|v\|_{X_t} = 0$. $\mathcal{M}$ equipped with the random norm N is a randomly normed space [17, Chapter 5].

Remark 5.4. Here we briefly explain the necessity of using the lattice supremum. Observe that $\|\Phi_t f(t)\|_{Y_t} \leq C\|f(t)\|_{X_t}$ a.e. does not imply an estimate $\|\Phi_t\|_{\mathcal{B}(X_t,Y_t)} \leq C$ a.e., because each f could have its own null-set. Otherwise, it does imply that $N(\Phi) \leq C$ a.e.

Lemma 5.5. *Let $\Phi_t : X_t \to Y_t$ be a measurable family of bounded linear operators, then its norm can be calculated in the following way*

$$N(\Phi)(t) = \bigvee \left\{ \frac{\|\Phi_t \zeta(t)\|_{Y_t}}{\|\zeta(t)\|_{X_t}} : \zeta \textit{ is a measurable section of } \{X_t\} \right\}. \tag{5.1}$$

Proof. Denote the right hand side of (5.1) as $K(t)$. It is clear that for any $v \in V$

$$\frac{\|\Phi_t v\|_{Y_t}}{\|v\|_{X_t}} \leq K(t). \tag{5.2}$$

Let $g : (0,T) \to \mathbb{R}$ be a measurable function such that for any $v \in V$ an inequality $\frac{\|\Phi_t v\|_{Y_t}}{\|v\|_{X_t}} \leq g(t)$ holds a.e. on $(0,T)$. For any simple section $f(t) = \sum_{k=1}^n \chi_{A_k}(t) \cdot v_k$, we have

$$\frac{\|\Phi_t f(t)\|_{Y_t}}{\|f(t)\|_{X_t}} = \sum_{k=1}^n \frac{\|\Phi_t v_k\|_{Y_t}}{\|v_k\|_{X_t}} \chi_{A_k}(t) \leq g(t) \quad \text{a.e. on } (0,T).$$

Now if $\zeta(t)$ is a measurable section then there is a sequence of simple sections which converges a.e. on $(0,T)$. Therefore, $\frac{\|\Phi_t \zeta(t)\|_{Y_t}}{\|\zeta(t)\|_{X_t}} \leq g(t)$ a.e. and

$$K(t) \leq g(t). \tag{5.3}$$

Thus, by Definition 5.3 equations (5.2), (5.3) imply (5.1). $\square$

A useful consequence from above lemma is that for any measurable section ζ

$$\|\Phi_t f(t)\|_{Y_t} \leq N(\Phi)(t)\|\zeta(t)\|_{X_t} \quad \text{a.e. on } (0,T).$$

5.2. Embeddings between edges

Let

$$\|v\|_T := \lim_{t\to T-} \|v\|_t \quad \text{and} \quad \|v\|_0 := \lim_{t\to 0+} \|v\|_t,$$

for $v \in V$. Define $(X_T, \|\cdot\|_T)$ and $(X_0, \|\cdot\|_0)$ as completions of corresponding quotient spaces. However, in critical cases, the first one could contain only zero, while the last one would be empty. Then, we have the following trivial embeddings.

Proposition 5.6. *The maps*

$$\begin{aligned} u(t) &\mapsto P(t,0)u(t) \quad \textit{from } W^{1,p}((0,T);X_0) \textit{ to } W^{1,p}((0,T);\{X_t\}) \\ u(t) &\mapsto P(T,t)u(t) \quad \textit{from } W^{1,p}((0,T);\{X_t\}) \textit{ to } W^{1,p}((0,T);X_T) \end{aligned}$$

are both bounded operators.

Proof. First, note that both families are measurable (in the sense of Definition 5.1). Let $u \in W^{1,p}((0,T);X_0)$. Then, $\|P(t,0)u(t)\|_t \le \|u(t)\|_0$ and thus $P(t,0)u(t) \in L^p((0,T);\{X_t\})$. By assumption, there is the derivative $u'(t) \in L^p((0,T);X_0)$. Then

$$\begin{aligned} \|P(t,0)u(t) - P(t,s)P(s,0)u(s)\|_t &\le \|P(t,0)u(t) - P(t,0)u(s)\|_t \\ &\le \|u(t)-u(s)\|_0 \le \int_s^t \|u'(\tau)\|_0 \, d\tau. \end{aligned}$$

Now, let $u \in W^{1,p}((0,T);\{X_t\})$. Then $\|P(T,t)u(t)\|_T \le \|u(t)\|_t$ for all $t \in (0,T)$. So $P(T,t)u(t) \in L^p((0,T);X_T)$. Check that there is a derivative in L^p

$$\begin{aligned} \|P(T,t)u(t) - P(T,s)u(s)\|_T &= \|P(T,t)u(t) - P(T,t)P(t,s)u(s)\|_T \\ &\le \|u(t) - P(t,s)u(s)\|_T \\ &\le \|u(t) - P(t,s)u(s)\|_t \le \int_s^t g(\tau)\, d\tau. \end{aligned}$$

Consequently, $P(T,t)u \in W^{1,p}((0,T);X_0)$. $\square$

5.3. Isomorphism with a standard Sobolev space

We say that $\{\Phi_t\}$ is *a measurable family of isomorphisms* between measurable families $\{X_t\}$ and $\{Y_t\}$ if $\Phi_t : X_t \to Y_t$ is an isomorphism for almost all $t \in (0,T)$, and families $\{\Phi_t\}$ and $\{\Phi_t^{-1}\}$ are measurable (in the sense of Definition 5.1). Under isomorphism between normed spaces, we mean a bijective bounded linear operator with bounded inverse.

Proposition 5.7. *Let $\Phi_t : X_t \to Y_t$ be a measurable family of isomorphisms. If $N(\Phi)$ is in $L^\infty((0,T))$ then operator $(\Phi u)(t) = \Phi_t u(t)$ is an isomorphism from $L^p((0,T);\{X_t\})$ to $L^p((0,T);\{Y_t\})$.*

Proof. It is clear that if Φ_t are bijective, then so is Φ. If $u \in L^p((0,T);\{X_t\})$, then

$$\begin{aligned}\|\Phi u\|^p_{L^p((0,T);\{Y_t\})} &= \int\limits_0^T \|\Phi_t u(t)\|_{Y_t}\,dt \le \int\limits_0^T N(\Phi)(t)\|u(t)\|_{X_t}\,dt \\ &\le \|N(\Phi)\|_{L^\infty}\|u\|^p_{L^p((0,T);\{X_t\})}.\end{aligned}$$

Thus operator Φ is bounded and by the Open Mapping theorem, we have that Φ^{-1} is bounded. □

Corollary 5.8. *Let $\Phi_t : X_t \to Y$ be a measurable family of isomorphisms, and Y possesses the Radon–Nikodým property. If $N(\Phi)$ is in $L^\infty((0,T))$, then $L^p((0,T);\{X_t\})$ has the Radon–Nikodým property.*

We want to know when the operator $\Phi : W^{1,p}((0,T);\{X(t)\}) \to W^{1,p}((0,T);Y)$ defined by the rule $(\Phi u)(t) = \Phi_t u(t)$ is an isomorphism. Necessary and sufficient conditions are given in the following.

Theorem 5.9. *Let $1 < p \le \infty$ and Y has the Radon–Nikodým property. Then a measurable family of isomorphisms $\Phi_t : X_t \to Y$ induces an isomorphism $\Phi : W^{1,p}((0,T);\{X_t\}) \to W^{1,p}((0,T);Y)$ defined by the rule $(\Phi u)(t) = \Phi_t u(t)$ if and only if $N(\Phi), N(\Phi^{-1}) \in L^\infty((0,T))$ and*

$$N(\Phi_t P(t,s) - \Phi_s)(s) \le M(t-s) \tag{5.4}$$

for almost all $s < t$, and some constant M.

Proof. Necessity. First, we show that $N(\Phi)$ is essentially bounded. Fix a cut-off function $\eta \in C_0^\infty(\mathbb{R})$ equal to 1 on $B(0,1)$ and 0 outside the ball $B(0,2)$. By substituting the functions $u_r(t) = \eta(\frac{t-z}{r})v$, where $v \in V$ and $B(z,2r) \subset (0,T)$, into the inequality $\|\Phi u_r\|_{L^p(Y)} \le K\|u_r\|_{W^{1,p}((0,T);\{X_t\})}$, derive

$$\int\limits_{|z-t|<r} \|\Phi_t v\|^p_Y\,dt \le CK \int\limits_{|z-t|<2r} \|v\|^p_{X_t}\,dt.$$

Applying the Lebesgue differentiation theorem, we infer for every $v \in V$

$$\frac{\|\Phi_z v\|_Y}{\|v\|_{X_z}} \le C_1 \text{ a.e. on } (0,T), \tag{5.5}$$

and this means $N(\Phi) \in L^\infty((0,T))$. Similarly $N(\Phi^{-1}) \in L^\infty((0,T))$.

Now, note that $(\Phi u_r)'(t) = \frac{1}{r}\eta'(\frac{t-z}{r})\Phi_t v + \eta(\frac{t-z}{r})(\Phi_t v)'$. In the same manner as in the previous step, we obtain for every $v \in V$

$$\|(\Phi_z v)'\|_Y \le C_1\|v\|_{X_z} \text{ a.e. on } (0,T). \tag{5.6}$$

Next, from (5.6) and the fact that $\Phi v \in W^{1,p}((0,T);Y)$, for $v \in V$ we have

$$\begin{aligned}\|(\Phi_t P(t,s) - \Phi_s)v\|_Y = \|\Phi_t v - \Phi_s v\|_Y &\le \int\limits_s^t \|(\Phi_\tau v)'\|_Y\,d\tau \\ &\le C_1\int\limits_s^t \|v\|_{X_\tau}\,d\tau \le C_1\|v\|_{X_s}(t-s),\end{aligned} \tag{5.7}$$

which holds for all $t > s$ from $(0,T)\setminus\Sigma$, and $|\Sigma| = 0$. Observe that for each $t \in (0,T)\setminus\Sigma$ the family of operators $\Phi_t P(t,s) - \Phi_s$ is measurable. Therefore, by (5.7) $N(\Phi_t P(t,s) - \Phi_s)(s) \le C_1(t-s)$ for almost all $t > s$.

Sufficiency. First, we prove that (5.8) ensues control for inverse

$$N(\Phi_t^{-1} - P(t,s)\Phi_s^{-1})(t) \le M'(t-s). \tag{5.8}$$

Indeed,

$$\begin{aligned}
\|\Phi_t^{-1} - P(t,s)\Phi_s^{-1} v\|_{X_t} &= \|\left(\Phi_t^{-1} - P(t,s)\Phi_s^{-1}\right)\Phi_s\Phi_s^{-1}v\|_{X_t} \\
&= \|\Phi_t^{-1}\left(\Phi_s - \Phi_t P(t,s)\right)\Phi_s^{-1}v\|_{X_t} \le N(\Phi^{-1})(t)\|\left(\Phi_s - \Phi_t P(t,s)\right)\Phi_s^{-1}v\|_Y \\
&\le N(\Phi^{-1})(t)N(\Phi_t P(t,s) - \Phi_s)(s)\|\Phi_s^{-1}v\|_{X_s} \\
&\le N(\Phi^{-1})(t)N(\Phi_t P(t,s) - \Phi_s)(s)N(\Phi^{-1})(t)\|v\|_Y \le \|N(\Phi^{-1})\|^2_{L^\infty} M(t-s)\|v\|_Y.
\end{aligned}$$

Due to Proposition 5.7, Φ is an isomorphism from $L^p((0,T);\{X_t\})$ to $L^p((0,T);Y)$. In particular, this implies that $L^p((0,T);\{X_t\})$ has the Radon–Nikodým property. Now, let $u \in W^{1,p}((0,T);\{X(t)\})$, then

$$\begin{aligned}
&\|\Phi_{s+h}u(s+h) - \Phi_s u(s)\|_Y \\
&\qquad \le \|\Phi_{s+h}(u(s+h) - P(s,s+h)u(s))\|_Y + \|\Phi_{s+h}P(s,s+h)u(s) - \Phi_s u(s)\|_Y \\
&\qquad \le C(s+h)\|u(s+h) - P(s,s+h)u(s)\|_{X_{s+h}} + N(\Phi_{s+h}P(s,s+h) - \Phi_s)(s)\cdot\|u(s)\|_{X_s} \\
&\qquad \le C(s+h)\int_s^{s+h} g(\tau)\,d\tau + Mh\|u(s)\|_{X_s}.
\end{aligned}$$

Calculating L^p-norm on $J \Subset (0,T)$ such that $|h| < \operatorname{dist}(J,\{0,T\})$, we obtain

$$\|\Phi_{\cdot+h}u(\cdot+h) - \Phi_\cdot u(\cdot)\|_{L^p(J;Y)} \le \left(\operatorname*{ess\,sup}_{(0,T)} N(\Phi)(t)\|g\|_{L^p((0,T))} + M\|u\|_{L^p((0,T),\{X_t\})}\right)h.$$

Due to [5, Theorem 2.2] or Proposition 4.7 $\Phi u \in W^{1,p}((0,T);Y)$.

In the same manner for operator $(\Phi^{-1}u)(t) = \Phi_t^{-1}u(t)$ we have inequality

$$\|\Phi^{-1}_{\cdot+h}u(\cdot+h) - \Phi^{-1}_\cdot u(\cdot)\|_{L^p(J;\{X_{t\vee(t+h)}\})} \le \left(\operatorname*{ess\,sup}_{(0,T)} N(\Phi^{-1})(t)\|\|u'\|_Y\|_{L^p((0,T))} + M'\|u\|_{L^p((0,T),Y)}\right)h,$$

for any $u \in W^{1,p}((0,T);Y)$. Applying Corollary 5.8 and Proposition 4.7, we conclude $\Phi^{-1}u \in W^{1,p}((0,T);\{X_t\})$. $\square$

We point out that the above proof significantly uses the fact that functions from $W^{1,p}((0,T);Y)$ possess weak derivatives. Thus, additional considerations should be made to replace Y by $\{Y_t\}$.

5.4. Examples

Let us apply the last theorem to the two examples from section 2.

Example 5.1 *(Composition operator).* First, we examine Example 2.3. Since all spaces X_t from this example consist of the same functions as space $W^{1,q}(\Omega_0)$, we want to check if the constructed space $W^{1,p}((0,T);\{X_t\})$

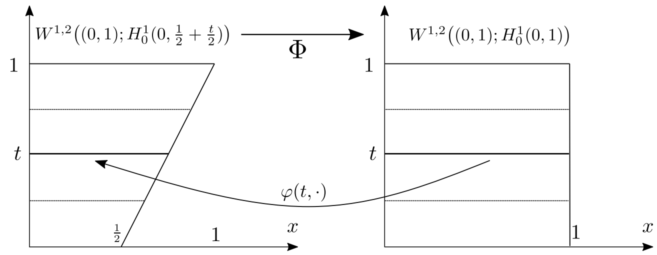

Fig. 3. ϕ maps a cylindrical domain to a non-cylindrical one, and it preserves the order of layers. Operator Φ acts between Sobolev spaces, but fails to be bounded.

is isomorphic to space $W^{1,p}((0,T);W^{1,q}(\Omega_0))$ under operator $(\Phi u)(t) = w(t)u(t)$ (here $w : (0,T) \to \mathbb{R}$ is some weight function, which we use just for the demonstration of Theorem 5.9). So we choose operators of multiplication by constant $w(t)I : W^{1,q}(\Omega_t) \to W^{1,q}(\Omega_0)$ as $\Phi_t : X_t \to Y$, $Y = W^{1,q}(\Omega_0)$. Then, applying 5.9, we obtain the conditions: w is Lipschitz function bounded from 0.

Example 5.2 *(Monotone family of Hilbert spaces).* As in Example 2.4, let $\{Q_t\}$ be a non-decreasing family in $\mathbb{R}^n$, and $W^{1,2}((0,T);\{H^1_0(Q_t)\})$ is a Sobolev space in a non-cylindrical domain (see Fig. 3). We want to ask the following question: if is it possible to construct isomorphism $\Phi u(t) = C_{\varphi(t,\cdot)}u(t)$ from this space to a Sobolev space in a cylindrical domain with the help of composition operators between inner spaces $C_{\varphi(t,\cdot)} : H^1_0(Q_t) \to H^1_0(Q_T)$? To answer this question, we consider the mapping $\varphi(t,x) : (0,T) \times Q_T \to \bigcup t \times Q_t$ with the property that for every $t \in (0,T)$ mapping $\varphi(t,\cdot) : Q_T \to Q_t$ is a quasi-isometry. Then, for each t, operator $C_{\varphi(t,\cdot)}$ is an isomorphism [29, Theorem 4].

Unfortunately, Theorem 5.9 gives us a negative answer to the above question. The essential obstacle is that assumption (5.4) in the case of composition operators is equivalent to the demand on additional derivatives. To demonstrate this, we consider a simple example. Let Q_t be line segments $[0, 1/2 + t/2]$, where $t \in (0,1)$, and $\varphi(t,x) = \frac{(1+t)x}{2}$.

We will show that (5.4) fails. To do this, we fix $0 < s < t < 1$ and a point $a \in (0, 1/2 + s/2)$. Take a sequence $f_n(x) = C|x-a|^{\frac{2}{3}}\eta_0(x)\eta_n(x)$, $n \in \mathbb{N}$, where:

- $\eta_0 \in C^\infty_0(Q_s)$ and $\eta_0 = 1$ on $[\delta, 1/2 + s/2 - \delta]$,
- $\eta_n \in C^\infty(Q_s)$ and $\eta_n(x) = 0$ for $x \in [a - 1/n, a + 1/n]$ and $\eta_n(x) = 1$ for $x \in Q_s \setminus [a - 2/n, a + 2/n]$,
- the constant C such that $\|f_n\|_{H^1_0(Q_s)} \le 1$.

Then L^2-norm of f''_n tends to infinity. Using this fact and the Taylor expansion in $1/2(1+s)x$ we obtain

$$\begin{aligned}
&\frac{1}{t-s}\big\|(C_{\varphi(t,\cdot)}P(t,s) - C_{\varphi(s,\cdot)})f_n\big\|_{H^1_0(Q_T)}\\
&\qquad = \frac{1}{t-s}\big\|f_n(1/2(1+t)x) - f_n(1/2(1+s)x)\big\|_{H^1_0(Q_T)}\\
&\qquad \ge \frac{1}{2}\frac{1}{t-s}\big\|(1+t)f'_n(1/2(1+t)x) - (1+s)f'_n(1/2(1+s)x)\|_{L^2(Q_T)}\\
&\qquad = \frac{1}{2}\frac{1}{t-s}\big\|f''_n(1/2(1+s)x)\frac{1}{2}(t-s)x + o((t-s)x)\\
&\qquad + (t-s)f'_n(1/2(1+s)x) + tf''_n(1/2(1+s)x)\frac{1}{2}(t-s)x + o(t(t-s)x)\big\|_{L^2(Q_T)}\\
&\qquad = \frac{1}{2}\big\|(1+t)f''_n(1/2(1+s)x)\frac{1}{2}x + f'_n(1/2(1+s)x) + o(x)\big\|_{L^2(Q_T)} \to \infty \text{ as } n \to \infty.
\end{aligned}$$

6. Scalar characterization

The space $L^p((0,T);\{X_t\})$ admits a scalar description: a measurable section $u(t)$ belongs to $L^p((0,T);\{X_t\})$ iff $\|u(t)\|_t \in L^p((0,T))$. In general case, there is no such characterization for the Sobolev space $W^{1,p}((0,T);\{X_t\})$. Nevertheless, if $u \in W^{1,p}((0,T);\{X_t\})$, then the norm $\|u(t)\|_t$ enjoys some regularity properties. For example, in general settings by rather standard methods one can prove the following inequality:

$$\|u(t)\|_{L^\infty((0,T);\{X_t\})} \le C\|u\|_{1,p} + \|\partial_t \rho(t,u(t))\|_{L^p((0,T);\{X_t\})}$$

for any $u \in W^{1,p}((0,T);\{X_t\})$, $1 \le p < \infty$.

We can obtain the scalar characterization in the simplest form when additional conditions are imposed on the norm function. Namely, the following theorem holds.

Theorem 6.1. *Let $u \in W^{1,p}((0,T);\{X_t\})$ and assume that*

(1) For any $v \in V$ function $t \mapsto \rho(t,v)$ belongs to $W^{1,p}((0,T))$;

(2) Weak derivatives $\partial_t \rho(t,v)$ have a majorant $H(t) \in L^p((0,T))$.

Then $\|u(t)\|_t \in W^{1,p}((0,T))$.

Proof. $\|u(t)\|_t \in L^p((0,T))$ by the definition.

Suppose that $t > s$. Then

$$\begin{aligned}\big|\|u(t)\|_t - \|u(s)\|_s\big| &\le \big|\|u(t)\|_t - \|u(s)\|_t\big| + \big|\|u(s)\|_t - \|u(s)\|_s\big| \\ &\le \|u(t)-u(s)\|_t + |\rho(t,u(s)) - \rho(s,u(s))| \\ &\le \int_s^t g_u(\tau)\,d\tau + \int_s^t |\partial_t \rho(\tau,u(s))|\,d\tau \le \int_s^t g_u(\tau) + H(\tau)\,d\tau. \quad \square\end{aligned}$$

Corollary 6.2. *Suppose (1) and (2) from Theorem 6.1 hold true. Then there exists a constant C such that*

$$\|u\|_{L^\infty((0,T);\{X_t\})} \le C\|u\|_{W^{1,p}((0,T);\{X_t\})}$$

for any $u \in W^{1,p}((0,T);\{X_t\})$.

The following theorem gives us another scalar characteristic and establishes a link with the approach of Yu. G. Reshetnyak to define Sobolev spaces of functions with values in a metric space [26].

Theorem 6.3. *If $u \in L^0((0,T);\{X_t\})$ and the following two assumptions hold:*

(A) for any $v \in V$ the function $\psi_v(t) = \|u(t)-v\|_t \in W^{1,p}((0,T))$,

(B) the family of derivatives $\{\psi_v'(t)\}_{v\in V}$ has a majorant $\psi' \in L^p((0,T))$,

then $u \in W^{1,p}((0,T);\{X_t\})$.

Proof. When $v = 0$, from (A) we derive $\|u(t)\|_t \in L^p((0,T))$, which implies $u \in L^p((0,T);\{X_t\})$. Next, (A) and (B) imply that for any $v \in V$

$$\big|\|u(t)-v\|_t - \|u(t_0)-v\|_{t_0}\big| \le \int_{t_0}^t |\psi_v'(s)|\,ds \le \int_{t_0}^t |\psi'(s)|\,ds. \tag{6.1}$$

Let $t \geq t_0$. Choose a sequence $\{v_k\} \subset V$ that $\|u(t_0) - v_k\|_{t_0} \to 0$ as $k \to \infty$. Then, proceeding to the limit in (6.1), we derive

$$\|u(t) - u(t_0)\|_t \leq \int\limits_{t_0}^{t} |\psi'(s)|\, ds. \quad \square$$

The converse to 6.3 does not hold in general. The following example shows that condition (A) fails in some cases.

Example 6.1. Let V be the vector space of all continuous functions defined on $(0,1)$, and $T = 1$. Define a family of norms on V

$$\|x\|_t = \begin{cases} \sup\limits_{(0,1)} |x(s)|, & 0 < t < 0.5 \\ \sup\limits_{(0,0.5)} |x(s)|, & 0.5 \leq t < 1, \end{cases}$$

where $x(s) \in C((0,1))$. Let D_t be a completion of $C(0,1)/\ker \|\cdot\|_t$ with respect to $\|\cdot\|_t$. Then, the function $u(t)(s) = s$ belongs to $W^{1,p}((0,1);\{D_t\})$, $u'(t) = 0$. On the other hand,

$$\|u(t)\|_t = \begin{cases} 1, & 0 < t < 0.5 \\ 0.5, & 0.5 \leq t < 1, \end{cases}$$

is obviously out of space $W^{1,p}((0,1))$.

Remark 6.4. An alternative way for applying Reshetnyak's approach is the following. Define the space $R^{1,p}((0,T);\{X_t\})$ is the class of all functions $f \in L^p((0,T);\{X_t\})$ such that:

(A) for every $x'(t) \in X_t'$, $\|x'(t)\|_t \leq 1$, the function $\psi_{x'}(t) := \langle x'(t), f\rangle_t$ belongs to $W^{1,p}((0,T))$; and
(B) there is a nonnegative function $g \in L^p((0,T))$ such that $|\psi_{x'}'| \leq g$ a.e. on $(0,T)$ for every $x'(t) \in X_t'$ with $\|x'(t)\|_t \leq 1$.

For correctness, one should introduce some notion of measurability for sections $x' : (0,T) \to \bigcup_t X_t'$. Then, a natural question arises: under what conditions spaces $W^{1,p}$ and $R^{1,p}$ coincides.

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